1 Fourier Transformation.

Before stating the inversion theorem for the Fourier transformation on $L_2(R^1)$ recall that this is the space of Lebesgue measurable functions whose absolute value is square integrable. Alternatively, according to Remark 3.15 of [6], it is the completion of the space of continuous functions with compact support, with respect to the norm given by the Riemann integral,

$$||f-g|| = \left(\int_{R^1} (f(y)-g(y))^2 dy\right)^{\frac{1}{2}}. \tag{1}$$

For the notion of the completion of a metric space, we refer to Exercise 3.24 of [5]. The present proof is, essentially, the one given in Section 113 of [4]. The main difference being that instead of a trigonometric integral we use a limiting case of the residue theorem; Theorem III.19 of [1], Theorem V.2.2 and Example and Example V.2.7 of [2], Theorem 10.29 of [6].

It is a pleasure to thank Math-8802 student Changhyeong Lee for his suggestion, which led to a selfcontained proof of Lemma 1.3 .

Theorem 1.1. *Let S denote the set of finite linear combinations of characteristic functions of subintervals of R^1 and define*

$$Ff(x) = (2\pi)^{-\frac{1}{2}} \int_{R^1} e^{-ixy} f(y)dy, \;\; f \in S. \tag{2}$$

Then,

$$F^*F = I, \;\; \textit{on} \;\; S. \tag{3}$$

Furthermore, the closure of this transformation is unitary, that is to say,

$$F^*F = I,, \;\; \textit{on all of} \;\; L_2(R^1). \tag{4}$$

and

$$FF^* = I, \;\; \textit{on all of} \;\; L_2(R^1). \tag{5}$$

Before proving this theorem we note that it is a version of Theorem 7.7 of [7]. In fact, a closer version is given in Theorem 9.13 of [6]

We start the proof of Theorem 1.1 with a proposition saying that F is an isometry on S.

Proposition 1.2. *Let*

$$f \in S \;\; (and) \;\; g \in S. \tag{6}$$

Then

$$(Ff, Fg) = (f, g)f. \tag{7}$$

We start the proof of conclusion (7) by showing that it holds for the characteristic function of a single interval. In other words, define

$$\chi_{(a.b]}(x) = \begin{cases} 1, & \text{if} a < x \leq b \\ 0, & \text{0therwise.} \end{cases} \tag{8}$$

Then,

$$(F\chi_{(a.b]}, F\chi_{(a.b]}) = (\chi_{(a.b]}, \chi_{(a.b]}). \tag{9}$$

As a first step of the proof of formula (9) we note that clearly,

$$(\chi_{(a.b]}, \chi_{(a.b]}) = b - a. \tag{10}$$

As a second step of the proof of formula (9) we observe that

$$\|F\chi_{(a.b]}\|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(1 - e^{-i(b-a)x}) + (1 - e^{-i(a-b)x})}{x^2} dx. \tag{11}$$

Indeed, combining the definitions (2) and (8) with the Fundamental Theorem of the calculus and with the chain rule, we find

$$F\chi_{(a.b]}(x) = (2\pi)^{-\frac{1}{2}} \frac{e^{-ibx} - e^{-iax}}{-ix}.$$

Combining the previous formulae, in turn, with the definition of the inner product, we find formula (11).

As a third step of the proof of formula (9) we employ the residue theorem, to evaluate the integral on the right of formula (11). For this purpose, we formulate a lemma. This lemma is, essentially, a version of the residue theorem.

Lemma 1.3. *Let $0 < \alpha$ and $0 < \delta$ be given and let S_δ^+ denote the semicircle in the upper z-half plane:*

$$S_\delta^+ = \{z : z = \delta e^{i\theta},\ \ \pi \geq \theta \geq 0\}. \tag{12}$$

Then,

$$\lim_{\delta\to 0} \int_{S_\delta^+} \frac{1 - e^{i\alpha z}}{z^2} dz = -\alpha\pi. \tag{13}$$

