Math 8802, Final Examination, Wednesday, May 10, 2000.

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Each problem worth 25 points, do any 4 out of 9.

1. As usual, let $\mathcal{D} = \mathcal{D}(\mathcal{R}^1)$ denote the set of those functions in $\mathcal{C}^\infty = \mathcal{C}^\infty(\mathcal{R}^1)$ which have compact support. First, define the family of seminorms

$$q_N(\varphi) = \|\varphi\|_N = \max\{|D^\alpha \varphi(x)| : x \in \mathcal{R}^1, |\alpha| \leq N\}, \ N = 0, 1, ... \tag{1}$$

Here, of course, D is the differentiation operator given by $Df(x) = f'(x)$. Second, using these seminorms define a toplogy on $\mathcal{D}$ given by the local basis:

$$V_N = q_N^{-1}[0, \frac{1}{N}), \ N = 0, 1, ... \tag{2}$$

Prove that $\mathcal{D}$ is not complete with respect to this topology.

2. Let $\mathcal{X}$ be a topological vector space and let Λ be a continous linear map of $\mathcal{X}$ into itself. Prove that Λ is bounded.

3. Assume that the topological vector space of the previous problem is metrizable. Prove that under this additional assumption the converse implication holds as well. That is to say, prove that a bounded linear map of $\mathcal{X}$ into itself is continuous.

4. Next redefine the topology on $\mathcal{D}$ as follows:

   (a) Recall that for a given compact set $\mathcal{K} \in \mathcal{R}^1$, $\mathcal{D}_K$ denotes the set of those functions in $\mathcal{C}^\infty = \mathcal{C}^\infty(\mathcal{R}^1)$ whose support lies in $\mathcal{K}$.

   (b) For each such compact set $\mathcal{K}$ let τ_K denote the Frechet space toplogy of $\mathcal{D}_K$. That is to say, the topology given by the local base:

$$U_N = q_N^{-1}[0, \frac{1}{N}) \cap \mathcal{D}_K, \ N = 0, 1, ... \tag{3}$$

   (c) Let β be the collection of all convex balanced sets $W \subset \mathcal{D}$ such that for each compact $\mathcal{K} \subset \mathcal{R}^1$ we have $\mathcal{D}_K \cap W \in \tau_K$.

   (d) Let τ be the collection of all unions of sets of the form $\varphi + W$, with $\varphi \in \mathcal{D}$ and $W \in \beta$.

   Prove that if a set E is bounded with respect to this topology τ then there is a comapct set $\mathcal{K}$ such that $E \cap \mathcal{D}_K$.

5. Prove that the differentiation operator D of Problem 1 is a continous mapping of $\mathcal{D}$ into itself.

6. Let $u \in \mathcal{D}$ and with the help of it define the linear funtional on $\mathcal{D}$ by,

$$\Lambda_u(\varphi) = \int_{\mathcal{R}^1} u(x)\varphi(x)dx. \tag{4}$$

   Prove that $\Lambda_u \in \mathcal{D}'$, that is to say, prove that Λ_u is continuous.

7. Prove that there is a continuous mapping of $\mathcal{D}$ into itself , say T_D, such that for each $u \in \mathcal{D}$,

$$\Lambda_{Du} = \Lambda_u T_D. \tag{5}$$

8. Define the space $\mathcal{S} = \mathcal{S}(\mathcal{R}^1)$, the space of rapidly decreasing functions. First, prove that the Fourier trasnformation, F, is a continous mapping of $\mathcal{S}$ into itself. Second, prove that there is another continuous mapping of $\mathcal{S}$ into itself, say T_F, such that for each $u \in \mathcal{S}$,

$$\Lambda_{Fu} = \Lambda_u T_F. \tag{6}$$

9. State and prove the Fredholm Alternative Theorem for operators acring on a Hilbert space.

HAVE A NICE VACATION.