Finite-wavelength stability of capillary-gravity solitary waves

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Abstract

We consider the Euler equations describing nonlinear waves on the free surface of a two-dimensional inviscid, irrotational fluid layer of finite depth. For large surface tension, Bond number larger than $1/3$, and Froude number close to 1, the system possesses a one-parameter family of small-amplitude, traveling solitary wave solutions. We show that these solitary waves are spectrally stable with respect to perturbations of finite wave-number. In particular, we exclude possible unstable eigenvalues of the linearization at the soliton in the long-wavelength regime, corresponding to small frequency, and unstable eigenvalues with finite but bounded frequency, arising from non-adiabatic interaction of the infinite-wavelength soliton with finite-wavelength perturbations.

1 Introduction

In this article, we study stability of solitary waves traveling at constant velocity on the free surface of a two-dimensional inviscid fluid layer of finite depth under the influence of gravity and surface tension. The equations of motion are the Euler equations for nonlinear surface waves. Solitary waves are among the most striking phenomena and appear to be stable in several parameter regimes. Both for large surface tension and in the absence of surface tension, solitary waves are known to exist as particular solutions. Together with the solitary waves, there exists a family of spatially periodic waves, which are known as Stokes waves in the absence of surface tension.

Phenomenologically, solitary waves appear to be stable in both parameter regimes mentioned, whereas Stokes waves are stable only for large enough wavelengths. At some critical finite wavelength, the periodic waves destabilize, an instability mechanism first discovered in [BF67, Be67, Wh67] and known as the Benjamin-Feir instability.

Mathematically, the water wave problem is an evolutionary partial differential equation and possesses a Hamiltonian structure [Za68]. Various symmetries and associated conservation laws are known; see [BO80]. The initial-value problem to this partial differential equation is well posed locally in time in the case of gravity waves [Na74, KN79, Yo82, Cr85, Wu97]. Both solitary waves and spatially periodic Stokes waves are particular equilibria of the Hamiltonian system. Their stability or instability is to first order determined by the spectrum of the linearization. Complete stability proofs would however have to take into consideration the effects of nonlinearity, as well. Throughout this paper, we focus on the spectrum of the linearization, the first and basic step towards stability of solitary waves.

Existence of free surface waves in the full Euler equations has attracted a lot of interest in the late 80's using bifurcation theory. For example, existence of solitary waves for large surface tension, Bond number larger than $1/3$, was shown in [Ki88, AK89, Sa91].

Stability of surface waves in the full Euler equation is, from a mathematical point of view, a completely open problem, for both cases of gravity and capillary-gravity waves. Although a tremendous amount of literature is devoted to stability and instability of surface waves, to our knowledge, the present work represents the first rigorous attempt to show stability of solitary waves. Below, we summarize part of the previous work on stability and instability.

Most detailed results are available for Stokes waves. In the absence of surface tension, a rigorous proof of the Benjamin-Feir instability of small-amplitude Stokes waves has been given in [BM95]. Rigorous stability proofs, even for the linearized problem, do not seem to be available. On the other hand, instability induced by critical eigenvalues leaving the imaginary axis of the linearized equations about a periodic wave upon variations of parameters has been extensively studied, both numerically and analytically; see, for example, [Mc82, LH84, Sa85, MS86, LHT97] and the references therein.

Solitary waves in shallow water in the absence of surface tension appear to be stable at small amplitude. This is suggested by the numerical results on eigenvalues of the linearized operator in the absence of surface tension in [Ta86]. An instability seems to occur at some critical, finite amplitude, see again [Ta86]. The nature of this crest instability has also been investigated in direct numerical simulations, in [LHT97].

As already mentioned, stability results for solitary waves in the full Euler equations are not known. However, for large-wavelength initial data, the evolution of the free surface is governed on large time scales by certain model equations. For example, both for zero and for large surface tension, a formal expansion of the solution in the large wavelength exhibits at leading order a Korteweg-de Vries equation [KdV, Bou]. In other parameter regimes, the fifth order Kawahara equation [Ka72], or nonlinear Schrödinger equations can be derived. Together with these model equations, there come two mathematical problems:

(i) What are the wave dynamics in the model equations?

