Math 4606 Test 1 Solutions February, 21, 2001.

Solutions by A. Wiandt and T. Wiandt

Additional Information:

The sequence $\{a_n\}$ is Cauchy if to every $\varepsilon > 0$ there is number N such that for every number n and m:

$$n > N \text{ and } m > N \text{ implies } |a_n - a_m| \leq \varepsilon.$$

(1) (20 pts.) Let the sequence $\{a_n\} \in R^2$ be given by:

$$a_n = (1 + \frac{1}{n}, 2 - \frac{1}{n^2}).$$

Find, $\lim_{n\to\infty} a_n$.

Solution. We know that $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$. Then

$$\lim_{n\to\infty} a_n = (\lim_{n\to\infty}(1 + \frac{1}{n}), \lim_{n\to\infty}(2 - \frac{1}{n^2})) = (1, 2).$$

(2) (20 pts.) Prove that a convergent sequence does have the Cauchy property.

Solution. The definition of Cauchy property is in the additional information part at the beginning. We have to prove that if a sequence $\{a_n\}$ is convergent, then it satisfies that condition. By the definition of convergence, $\{a_n\}$ is convergent if there exists a number a such that for every $\varepsilon > 0$ there is a number N such that if $n > N$ then $|a_n - a| \leq \varepsilon$. Fix an $\varepsilon > 0$. By the definition of convergence there exists a number a such that for $\varepsilon/2$ there is a number N such that if $n > N$ then $|a_n - a| \leq \varepsilon/2$. But then for this N if $n > N$, $m > N$ then by the triangle inequality

$$|a_n - a_m| \leq |a_n - a| + |a - a_m| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

and this is what we wanted to show.

(3) (20 pts.) State and prove the Schwarz inequality.

Solution. Schwarz inequality:

$$\mathbf{p} \cdot \mathbf{q} \leq |\mathbf{p}||\mathbf{q}|$$

The proof can be found in the book.

Remark. The majority of textbooks state the Schwarz inequality as

$$|\mathbf{p} \cdot \mathbf{q}| \leq |\mathbf{p}||\mathbf{q}|$$

This might seem to be a stronger statement, but in fact it is equivalent to the first, as we can see if we change $\mathbf{p}$ into $-\mathbf{p}$ in the first inequlity.

(4) (20 pts.)
  (a) (10pts.) Define that a given subset, say C, of an abstract vector space, say V, is convex.
  (b) (5pts.) Give an example of two convex sets whose union is not convex.
  (c) (5 pts.) Prove that the intersection of two convex sets is convex.

Solution. (a) $C \subset V$ is convex if for any $\mathbf{p}, \mathbf{q} \in C$

$$\lambda \mathbf{p} + (1 - \lambda)\mathbf{q} \in C$$

whenever $0 < \lambda < 1$.
(b) Let $V = R^2$, $A = \{(x, y) : y \geq 0\}$ and $B = \{(x, y) : x \geq 0\}$. A and B are clearly convex. $A \cup B = \{(x, y) : x \geq 0 \text{ or } y \geq 0\}$. This is not convex, because

$$\frac{1}{2}(-3, 1) + \frac{1}{2}(1, -3) = (-1, -1) \notin A \cup B.$$

(c) Suppose that $A \subset V$ and $B \subset V$ are convex. We want to prove that if $\mathbf{p}, \mathbf{q} \in A \cap B$ then

$$\lambda \mathbf{p} + (1 - \lambda)\mathbf{q} \in A \cap B$$

whenever $0 < \lambda < 1$. So suppose $\mathbf{p}, \mathbf{q} \in A \cap B$. Then $\mathbf{p}, \mathbf{q} \in A$, A is convex, so

$$\lambda \mathbf{p} + (1 - \lambda)\mathbf{q} \in A$$

whenever $0 < \lambda < 1$. On the other hand, $\mathbf{p}, \mathbf{q} \in B$, B is convex, so

$$\lambda \mathbf{p} + (1 - \lambda)\mathbf{q} \in B$$

whenever $0 < \lambda < 1$. This implies that

$$\lambda \mathbf{p} + (1 - \lambda)\mathbf{q} \in A \cap B$$

whenever $0 < \lambda < 1$ and the statement is proved.

(5) (20 pts.) (Bolzano – Weierstrass Theorem in R^1)
  Let the sequence $\{a_n\} \in R^1$ be bounded in the sense that

$$\sup_{n<\infty} |a_n| < \infty.$$

Prove that $\{a_n\} \in R^1$ has a convergent subsequence.

Solution. $\sup_{n<\infty} |a_n| < \infty$, so $\sup_{n<\infty} |a_n| = L$ for some finite number L. This means the sequence $\{a_n\} \subset [-L, L]$. Cut the interval $[-L, L]$ into two equal halves. We claim that at least one of the halves contains an infinite number of terms from the sequence. Suppose not: then each halves contain only finitely many, so $[-L, L]$ contains only finitely many, which contradicts the fact $\{a_n\}$ is a sequence. Choose one of the halves which contains infinitely many terms and rename these terms as $\{a_{1,n}\}$. Now repeat this process, in step i we cut the given interval into two equal halves and choose one with infinitely many terms from the remaining sequence from the $(i-1)$th step. This way we get a sequence of intervals $I_1 \supset I_2 \supset I_3 \supset \ldots$. Now let $\{b_n\} = \{a_{n,n}\}$ for every n. We claim this subsequence is convergent. It is enough to prove it has the Cauchy property, as every Cauchy sequence is convergent in R^1. The length of the interval in the ith step is $\frac{2L}{2}$, a fact easily proved by induction. Fix now an $\varepsilon > 0$. There exists N, an integer number, such that $\frac{2L}{2} < \varepsilon$. Now if $n, m > N$ then

$$|b_n - b_m| \leq \frac{2L}{2^N} < \varepsilon$$

because $b_n, b_m \in I_N$ and we verified the Cauchy property for $\{b_n\}$ and completed the proof.