Math 1272 Calculus II, Fall 2007
TA: Ben Rosenfield

Trig integrals and trig substitution

1. Procedure for $\int \sin^m(x)\cos^n(x)\ dx$

We will use the following formulas

$$\sin^2(x) + \cos^2(x) = 1$$

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$

$$\sin(x)\cos(x) = \frac{\sin(2x)}{2}$$

Let's say we want to calculate $\int \sin^m(x)\cos^n(x)\ dx$. There are three cases.

First case: the power of cos (in this case n) is *odd*, so $n = 2k+1$. Rewrite $\cos^n(x)$ as

$$\cos^n(x) = \cos^{2k+1}(x) = \cos^{2k}(x)\cos(x) = \big(\cos^2(x)\big)^k\cos(x)$$

Plug $\cos^2(x) = 1 - \sin^2(x)$ into this formula to turn the integral into

$$\int \sin^m(x)\cos^n(x)\ dx = \int \sin^m(x)\big(1-\sin^2(x)\big)^k\cos(x)\ dx$$

Now we calculate this integral using u-substitution. Let $u = \sin(x)$, so $du = \cos(x)dx$. Then the integral is

$$\int u^m\big(1-u^2\big)^k\ du$$

Now, we have to actually multiply out $(1-u^2)^k$, which is sort of a pain, but there isn't really any way around it. Once this is done, the integral in u is easy to evaluate (it is just powers of u), so do it, and plug $u = \sin(x)$ into the resulting formula.

Second case: the power of sin (in this case m) is *odd*, so $m = 2k+1$. The procedure is nearly identical to the first case. Rewrite $\sin^m(x)$ as

$$\sin^m(x) = \sin^{2k+1}(x) = \sin^{2k}(x)\sin(x) = \big(\sin^2(x)\big)^k\sin(x)$$

Plug $\sin^2(x) = 1 - \cos^2(x)$ into this formula to turn the integral into

$$\int \sin^m(x)\cos^n(x)\ dx = \int \big(1-\cos^2(x)\big)^k\cos^n(x)\sin(x)\ dx$$

Now we calculate this integral using u-substitution. Let $u = \cos(x)$, so $du = -\sin(x)dx$. Then the integral is

$$\int -\big(1-u^2\big)^k u^n\ du$$

As above, we have to actually multiply out $(1-u^2)^k$. And, as above, once this is done, the integral in u is easy to evaluate (it is just powers of u), so do it, and plug $u = \cos(x)$ into the resulting formula.

Third case: the powers of both sin and cos are even, say $n = 2k$ and $m = 2l$. This case can be pretty awful if n and m are even a little big, but here's what you do. Use $\sin^2(x) = \frac{1-\cos(2x)}{2}$ and $\cos^2(x) = \frac{1+\cos(2x)}{2}$ to turn

$$\int \sin^{2k}(x)\cos^{2l}(x)\ dx = \int \left(\frac{1-\cos(2x)}{2}\right)^k \left(\frac{1+\cos(2x)}{2}\right)^l\ dx$$

Now, you have to multiply these out, *and* re-use $\cos^2(x) = \frac{1+\cos(2x)}{2}$ until there are no powers of cos left (if $n + m \geq 4$, you will have to do this at least once). This done, are left with an integral of cos's, which you can calculate directly.
Note that if $n = m = 2k$, you can use $\sin(x)\cos(x) = \frac{\sin(2x)}{2}$ to turn the integral into

$$\int \sin^{2k}(x)\cos^{2k}(x)\ dx = \int \left(\frac{\sin(2x)}{2}\right)^{2k}$$

and then begin the above process.

Finally, note that if both n and m are odd, you can use either the first or the second case.

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2. sin-cos examples

(1) $\int \sin^5(x)\cos^2(x)\ dx$

The power of sin is odd, so we are in the second case. Write

$$\sin^5(x) = \sin^4(x)\sin(x) = \left(1-\cos^2(x)\right)^2 \sin(x)$$

Then the integral becomes

$$\int \left(1-\cos^2(x)\right)^2 \cos^2(x)\sin(x)\ dx$$

Now we use the substitution $u = \cos(x)$, $du = -\sin(x)\ dx$. Then the integral is

$$-\int (1-u^2)^2u^2\ du = -\int (1-u^2)(1-u^2)u^2\ du = -\int (1-2u^2+u^4)u^2\ du = -\int u^2 - 2u^4 + u^6\ du$$

$$= -u^3/3 + 2u^5/5 - u^7/7 + C = -\cos^3(x)/3 + 2\cos^5(x)/5 - \cos^7(x)/7 + C$$

(2) $\int \sin^6(x)\cos^3(x)\ dx$

The power of cos is odd, so we are in the first case. Write

$$\cos^3(x) = \cos^2(x)\cos(x) = \left(1-\sin^2(x)\right)\cos(x)$$

Then the integral becomes

$$\int \left(1-\sin^2(x)\right)\sin^6(x)\cos(x)\ dx$$

Now we use the substitution $u = \sin(x)$, $du = \cos(x)\ dx$. Then the integral is

$$\int (1-u^2)u^6\ du = \int u^6 - u^8\ du = u^7/7 - u^9/9 + C = \sin^7(x)/7 - \sin^9(x)/9 + C$$

