

### Solution to Problem 3.1#86.

Step 1.

As usual the first step of a solution of a word problem is to translate it into mathematical language. Let  $y$  denote the length of each of the two other sides in the figure given right after this problem. Then, using the hint concerning the area of an equilateral triangle, we find that the area of the window is given by

$$A(x, y) = xy + \frac{\sqrt{3}}{4}x^2. \quad (0.1)$$

We are also given that the perimeter of the window is 16 feet. Using the figure, this leads to

$$3x + 2y = 16. \quad (0.2)$$

Equation 0.2 allows us to express  $y$  in terms of  $x$ . Indeed, it yields,

$$y = \frac{16x - 3x}{2}. \quad (0.3)$$

Substituting formula (0.3) into formula (0.1) we arrive at

$$A(x) = x \frac{16x - 3x}{2} + \frac{\sqrt{3}}{4}x^2. \quad (0.4)$$

Rearranging terms in formula (0.4) we see that  $A(x)$  is the quadratic function given by

$$A(x) = \left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)x^2 - 8x. \quad (0.5)$$

Thus our word problem in mathematical language means to find the value of  $x$  for which quadratic function given by formula (0.5) is maximum.

Step 2.

Finding the the maximum of the quadratic function given by formula (0.5). We follow very closely the procedure of Example 1 of Section 3.1 of our Precalculus text. Of course, this procedure is to "complete the square", which many of you had in High School.

We know from Section 3.1 that such a quadratic function is a parabola. Note that the coefficient of  $x^2$  is negative,

$$\left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right) < 0. \quad (0.6)$$

Therefore the parabola opens down and the maximum is attained at the vertex. In other words, the problem is to find the vertex of this parabola.

To find the  $x$  - *coordinate* of the vertex of this parabola, first we factor out the coefficient of  $x^2$ . In other words we write,

$$\left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)x^2 + 8x = \left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right) \left(x^2 + \frac{8}{\left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)}\right). \quad (0.7)$$

To find the  $x$  - *coordinate* of the vertex of this parabola, second we "complete the square" of the second parenthesis of formula (0.7). For this pupose note that

$$\frac{1}{2} \frac{8}{\left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)} = \frac{4}{\left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)}. \quad (0.8)$$

Next we calculate

$$\left[ x + \frac{4}{\left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)} \right]^2 = x^2 + \frac{8}{\left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)} + \left( \frac{4}{\left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)} \right)^2. \quad (0.9)$$

If formula (0.9) looks too complicated please FOIL out the left side. Now comes the key observation of the "complete the square" procedure: In other words, we observe that the sum of the first two terms on the right of formula (0.9) is the second parenthesis on the right of formula (0.7). Therefore subtracting the third term on the right formula (0.9) from both sides of this formula, we arrive at

$$\left[ x + \frac{4}{\left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)} \right]^2 - \left( \frac{4}{\left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)} \right)^2 = x^2 + \frac{8}{\left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)}. \quad (0.10)$$

To find the  $x$  - *coordinate* of the vertex of this parabola, third we substitute formula (0.10) into formula (0.7). Then, we arrive at

$$\left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)x^2 + 8x = \left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right) \left( \left[ x + \frac{4}{\left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)} \right]^2 - \left( \frac{4}{\left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)} \right)^2 \right). \quad (0.11)$$

Step 3.

Combining formula (0.10) with the horizontal and vertical shift formulas of Section 2.5 we see that the  $x$  - *coordinate* of the vertex of this parabola is given by requirement that the square bracket is zero. In other words,

$$\left[ x + \frac{4}{\left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)} \right] = 0. \quad (0.12)$$

Clearly, this yields

$$x = -\frac{4}{\left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)}. \quad (0.13)$$

Note that there is no need to simplify this fraction.

Note also that the Problem 3.1#86 did not ask for a decimal approximation. At the same time, for a practicing architect this decimal number is important. But that is another issue.