<u>Another Proof of Heron's Formula</u>

By Justin Paro

In our text, <u>Precalculus</u> (fifth edition) by Michael Sullivan, a proof of Heron's Formula was presented. This proof invoked the Law of Cosines and the two half-angle formulas for sin and cos. Upon inspection, it was found that this formula could be proved a somewhat simpler way.

This alternate proof for Heron's Formula was first conceived from the task of finding a function of the Area of the triangle in terms of the three sides of the triangle. So, this proof will use a fundamental formula for the Area of a triangle, it will also use the Law of Cosines, and it will use the simple formula for the Difference of Two Squares.

From the Formula of the Area of a Triangle,

(1) $$\mathrm{A} = \frac{1}{2} ba \sin\gamma$$

and since $\sin\gamma$ can be expressed in terms of $\cos\gamma$ to get the equation,

(2) $$\sin\gamma = \sqrt{1-\cos^2\gamma}$$

by replacing $\sin\gamma$ in equation (1) with the right side of equation (2) we receive the equation,

(3) $$\mathrm{A} = \frac{1}{2} ba\sqrt{1-\cos^2\gamma}$$

From the Law of Cosines,

(4) $$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

we can solve for $\cos\gamma$ in the Law of Cosines and receive,

(5) $$\cos\gamma = \frac{a^2+b^2-c^2}{2ab}$$

And, by replacing the value for $\cos\gamma$ in equation (3) with the right side of equation (5), we receive an equation that is the Area of a triangle as a function of the three sides of the triangle,

$$(6) \qquad \mathrm{A} = \frac{1}{2}ba\sqrt{1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2}$$

While this proof so far is more elegant than the proof presented in our text, the formula is not stated as elegantly as Heron's formula, which says,

$$\mathrm{A} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = 1/2\,(a+b+c)$. To get from equation (6) to Heron's Formula is relatively simple when you invoke the simple formula for the Difference of Two Squares,

$$(7) \qquad x^2 - y^2 = (x+y)(x-y)$$

and also apply simple algebra.

From equation (6) we get,

$$(8) \qquad \mathrm{A} = \frac{1}{2}ba\sqrt{\frac{(2ab)^2-(a^2+b^2-c^2)^2}{(2ab)^2}}$$

and from (7), the terms x^2 and y^2 are replaced by the terms $(2ab)^2$ and $(a^2+b^2-c^2)$ in equation (8) to get,

$$(9) \qquad \mathrm{A} = \frac{1}{2}ba\sqrt{\frac{(2ab-a^2-b^2+c^2)(2ab+a^2+b^2-c^2)}{(2ab)^2}}$$

In order to factor $2ab-a^2-b^2$ into $(a-b)^2$, we need to factor out the negative one from the term $2ab-a^2-b^2+c^2$ in equation (9),

$$(10) \qquad \mathrm{A} = \frac{1}{2}ba\sqrt{\frac{-(a^2-2ab+b^2-c^2)(a^2+2ab+b^2-c^2)}{(2ab)^2}}$$

As was stated above, we can factor the terms $a^2 - 2ab + b^2$ and $a^2 + 2ab + b^2$ into $(a-b)^2$ and $(a+b)^2$. We then get the equation,

(11) $$\mathrm{A} = \frac{1}{2}ba\sqrt{\frac{-[(a-b)^2 - c^2][(a+b)^2 - c^2]}{(2ab)^2}}$$

The formula for The Difference of Two Squares can be invoked again and applied to equation (11). From equation (7), the x^2 and y^2 terms are replaced by $(a-b)^2 - c^2$ and $(a+b)^2 - c^2$ in equation (11) to get,

(12) $$\mathrm{A} = \frac{1}{2}ba\sqrt{\frac{-(a-b-c)(a-b+c)(a+b-c)(a+b+c)}{(2ab)^2}}$$

Now we multiply the negative one into the term $(a-b-c)$ to get,

(13) $$\mathrm{A} = \frac{1}{2}ba\sqrt{\frac{(b+c-a)(b+c-a)(a+b-c)(a+b+c)}{(2ab)^2}}$$

From Heron's Formula, $s = 1/2\,(a+b+c)$, and this implies that $2s = a+b+c$. In order to substitute 2s into equation (13), we need to express equation (13) as,

(14) $$\mathrm{A} = \frac{1}{2}ba\sqrt{\frac{(a+b+c-2a)(a+b+c-2b)(a+b+c-2c)(a+b+c)}{(2ab)^2}}$$

Since $2s = a+b+c$, we can substitute 2s into equation (14) to get,

(15) $$\mathrm{A} = \frac{1}{2}ba\sqrt{\frac{2s(2s-2a)(2s-2b)(2s-2c)}{(2ab)^2}}$$

and by factoring out the numeral 2 from the term $(2s-2a)(2s-2b)(2s-2c)$, we get,

(16) $$\mathrm{A} = \frac{1}{2}ba\sqrt{\frac{16s(s-a)(s-b)(s-c)}{(2ab)^2}}$$

Now, by performing the square root operation on the terms 16 and $(2ab)^2$ in equation (16), we get,

$$(17) \qquad A = \frac{4ba}{4ba}\sqrt{s(s-a)(s-b)(s-c)}$$

and by the act of cancellation, we receive,

$$(18) \qquad A = \sqrt{s(s-a)(s-b)(s-c)}$$

which is Heron's Formula.

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The idea behind trying a different approach to the proof of Heron's Formula was suggested by Professor Rejto, who was convinced that the complex method behind the proof in Precalculus (fifth edition) by Michael Sullivan was not necessary. Professor Rejto suggested that the Area of a triangle could be expressed as a function of its three sides using a formula for the Area and the Law of Cosines. The formula he came up with was valid, however it was not nearly as elegantly stated as in Heron's Formula.

Professor Rejto then approached me and asked me if I was willing to try to get to Heron's Formula from equation (6). I said I would attempt it, and I came up with this alternative means of getting to Heron's Formula. So, this proof was a joint effort between Professor Rejto and myself.