(3) $\int \sin^2(x)\cos^4(x)\ dx$

The powers of both sin and cos are even, so we are in the third case. We have

$$\sin^2(x) = \left(\frac{1-\cos(2x)}{2}\right) \quad \text{and} \quad \cos^4(x) = \left(\frac{1+\cos(2x)}{2}\right)^2$$

So the integral becomes

$$\int \left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1+\cos(2x)}{2}\right)^2 dx = \frac{1}{8}\int \big(1-\cos(2x)\big)\big(1+\cos(2x)\big)^2\ dx$$

$$= \frac{1}{8}\int \big(1-\cos(2x)\big)\big(1+2\cos(2x)+\cos^2(2x)\big) = \frac{1}{8}\int 1+\cos(2x)-\cos^2(2x)-\cos^3(2x)\ dx$$

$$= \frac{1}{8}\int 1+\cos(2x)-\frac{1}{2}\big(1+\cos(4x)\big)-\big(1-\sin^2(2x)\big)\cos(2x)\ dx$$

$$= \frac{1}{8}\int 1+\cos(2x)-\frac{1}{2}-\frac{1}{2}\cos(4x)-\cos(2x)+\sin^2(2x)\cos(2x)\ dx$$

$$= \frac{1}{8}\big(\frac{1}{2}x-\frac{1}{8}\sin(4x)+\frac{1}{6}\sin^3(2x)\big)+C = x/16-\sin(4x)/64+\sin^3(2x)/48+C$$

(4) $\int \sin^4(x)\cos^4(x)\ dx$

I'll come back to this one later...

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3. Procedure for $\int \tan^m(x)\sec^n(x)\ dx$ and similar

For evaluating integrals of the form $\int \tan^m(x)\sec^n(x)\ dx$, we only need

$$\sec^2(x) = 1+\tan^2(x)$$

Similarly, for evaluating integrals of the form $\int \cot^m(x)\csc^n(x)\ dx$, we only need

$$\csc^2(x) = 1+\cot^2(x)$$

The procedures for evaluating integrals with powers of tan and sec (or cot and csc) are not as clean cut as for powers of sin and cos. We only have clear rules when either the power of sec is even or the power of tan is odd.

First case: The power of sec is even, so $n = 2k$. Our integral is

$$\int \tan^m(x)\sec^{2k}(x)\ dx$$

Using our formula, we convert this to

$$\int \tan^m(x)\big(\sec^2(x)\big)^{k-1}\sec^2(x)\ dx = \int \tan^m(x)\big(1+\tan^2(x)\big)^{k-1}\sec^2(x)\ dx$$

Substitute $u = \tan(x)$, $du = \sec^2(x)\ dx$. Then the integral is

$$\int u^m \big(1+u^2\big)^{k-1}\ du$$

As above, the integrand must be multiplied out to get a polynomial in u, then the integration is simple.

Second case: The power of tan is odd, so $m = 2k+1$. It is important that powers of sec appear. We rewrite

$$\begin{aligned}\tan^{2k+1}(x)\sec^n(x) = \tan^{2k}(x)\sec^{n-1}(x)\sec(x)\tan(x) &= \big(\tan^2(x)\big)^k \sec^{n-1}(x)\sec(x)\tan(x)\\ &= \big(\sec^2(x)-1\big)^k \sec^{n-1}(x)\sec(x)\tan(x)\end{aligned}$$

Thus our integral is

$$\int \tan^m(x)\sec^n(x)\ dx = \int \tan^{2k+1}(x)\sec^n(x)\ dx = \int \big(\sec^2(x)-1\big)^k \sec^{n-1}(x)\sec(x)\tan(x)\ dx$$

Substitute $u = \sec(x)\ dx$, $du = \sec(x)\tan(x)\ dx$. Then the integral is

$$\int \big(u^2-1\big)^k u^{n-1}\ du$$

Again, the integrand must be multiplied out, at which point the integral is easy to compute.

Of course, many other configurations can occur, but they must be tackled on a case by case basis.

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4. sec-tan integrals

(1) $\int \tan^2(x)\sec^4(x)\ dx$

Since the power of sec is even, this is the first case. We rewrite it as

$$\int \tan^2(x)\sec^4(x)\ dx = \int \tan^2(x)\sec^2(x)\sec^2(x)\ dx = \int \tan^2(x)\big(1+\tan^2(x)\big)\sec^2(x)\ dx$$

Set $u = \tan(x)$, $du = \sec^2(x)\ dx$, so the integral is

$$\int u^2(1+u^2)\ du = \int u^2 + u^4\ du = u^3/3 + u^5/5 + C$$

So the answer is

$$\tan^3(x)/3 + \tan^5(x)/5 + C$$

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5. Procedure for integrals with sin(nx)'s and the like

For evaluating integrals like $\int \sin(nx)\cos(mx)\ dx$, $\int \sin(nx)\sin(mx)\ dx$, we need the following formulas

$$\sin(nx)\cos(mx) = \frac{1}{2}\Big(\sin\big((n-m)x\big) + \sin\big((n+m)x\big)\Big)$$

$$\sin(nx)\sin(mx) = \frac{1}{2}\Big(\cos\big((n-m)x\big) - \cos\big((n+m)x\big)\Big)$$

$$\cos(nx)\cos(mx) = \frac{1}{2}\Big(\cos\big((n-m)x\big) + \sin\big((n+m)x\big)\Big)$$