1 A special case of Theorem 13.19; the only if part

In this note we prove a special case of Theorem 13.19 of [3]. See also [1].

Theorem 1.1. *Let H be a Hilbert space and let U be a given unitary operator on it;*

$$UU^* = U^*U = I. \tag{1}$$

Suppose that the nullspace of U is trivial,

$$N(I - U) = \{0\} \tag{2}$$

and define

$$T = i(I + U)(I - U)^{-1} \tag{3}$$

Then, this operator is selfadjoint,

$$T = T^*. \tag{4}$$

We start the proof of conclusion (4) by showing that

$$T^* = -i(I + U^*)(I - U^*)^{-1}. \tag{5}$$

To see this formula, recall Theorem 13.2 of [3] which says that for any two operators, say Q and R,

$$Q^*R^* \subset (QR)^*.$$

Furthermore, if in addition $R \in B(H))$, then

$$Q^*R^* = (QR)^*.$$

Clearly, the operators $(I + U)$ and $(I - U)^{-1}$ commute,

$$(I + U)(I - U)^{-1} = (I - U)^{-1}(I + U), \;\; x \in R(I - U). \tag{6}$$

Formula (6) allows us to apply formula (1) to the operator $(I - U)^{-1}$ in place of Q and to the operator $(I + U)$ in place of R. Then using that $i^* = -i$, we find,

$$T^* = -i(I + U^*)[(I - U)^{-1}]^*. \tag{7}$$

Another application of formula (1) to the operator $(I - U)^{-1}$ in place of Q and to the operator $(I - U)$ in place of R yields,

$$[(I - U)^{-1}]^* = (I - U^*)^{-1}. \tag{8}$$

Combining formulae (8) and (7) we obtain formula (5).

We continue the proof of conclusion (4) by showing that

$$(I + U)(I - U)^{-1} = -(I + U^*)(I - U^*)^{-1}. \tag{9}$$

Indeed, we see from the assumption (1) that

$$(I + U) \cdot (I - U^*) = -(I + U^*) \cdot (I - U)$$

Next we multiply this formula by

$$(I - U^*)^{-1} \cdot (I - U)^{-1}.$$

Then using that each of the two factors of this product commutes with each of the four factors of formula (1), we obtain formula (9).

We complete the proof of conclusion (4) by inserting formula (9) into formula (5). At the same time, this completes the proof of Theorem 1.1.

References

[1] John B. Conway, *A course in functional analysis*, second ed., Springer-Verlag, New York, 1990.

[2] Frigyes Riesz and Béla Sz.-Nagy, *Functional analysis*, Dover Publications Inc., New York, 1990, Translated from the second French edition by Leo F. Boron, Reprint of the 1955 original.

[3] Walter Rudin, *Functional analysis*, second ed., McGraw-Hill Book Co., New York, 1991, McGraw-Hill Series in Higher Mathematics.