3. ~~$S(i,j,n)$~~

the phase diagram for this investment process is

invest	0	1	1	1	1
time period	0	1	2	3	n
interest			i	$2i$	$(n-1)i$

then $$S(i,j,n) = n + \sum_{t=1}^{n-1} (ti)(1+j)^{n-1-t}$$
$$= n + i \sum_{t=1}^{n-1} t(1+j)^{n-1-t}$$

Now we can adopt result from Q2.

We know that $\displaystyle\sum_{t=0}^{n-1} (t+1) q^t = \frac{1-(n+1)q^n + nq^{n+1}}{(1-q)^2}$

we need to find $\displaystyle\sum_{t=1}^{n-1} t(1+j)^{n-1-t}$,

actually, $\displaystyle\sum_{t=1}^{n-1} t(1+j)^{n-1-t}$

$$= \sum_{t=0}^{n-2} (t+1)(1+j)^{n-1-(t+1)}$$
$$= \sum_{t=0}^{n-2} (t+1)(1+j)^{n-t}$$
$$= (1+j)^n \sum_{t=0}^{n-2} (t+1)\left(\frac{1}{1+j}\right)^t$$

$$= (1+j)^n \cdot \frac{1 - (n-2+1)\left(\frac{1}{1+j}\right)^{n-2} + (n-2)\left(\frac{1}{1+j}\right)^{n-2+1}}{\left(1 - \frac{1}{1+j}\right)^2}$$

$$= (1+j)^{n+2} \cdot \frac{1 - (n-1)\left(\frac{1}{1+j}\right)^{n-2} + (n-2)\left(\frac{1}{1+j}\right)^{n-1}}{\frac{j^2}{\cancel{(1+j)^2}}}$$

$$= \frac{(1+j)^{n+2} - (n-1)(1+j)^4 + (n-2)(1+j)^3}{j^2}$$

therefore, we can have a closed form for

~~ES~~ $S(i, j, n)$

$$= n + \left(i/j^2\right)\left[(1+j)^{n+2} - (n-1)(1+j)^4 + (n-2)(1+j)^3\right]$$