Similarly, let $\alpha < 0$ and $0 < \delta$ be given and let S_δ^- denote the semicircle in the lower z-half plane:

$$S_\delta^- = \{z : z = \delta e^{i\theta},\ \ \pi < \theta < 2\pi\}.$$

Then,

$$\lim_{\delta\to 0} \int_{S_\delta^-} \frac{1 - e^{i\alpha z}}{z^2} dz = -i\alpha\pi. \tag{14}$$

To prove this lemma recall that e^z is an entire function, and so it does admit a convergent power series expansion in the entire z plane. Hence, there are constants a_n such that

$$e^z = \sum_{n=0}^{n=\infty} a_n z^n. \tag{15}$$

Next we apply this formula to $i\alpha z$ in place of z and subtract the resulting formulae from 1. Then using that the constant term of the resulting power series is zero, we find

$$\frac{1 - e^{i\alpha z}}{z^2} = -\frac{i\alpha}{z} - \sum_{n=2}^{n=\infty} a_n \left[(i\alpha^{n-2})^{n-2}\right] z^{n-2} \tag{16}$$

Since the power series (15) converges in the entire plane, for each complex number z_0,

$$\sup_n |a_n(z_0)^n| < \infty. \tag{17}$$

This implies that

$$\sup_n |a_n(i\alpha)^n| < \infty. \tag{18}$$

Hence,

$$\sup_{|z|<1} |\sum_{n=2}^{n=\infty} a_n (i\alpha)^{n-2} z^{n-2}| < \infty.$$

We see from the previous estimate that

$$\lim_{\delta\to 0} \int_{S_\delta^+} \sum_{n=2}^{n=\infty} a_n \left[(i\alpha)^{n-2}\right] z^{n-2} dz = 0. \tag{19}$$

Combining relations (19) and (16) we find,

$$\lim_{\delta\to 0} \int_{S_\delta^+} \frac{1-e^{i\alpha z}}{z^2} dz = -\lim_{\delta\to 0} \int_{S_\delta^+} \frac{i\alpha}{z} dz. \tag{20}$$

Next we claim that

$$\int_{S_\delta^+} \frac{i\alpha}{z} dz = i\alpha[i(Arg0 - Arg\pi)] = -\alpha\pi. \tag{21}$$

Indeed, we apply the definition of the integral of a complex valued function over a given path of the complex plane, Definitions 10.8 of [6], to the path of the definition (12). Then using that along this path,

$$\frac{dz}{d\theta}(\theta) = i\delta e^{i\theta} \tag{22}$$

we find,

$$-\int_{S_\delta^+} \frac{i\alpha}{z} dz = \alpha \int_\pi^0 d\theta. \tag{23}$$

Hence, formula (21) follows. Then, inserting formula (21) into formula (20) we find conclusion (13). Since the proof of conclusion (14) is similar, we do not repeat it. This completes the proof of Lemma 1.3.

As a fourth step of the proof of formula (9), we note that clearly

$$(F\chi_{(a.b]}, F\chi_{(a.b]}) = \frac{1}{2\pi} \lim_{\delta\to 0} \int_{\delta<|x|<\frac{1}{\delta}} \frac{(1-e^{i(b-a)x}) + (1-e^{-i(a-b)x})}{x^2} dx. \tag{24}$$

At the same time we see that $\alpha > 0$ implies,

$$\lim_{\delta\to 0} \int_{S_{\frac{1}{\delta}}^+} \frac{1-e^{i\alpha z}}{z^2} dz = 0 \tag{25}$$

Next we apply conclusion (13) of Lemma 1.3 to $b-a$ in place of α. Then combining the result with formulae (25) , (26) and with the Cauchy Theorem on holomorphic functions, we obtain,

$$\lim_{\delta\to 0} \int_{\delta<|x|<\frac{1}{\delta}} \frac{1-e^{i(b-a)z}}{z^2} dz = \pi(b-a). \tag{26}$$

Similarly, applying conclusion (14) of Lemma 1.3 to $a-b$ in place of α we obtain

$$\lim_{\delta\to 0} \int_{\delta<|x|<\frac{1}{\delta}} \frac{1-e^{i(a-b)z}}{z^2} dz = \pi(b-a). \tag{27}$$

Finally, adding formulae (27) and (26) together and using formula (24), we arrive at formula (9).

