A more detailed proof of Rudin's Theorem 13.2

Theorem 0.1. *Let S and T be possibly unbounded operators acting in the Hilbertspace H. Then*

$$T^*S^* \subset (ST)^*. \tag{0.1}$$

Furthermore, if in addition $S \in B(H)$ then

$$T^*S^* = (ST)^*. \tag{0.2}$$

Since the proof of this theorem is based on the definition of the adjoint of a given aperator, we start this proof by recalling this definition. It is given in equations (13.2) and (13.3) of Rudin. An equivalent formulation is the following:

Definition 0.2. *Let A be a given densely defined, possibly unbounded operator. Then its adjoint, A^* is defined by,*

$$D(A^*) = \{y : \text{there is a } y^* \text{ such that for every } x \in D(A), (Ax, y) = (x, y^*)\} \tag{0.3}$$

and

$$A^*y = y^*. \tag{0.4}$$

To prove of conclusion (0.1) assume that

$$y \in D(T^*)S^*. \tag{0.5}$$

Then, we see from assumption (0.5) that $y \in D(S^*)$. This fact allows us to apply the definitions (0.3) and (0.4) to the operator S in place of the operator A and to the vector Tx in place of the vector x. This application yields,

$$(STx, y) = (Tx, S^*y), \;\; x \in D(ST).$$

We also see from assumption (0.5) that $S^*y \in T(S^*$. This fact allows us to apply the definitions (0.3) and (0.4) to the operator T in place of the operator A and to the vector $^*y$ in place of the vector y. This application yields,

$$(Tx, S^*y) = (x, T^*S^*y)), \;\; x \in D(ST).$$

Combining the previous to formulae we find

$$(STx, y) = (x, T^*S^*y)), \;\; x \in D(ST). \tag{0.6}$$

On the other hand, applying the definitions (0.3) and (0.4) to the operator ST in place of the operator A, we find

$$(STx, y) = (x, T^*S^*y)), \;\; x \in D(ST). \tag{0.7}$$

Finally, combining formulae (0.3) and (0.4) that these operators are densely we arrive at conclusion (0.1).

Having established conclusion (0.1), to prove conclusion (0.2) all we have to do, is to show that

$$y \in D((ST)^*) \;\text{ implies }\; y \in D(T^*S^*). \tag{0.8}$$

Since by assumption $S \in B(H)$, we see from Section 12.9 that this also holds for the adjoint, $S^* \in B(H)$. Hence, $D(S^*) = H$ and so, we can prove this implication by proving that

$$S^*y \in D(T^*). \tag{0.9}$$

Applying the definition (0.3) to the operator T in place of the operator A, and to the vector S^*y in place of the vecrtor y, we see that the implication (0.6) is equivalent to:

$$y \in D((ST)^*) \text{ implies that there is a } y^* \text{ such that for every } x \in D((T)), (Tx, s^*y) = (x, y^*) \qquad (0.10)$$

Indeed, we claim that for $y \in D((TS)^*)$ the vector given by,

$$y^* = (TS)^* y, \qquad (0.11)$$

is such a vector. To see this, first we apply the definition (0.3) to the operator S in place of the operator A and to the vector Tx in plac of the vector x. This application yields,

$$(STx, y) = (Tx, S^* y).$$

Second, we to apply the the definitions (0.3) and (0.2) to the operator ST in place of the operator A. This application yields,

$$(STx, y) = (x, (ST)^* y), \;\; x \in D(ST)$$

Combining the previous two formulae and using the definition (0.11), we find

$$(Tx, S^* y) = (x, y^*), \;\; x \in D(ST). \qquad (0.12)$$

Using the assumption $S \in B(H)$ again, we see that $D(ST) = D(T)$. Combining this fact with formula (0.12) we find the implication (0.10). Hence relation (0.9) follows and so does the implication (0.8). This completes the proof of conclusion (0.2).