1 A special case of Theorem 13.19

Theorem 1.1. *Let H be a Hilbert space and let T be a selfadjoint operator in it;*

$$T = T^*. \tag{1}$$

Then,

$$x \in D(T) \ \text{and} \ (T + iI)x = 0 \ \text{ipmlies} \ x = 0 \tag{2}$$

and

$$R(T + i) = H. \tag{3}$$

Furthermore, the operator defined by

$$U = (T - iI)(T + iI)^{-1} \tag{4}$$

is such that,

$$U^*U = I \tag{5}$$

and,

$$UU^* = I \tag{6}$$

and,

$$(I - U)y = 0 \quad \text{ipmlies} \quad y = 0 \tag{7}$$

We prove conclusion (2) by observing that,

$$((T + iI)x, (T + iI)x) \geq (x, x), \ \ x \in D(T). \tag{8}$$

Indeed, we see from assumption (1) that

$$((T + iI)x, (T + iI)x) = (Tx, Tx) + (x, x), \ \ x \in D(T). \tag{9}$$

In fact, for this formula we do not need the full force of assumption (1). All that we need is that T is symmetric, i.e., $T \subset T^*$.

We start the proof of conclusion (3) by showing that this range is closed,

$$R(T + i) = \overline{R(T + i)} \tag{10}$$

To see this relation, let r_n be a Cauchy sequence in this range. That is to say, let,

$$r_n = (T + iI)x_n, \ \ x \in D(T), \ \ \text{and} \ \lim_{(m,n)\to\infty} \|r_n - r_m\| = 0. \tag{11}$$

Then applying the lower estimate (9) to the vector $x_n - x_m$ in place of the vector x, we find

$$\lim_{(m,n)\to\infty} \|x_n - x_m\| = 0.$$

Since H is complete, there is an x such that

$$\lim_{n\to\infty} x_n = x \tag{12}$$

According to Theorem 13.9 of [3] assumption (1) implies that the operator T is closed. Thererfore, combining relations (12) and (11) we arrive at

$$\lim_{n\to\infty} r_n = Tx. \tag{13}$$

That is to say, $R(T + iI)$ is sequentially closed and so relation (10) follows.

We continue the proof of conclusion (3) by showing that for any densely defined operator say, T,

$$R(T)^{\perp} = N(T^*). \tag{14}$$

Note that under the additional assumption $T \in B(H)$ relation (14) is the first conclusion of Theorem 12.10 of [3]. In fact, our proof is slight generalization of that proof. Accordingly, assume that the given vector y is orthogonal to this range. That is to say, assume that,

$$(Tx, y) = 0, \ \ x \in D(T). \tag{15}$$

Then clearly, the vector $y^* = 0$, is such that

$$(Tx, y) = (x, y^*), \ \ x \in D(T).$$

Therefore, we see from the definiton of the adjoint of an unbounded operator, (Definitions 13.1 of [3]), that

$$y \in D(T^*) \ \text{ and } \ T^*y = 0.$$

In other words, assumption (15) implies

$$y \in N((T^*) \tag{16}$$

Conversly, assume that relation (16) holds. Then clearly, this imples relation (15). That is to say,

$$y \in R((T^{\perp}) \tag{17}$$

Thus, relation (17) implies and is implied by relation (16). This completes the proof of relation (14).

We complete the proof of conclusion (3) by applying relation (14) to the operator $T + iI$ in place of the operator T. This yields,

$$R(T + iI)^{\perp} = N((T + iI)^*). \tag{18}$$

We see from assumption (1) that

$$(T + iI)^*) = T - iI.$$

The proof of the already established conclusion (2) shows that we can replace the complex number i by $-i$ in that conclusion. This, in turn, shows that

$$N((T + iI)^*) = \{0\}.$$

Inserting this relation into relation (18) we obtain

$$R(T + iI)^{\perp} = \{0\}. \tag{19}$$

It is a simple consequence of the Projection Theorem (Theorem 12.4 and Corollary of [3]) that

$$(R(T + iI)^{\perp})^{\perp} = \overline{R(T + iI)}.$$

Hence, combining relations (19) and (14), we arrive at conclusion (3).

We prove conclusion (5) by showing that

$$\|(T - iI)(T + iI)^{-1}x\| = \|x\|. \tag{20}$$

Indeed, according to the already established conclusion (3) to the given vector x there is a vector y such that

$$x = (T + iI)y.$$

For such a vector x, clearly,

$$\|(T - iI)(T + iI)^{-1}x\| = \|(T - iI)y\|.$$

On the other hand, we see from formula (9) that

$$\|(T + iI)y\| = \|(T - iI)y\|.$$

Hence formula (20) follows. That is to say, the operator U of definition (4) is an isometry. This completes the proof of conclusion (5).

We start the proof of conclusion (6) by showing that

$$R(U) = H. \tag{21}$$

Indeed, we see from the already established concluions (2) and (3) that

$$R(T - iI)^{-1} = D$$

and that

$$(T + iI)D = H.$$

Combining these two relations with definition (2) we arrive at relation (21).

We complete the proof of conclusion (6) by noting that it is a general fact that relation (21) and conclusion (5) together imply conclusion (6).

We start the proof of conclusion (7) by showing that

$$I - U = 2i(T + iI)^{-1}. \tag{22}$$

Indeed, we see from definition (4) that

$$I - U = I - (T - iI)(T + iI)^{-1}.$$

Clearly,

$$I - (T - iI)(T + iI)^{-1} = [(T + iI) - (T - iI)](T - iI)^{-1}.$$

Now inserting the previous formula into the one preceeding it we find formula (22).

We complete the proof of conclusion (7) by combining formula (22) with the already established conclusions (2) and (3).

This completes the proof of Theorem 1.1.

References

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[3] ______, *Functional analysis*, second ed., McGraw-Hill Book Co., New York, 1991, McGraw-Hill Series in Higher Mathematics.