We continue the proof of conclusion (7) by showing that it holds for the characteristic functions of two disjoint interval. In other words,

$$a < b \leq c < d \tag{28}$$

implies,

$$(F\chi(a,b],F\chi(c,d])=0. \tag{29}$$

Indeed, similarly to formula (24), we see that

$$(F\chi_{(a,b]},F\chi_{(c,d]})=\frac{1}{2\pi}\lim_{\delta\to 0}\int_{\delta<|x|<\frac{1}{\delta}}\frac{(e^{i(d-b)x}-e^{i(d-a)x})+(e^{i(c-a)x}-e^{i(c-b)x})}{x^2}dx. \tag{30}$$

Next we subtract 1 from each of the two negative terms and then add 1 to each of the two negative terms of the numerator. Then assumption (28) allows us to apply conclusion (13) of Lemma 1.3 to each of the four terms of the resulting integral. This yields,

$$\begin{aligned}\frac{1}{2\pi}\lim_{\delta\to 0}\int_{\delta<|x|<\frac{1}{\delta}}\frac{(e^{i(d-b)x}-e^{i(d-a)x})+(e^{i(c-a)x}-e^{i(c-b)x})}{x^2}dx=\\ =\pi[((d-b)-(d-a)+(c-a)-(c-b)].\end{aligned} \tag{31}$$

Since the right side of formula (31) adds up to zero, we obtain formula (30).

We complete the proof of conclusion (7) by observing that any function which is a finite linear combination of characteristic functions of intervals can also be written as a finite linear combination of characteristic functions of disjoint intervals. Combining this fact with formulae (30) and (9) and with the linearity of the inner product, we arrive at conclusion (7). This completes the proof of Proposition 1.2.

We continue the proof of Theorem 1.1 by showing that F^*, the adjoint of F is also an isometry. Indeed, we claim that F^* is the closure of the operator:

$$F^*f(x)=(2\pi)^{-\frac{1}{2}}\int_{R^1}e^{ixy}f(y)dy,\ \ f\in S. \tag{32}$$

We prove formula (32) by showing that

$$(F\chi_{(a,b]},\chi_{(c,d]})=(\chi_{(a,b]},F^*\chi_{(c,d]}) \tag{33}$$

Indeed we see from definitions (2) and (8) that

$$(F\chi_{(a,b]},\chi_{(c,d]})=(2\pi)^{-\frac{1}{2}}\int_{(c,d]}\int_{(a,b]}e^{-ixy}dydx.$$

Similarly, we see from definition (32) that

$$(\chi_{(a,b]},F^*\chi_{(c,d]})=(2\pi)^{-\frac{1}{2}}\int_{(a,b]}\int_{(c,d]}e^{-ixy}dydx.$$

It follows from Fubini's Theorem (Theorem 7.8 of [6]) that the right sides of the previous two formulae are equal. Actually, we do not need the full force of Fubini's Theorem. All that we need is the rather special case of a rectangle and a continuous function defined on it. For this special case we refer to Corollary 18.26 of [3]. Hence formula (33) follows and so does formula (32), by linearity.

We complete the proof of Theorem 1.1 by observing that if we replace the complex number i by the complex number $-i$ in the definition (33) then we find the definition (31). This fact allows us to apply Proposition 1.2 to the operator of definition (31). This application shows that F^* is an isometry on S. We say that an operator, say, T admits a closure, if the closure of the graph of T, $\overline{G(T)}$ is a graph. Then we define the operator $\overline{T}$, closure of the operator T, by

$$G(\overline{T})=\overline{G(T)}.$$

For the definition of the graph we refer to Definitions 13.1 of [7]. It s clear from this definition that the closure of a densely defined isometry is an isometry defined on the entire space. Hence the nullspace of the closure of the operator F^* is trivial. Applying Theorem 4.12 of [7] to the operator F in place of T, we find

$$N(F^*) = R(F)^{\perp}.$$

It is a simple consequence of the Projection Theorem (Theorem 12.4 and Corollary of [7]) that

$$(M^{\perp})^{\perp} = \overline{M}.$$

Since F^* is an isometry, this shows that the range of F is dense. Since F is an isometry as well, this range is closed. Thus F is unitary and Theorem 1.1 follows.

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