(ii) What can we conclude from the dynamics in the model equations for the dynamics of the full equations?

For the particular question of stability of solitary waves, we are interested in, these two problems reduce to first, the question of stability of solitary waves in the Korteweg-de Vries equation, and second, the question of validity of the approximation. Stability of solitary waves in the Korteweg-de Vries equation is fairly well understood. Orbital stability of the two-parameter family of solitary waves in this infinite-dimensional, integrable Hamiltonian system has been shown in [Be72, BSS87]. More towards the spirit of the present work, asymptotic stability of solitary waves has been shown in [PW96]. The proof there relies on a very careful understanding of the linearized problem using a scattering-type analysis. Convergence then is, necessarily, established in an exponentially weighted function space, where the Korteweg-de Vries equation is not Hamiltonian. Deviating from the primary objective of this work, we also mention stability results for the Kawahara equation [Ka72]. This fifth order partial differential equation describes the dynamics of surface waves in critical case of moderate surface tension, that is, for Bond numbers close to $1/3$. For Bond numbers slightly larger than $1/3$, the Kawahara equation supports solitary wave solutions just like the Korteweg-de Vries equation. Again, existence and orbital stability of these waves have been proved; see [IS92].

These stability results for the model equations let us believe that the solitary waves of the full Euler equations are stable at low amplitudes. However, the question to which extent solutions of the full system are well approximated by solutions of the model equations has not received a satisfactory answer that would allow us to conclude the stability of the solitary waves of the full system from only the stability of the corresponding waves of the model equation. Moreover, results on the validity of the model equations exist only in the case of gravity waves [KN79, KN86, Cr85, SW00]. In the presence of surface tension, the reduction method in [Ha96] permits to derive, in a rigorous and systematic manner, reduced systems that are nonlocal in the unbounded space variable and local in time, for different regions in the parameter plane (λ, b). The model equations, such as the Korteweg-de Vries and Kawahara equations, appear as the lowest order part in these reduced systems, but the connection between the solutions of the model equations and those of the reduced systems is still not clear.

If we want to infer stability of solitary waves in the full Euler equations from stability of the soliton in the Korteweg-de Vries equation, two major problems arise. First, the Korteweg-de Vries equations are valid on large, but finite time scales. Instabilities beyond these time scales are invisible in this leading order approximations. The second difficulty are non-adiabatic interactions between the infinite-wavelength solitary wave and finite-wavelength perturbations. In the long-wavelength approximation of the Korteweg-de Vries equation, these perturbations are ignored. However, even at the linear level, these types of interaction may produce unstable eigenvalues, as has been shown, in a different context, in [KS98].

We give an outline of our results. In the case of large surface tension, we use bifurcation theory to deduce spectral stability of small-amplitude solitary waves for eigenvalues of finite frequencies, corresponding to finite wave numbers of the perturbations; see Theorem 2. As a first step, we reformulate the Euler equations as an abstract, first-order differential equation in the spatial variable x; Section 2. The existence of solitary waves, Section 3, is described by a four-dimensional differential equation, which, due to symmetries reduces at leading order to a one-degree of freedom Hamiltonian system. The homoclinic orbit of this Hamiltonian system represents the solitary wave solution. This part of the analysis is similar to [Ki88]. The formulation of the Euler equations as a dynamical system in the spatial variable x in [Ki88] is slightly simpler, but does not generalize to the time-dependent case. We then linearize the Euler equations about this solitary wave solution and look for eigenfunctions with temporal growth $\mathrm{e}^{\sigma t}$. We obtain a generalized eigenvalue problem for the linearized operator $L(\sigma)$, depending on the spectral parameter σ. We formulate the stability problem in terms of the spectrum of this generalized eigenvalue problem and state our main results in Section 4. Stability of the continuous spectrum then follows from general perturbation arguments together with an explicit computation of the dispersion relation; Section 5. The main body of the proof is contained in Section 6, where point spectrum off the imaginary axis is excluded. It is here, that we crucially rely on the dynamical systems formulation of the problem. We define a complex analytic function, depending on the spectral parameter σ, which we call the Evans function of the full water-wave problem. Its zeroes σ coincide with the point spectrum. Stability of the solitary wave decomposes into stability in three different regimes, depending on the magnitude of the frequency of the eigenvalue, given by imaginary part of the spectral parameter σ:

(I) the long-wavelength,

(II) the intermediate-wavelength, and

(III) the short-wavelength regime.

