A Note on Solution of Problem #3

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Problem 3: Let the matrix A be given by

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -6 & -2 \\ -2 & 7 & 5 \end{bmatrix}$$

(a) Find an orthonormal basis for the column space of A.
(b) Next, let the vector b be given by

$$b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Find the orthogonal projection of this vector, b, onto column space of A.

Solution: The second part of this problem asks to find the projection of vector b onto the column space of matrix A. In the following we solve this problem based on two methods. In the first method, we compute the orthonormal basis of the column space of matrix A and then project vector b onto the computed orthonormal basis. In the second method, we use Theorem $3L$ in our text [1] to find the projection matrix P and then use this matrix to find the projection of vector b onto the column space of matrix A.

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1 First Method

The orthonormal basis for the column space of matrix A is

$$q_1 = \frac{1}{\sqrt{24}} \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}, q_2 = \frac{1}{\sqrt{32}} \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}$$

Thus the projection of vector b onto the column space of A is

$$b_p = (b^T q_1)q_1 + (b^T q_2)q_2$$

$$b_p = \frac{6}{\sqrt{24}} q_1 + \frac{4}{\sqrt{32}} q_2 = \frac{6}{24} \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + \frac{4}{32} \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}$$

$$b_p = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

2 Second Method

Theorem 3L in our text says that **if the columns of matrix A are linearly independent,** the projection of a vector, b, onto the column space of A can be computed as

$$P = A(A^T A)^{-1} A^T$$

$$b_p = Pb$$

However, in this problem columns of matrix A are not linearly independent and therefore Theorem $3L$ cannot be applied directly. We should notice that projection depends only on the span of the columns of matrix A, therefore we can compute the projection matrix by considering the independent columns of matrix A. To do so, we define matrix B to be,

$$B = \begin{bmatrix} 2 & 1 \\ 4 & -6 \\ -2 & 7 \end{bmatrix}$$

whose columns are the independent columns of matrix A. Now the projection matrix can be computed as

$$P = B(B^T B)^{-1} B^T = \begin{bmatrix} 2 & 1 \\ 4 & -6 \\ -2 & 7 \end{bmatrix} \left(\begin{bmatrix} 2 & 4 & -2 \\ 1 & -6 & 7 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & -6 \\ -2 & 7 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & 4 & -2 \\ 1 & -6 & 7 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{-1}{3} \\ \frac{1}{3} & \frac{-1}{3} & \frac{2}{3} \end{bmatrix}$$

Therefore, the projection of vector b onto the column space of matrix A is

$$b_p = Pb = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

References

[1] G. Strang, Linear Algebra and Its Applications, Third Edition, Harcourt Brace Jovanovich College Publishers, 1988.