Math 4606 Test 3 Solutions April, 27, 2001.

Solutions by Aniko and Tamas Wiandt

(1) (25 pts.) Let the function f mapping R^1 into R^1 be differentiable at the point $x \in R^1$ and let $f'(x)$ denote its derivative. Prove that

$$\lim_{|h|\to 0} \frac{|f(x+h) - f(x) - f'(x)h|}{|h|} = 0.$$

Solution. If f is differentiable at x and the derivative is denoted by $f'(x)$, then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

This implies that

$$\lim_{h\to 0}(\frac{f(x+h) - f(x)}{h} - f'(x)) = 0$$

and then

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - f'(x)h}{h} = 0$$

which implies

$$\lim_{|h|\to 0} \frac{|f(x+h) - f(x) - f'(x)h|}{|h|} = 0$$

and the proof is complete.

(2) (25 pts.) Let the function f map R^2 into R^1. Define that $f'(x)$ is a derivative of the function f at the point $x \in R^2$.

Solution. Use the formula from the previous problem: $x \in R^2$, $h \in R^2$, $f'(x) \in R^2$ and then $f'(x)$ is the derivative at x if

$$\lim_{|h|\to 0} \frac{|f(x+h) - f(x) - f'(x) \cdot h|}{|h|} = 0.$$

(3) (25 pts.) State and sketch the proof of the Mean Value theorem.

Solution. MVT: If f is continuous on an interval $[a, b]$ and $f'(x)$ exists for $a < x < b$, then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof: Consider the function $g(x) = f(x) + \frac{f(b)-f(a)}{a-b}x$. This is clearly continuous on $[a, b]$, differentiable on (a, b),

$$g(a) = g(b) = \frac{f(b)a - bf(a)}{a - b}$$

so by Rolle's Theorem there exists $c \in (a, b)$ such that $g'(c) = 0$.

$$g'(c) = f'(c) + \frac{f(b) - f(a)}{a - b}$$

and the statement follows.

(4) (25 pts.) Suppose the function f mapping R^2 into R^1 has continuous partial derivatives and at the point $x = (x_1, x_2) \in R^2$ define

$$Df(x) = (\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}).$$

Prove that $Df(x)$ is a derivative of the funtion f at the point $x \in R^2$.

Solution. By Problem (2) the definition of the derivative $f'(x)$ is

$$\lim_{|h| \to 0} \frac{|f(x + h) - f(x) - f'(x) \cdot h|}{|h|} = 0.$$

We have to prove that this statement is true for the above defined $Df(x)$. Let $h = (h_1, h_2)$, then

$$f(x + h) - f(x) = f(x_1 + h_1, x_2 + h_2) - f(x_1, x_2) =$$

$$= f(x_1 + h_1, x_2 + h_2) - f(x_1 + h_1, x_2) + f(x_1 + h_1, x_2) - f(x_1, x_2).$$

By the MVT

$$f(x_1 + h_1, x_2 + h_2) - f(x_1 + h_1, x_2) = \frac{\partial f(x_1 + h_1, x_2^*)}{\partial x_2} h_2$$

for some $x_2^* \in (x_2, x_2 + h_2)$ and

$$f(x_1 + h_1, x_2) - f(x_1, x_2) = \frac{\partial f(x_1^*, x_2)}{\partial x_1} h_1$$

for some $x_1^* \in (x_1, x_1 + h_1)$.

This implies that

$$f(x+h)-f(x)-Df(x)\cdot h=$$

$$\frac{\partial f(x_1+h_1,x_2^*)}{\partial x_2}h_2+\frac{\partial f(x_1^*,x_2)}{\partial x_1}h_1-(\frac{\partial f(x)}{\partial x_1},\frac{\partial f(x)}{\partial x_2})\cdot(h_1,h_2)=$$

$$=(\frac{\partial f(x_1+h_1,x_2^*)}{\partial x_2}-\frac{\partial f(x)}{\partial x_2})h_2+(\frac{\partial f(x_1^*,x_2)}{\partial x_1}-\frac{\partial f(x)}{\partial x_1})h_1.$$

Now clearly $0\leq |h_1|/|h|\leq 1$ and $0\leq |h_2|/|h|\leq 1$ and then by the triangle inequality we obtain that

$$0\leq\frac{|f(x+h)-f(x)-Df(x)\cdot h|}{|h|}\leq$$

$$\leq|\frac{\partial f(x_1+h_1,x_2^*)}{\partial x_2}-\frac{\partial f(x)}{\partial x_2}|+|\frac{\partial f(x_1^*,x_2)}{\partial x_1}-\frac{\partial f(x)}{\partial x_1}|.$$

The partial derivatives are continuous at $x=(x_1,x_2)$, so by the squeeze theorem we obtain that

$$\lim_{|h|\to 0}\frac{|f(x+h)-f(x)-Df(x)\cdot h|}{|h|}=0$$

and we established that $f'(x)=Df(x)$ and the proof is complete.