Our main result claims stability in (I) and (II). Stability in the short-wavelength regime (III) remains open.

In the intermediate-wavelength regime (II), we exclude eigenvalues popping out of the essential spectrum by analytically continuing the Evans function into the essential spectrum and explicitly computing its value from the linear dispersion relation about the flat surface.

The long-wavelength regime (I) requires a more subtle analysis. In appropriate scalings, we find the Korteweg-de Vries equation and the Evans function associated to the Korteweg-de Vries soliton, already computed explicitly in [PW92]. The major difficulty then is associated to the fact that the linear dispersion relation about the trivial surface in the long-wavelength limit is the dispersion relation of the wave equation and not the dispersion relation of the Korteweg-de Vries equation. Technically, the problem appears when we formulate the Euler equations for the potential of the velocity field, whereas we derive the Korteweg-de Vries equation for the derivative of the potential. In particular, at bifurcation, we have four critical modes with zero group velocity. Only three are represented in the third order Korteweg-de Vries equation. The central argument relies on the symmetry of the dispersion relation induced by reflection in physical space. The symmetry is exploited in Section 6.2.4, where we show that the additional critical mode does not couple to the three other modes. More precisely, we show that we can continue the Evans function for the full water-wave problem problem analytically in the KdV-scaled spectral parameter σ. At leading order, we are able to compute the Evans function explicitly and find the Evans function of the KdV-soliton, multiplied by σ. The additional factor σ precisely accounts for the fourth critical mode induced by translation of the velocity potential by constants. The stability proof is concluded by a perturbation argument, which shows that all roots of the Evans function are located in the origin, even for higher order perturbations, since they are induced by symmetries of the full water-wave problem.

The method developped here for the case of large surface tension can be applied to the case of zero surface tension, as well. Although, the formulation of the problem, Section 2, has to be adapted, most of the consequent analysis is very similar. In particular, Theorem 2 on spectral stability holds in absence of surface tension, as well. An important difference arises when proving absence of unstable point spectrum with small frequency. The fourth critical mode, which appears in addition to the KdV-spectrum, carries a group velocity with the opposite sign when compared with the case of large surface tension. This actually simplifies the stability proof substantially in allowing for a continuation of the Evans function across the essential spectrum by means of exponentially weighted spaces, just like in the Korteweg-deVries approximation; see [PW97] and [HS01] for solitary waves in different contexts, where a similar situation arises.

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2 The Euler equations and spatial dynamics

Consider nonlinear waves propagating at a constant speed c on the free surface of an inviscid fluid layer of mean depth h and constant density ρ. Assume that both gravity and surface tension are present, and denote by g the acceleration due to gravity and by T the coefficient of surface tension. In a coordinate system (X, Y) moving with the waves the bottom lies at $Y = 0$ and the free surface is described by $Y = Z(X, t)$, where t is the time variable. The flow is supposed to be irrotational, so the velocity field has a potential $\Phi = \Phi(X, Y, t)$. Introduce dimensionless variables by choosing the unit length to be h and the unit velocity to be c. The Euler equations of motion become

$$\Phi_{XX} + \Phi_{YY} = 0, \quad \text{for } 0 < Y < 1 + Z(X, t), \tag{2.1}$$

with the boundary conditions

$$\Phi_Y = 0 \tag{2.2}$$

at the bottom $Y = 0$, and

$$Z_t + Z_X + Z_X\Phi_X = \Phi_Y \tag{2.3}$$

$$\Phi_t + \Phi_X + \frac{1}{2}(\Phi_X^2 + \Phi_Y^2) + \lambda Z - \frac{bZ_{XX}}{(1 + Z_X^2)^{3/2}} = 0 \tag{2.4}$$

on the free surface $Y = 1 + Z(X, t)$. The dimensionless numbers

$$\lambda = gh/c^2 \quad \text{and} \quad b = T/\rho hc^2$$

are the inverse square of the Froude number and the Bond number. The analysis is made for capillary-gravity waves, so we fix $b \neq 0$.

The goal of this section is to write the system (2.1)–(2.4) in the abstract form

$$D\boldsymbol{w}_t = \boldsymbol{w}_x + F(\boldsymbol{w}; \lambda), \tag{2.5}$$

with boundary conditions

$$0 \ = \ f(\boldsymbol{w}), \quad \text{on } y=0, \tag{2.6}$$

$$B\boldsymbol{w}_t \ = \ f(\boldsymbol{w}), \quad \text{on } y=1, \tag{2.7}$$

where D, B are linear and F, f nonlinear maps acting on a Hilbert space of functions defined on the bounded cross-section of the domain.

Consider the new variables

$$u = \Phi_X, \quad \eta = \frac{bZ_X}{\sqrt{1+Z_X^2}}.$$

and the change of coordinates

$$x = X, \quad y = \frac{Y}{1+Z(X,t)}, \tag{2.8}$$

which transforms the moving domain $\{(X,Y) \in \mathbb{R}^2 \,|\, 0 \leq Y \leq 1 + Z(X,t)\}$ into $\mathbb{R} \times [0,1]$. Then, (2.1), (2.4) lead to the system

$$0 = \Phi_x - u - \frac{y\eta}{(1+Z)\sqrt{b^2-\eta^2}}\Phi_y, \quad \text{in } \mathbb{R} \times (0,1), \tag{2.9}$$

$$0 = u_x + \frac{1}{(1+Z)^2}\Phi_{yy} - \frac{y\eta}{(1+Z)\sqrt{b^2-\eta^2}}u_y, \quad \text{in } \mathbb{R} \times (0,1), \tag{2.10}$$

$$0 = Z_x - \frac{\eta}{\sqrt{b^2-\eta^2}}, \tag{2.11}$$

$$\Phi_t = \eta_x - \lambda Z - u - \frac{u^2}{2} + \frac{1}{2(1+Z)^2}\Phi_y^2 - \frac{\eta(1+u)}{(1+Z)\sqrt{b^2-\eta^2}}\Phi_y, \quad \text{on } y=1, \tag{2.12}$$

with boundary conditions

$$0 = \Phi_y, \quad \text{on } y=0, \tag{2.13}$$

$$Z_t = \frac{1}{1+Z}\Phi_y - \frac{\eta(1+u)}{\sqrt{b^2-\eta^2}}, \quad \text{on } y=1, \tag{2.14}$$

obtained from (2.2) and (2.3).

Equations (2.9)–(2.12) are of the form (2.5) in which the independent variable $\boldsymbol{w}$, the linear operator D and the map F are defined through

$$\boldsymbol{w} = (\Phi, u, Z, \eta)^T, \quad D\boldsymbol{w} = (0,0,0,\Phi_{|y=1})^T,$$

and

$$F(\boldsymbol{w};\lambda) = \begin{pmatrix} -u - \frac{y\eta}{(1+Z)\sqrt{b^2-\eta^2}}\Phi_y \\ \frac{1}{(1+Z)^2}\Phi_{yy} - \frac{y\eta}{(1+Z)\sqrt{b^2-\eta^2}}u_y \\ -\frac{\eta}{\sqrt{b^2-\eta^2}} \\ -\lambda Z - \left[u + \frac{u^2}{2} - \frac{1}{2(1+Z)^2}\Phi_y^2 + \frac{\eta(1+u)}{(1+Z)\sqrt{b^2-\eta^2}}\Phi_y\right]_{|y=1} \end{pmatrix}.$$

The boundary conditions (2.13), (2.14) are of the form (2.6), (2.7) in which

$$B\boldsymbol{w} = Z, \quad f(\boldsymbol{w}) = \frac{1}{1+Z}\Phi_y - \frac{y\eta(1+u)}{\sqrt{b^2-\eta^2}}.$$

We consider (2.5) as an abstract differential equation on the phase space

$$X := H^1(0,1) \times L^2(0,1) \times \mathbb{R}^2.$$

Set $U = \{(\Phi, u, Z, \eta) \in X \mid Z > -1, |\eta| < b\}$, and define

$$X^1 := H^2(0,1) \times H^1(0,1) \times \mathbb{R}^2,$$

and $V = U \cap X^1$. The properties of D, F, B and f are summarized in the following lemma.

Lemma 2.1 *The following statements hold:*

(i) D *is bounded linear operator from* X *(resp.* X^1*) into* X *(resp.* X^1*).*

(ii) B *is bounded linear operator from* X *(resp.* X^1*) into* $\mathbb{R}$*.*

(iii) $F \in C^k(V \times \mathbb{R}, X)$ *and* $f \in C^k(U, L^2(0,1)) \cap C^k(V, H^1(0,1))$*, for any* $k \geq 0$*.*

The proof is an easy consequence of the definition of D, B, f, and F and the function spaces X, X^1 and left to the reader.

Remark 2.2 *The Euler equations* (2.1)–(2.4) *possess a reversibility symmetry. For any solution* $(Z(X,t), \Phi(X,t))$*, reversibility yields a different solution* $(Z(-X,-t), -\Phi(-X,-t))$*. For the system* (2.5) *this means that* D *commute and* F *anticommute with the* $R = \mathrm{diag}(-1,1,1,-1)$*, and for the boundary conditions* (2.6)–(2.7) *that* $BR = B$ *and* $g(R\boldsymbol{w}) = -g(\boldsymbol{w})$*, for any* $\boldsymbol{w} \in U$*.*

3 Steady solitary waves

The Euler equations (2.1)–(2.4) possess steady solitary-wave solutions for any $b > 1/3$ and $\lambda = 1 + \varepsilon^2$ for ε sufficiently small. Mathematical proofs go back to [Ki88, AK89, Sa91]. Our main purpose is a study of the temporal stability properties of these solitary waves. As we explained in the previous section, our approach to the stability problem is technically based on a spatial dynamics formulation of the eigenvalue problem — similar to the existence proof given in [Ki88]. However, our formulation slightly differs from the one exploited there. For the convenience of the reader, and in order to exhibit the main technical tools in the slightly simpler steady problem, we sketch the proof of existence of solitary waves in this section. In particular, we describe the most important properties of the steady solitary wave solutions of (2.5)–(2.7) that exist for $b > 1/3$ and $\lambda > 1$, λ close to 1.

From now on we fix $b > 1/3$ and set $\lambda = 1 + \varepsilon^2$. The solitary waves are not unique, due to the invariance of the equations under translations in X, Φ, and due to Galilean invariance. Translational symmetry is ruled out by restriction to symmetric waves, that is reversible solutions of the spatial dynamics formulation, satisfying $Z(X) = Z(-X)$ and $\Phi(X,Y) = -\Phi(-X,Y)$. In the steady problem the mean flow m is conserved and can be used to select a unique solitary wave from the family of solitary waves obtained by Galilean invariance. Fixing the mean flow through a cross section to one amounts to the condition

$$m = 1 + Z(X) + \int_0^{1+Z(X)} \Phi_X(X,Y)\,\mathrm{d}Y = 1. \tag{3.1}$$

We consider the steady water-wave problem (2.5) with $\boldsymbol{w}_t = 0$

$$\boldsymbol{w}_x + F(\boldsymbol{w};\lambda) = 0, \tag{3.2}$$

with boundary conditions

$$f(\boldsymbol{w}) = 0, \quad \text{on } y = 0, 1. \tag{3.3}$$

The proof of existence of solitary waves for this system is, as the one in [Ki88], based on a center manifold reduction. However, the reduction procedure cannot be applied directly to this system because of the nonlinear boundary condition on $y = 1$. We therefore consider first a nonlinear change of variables on U which transforms this boundary condition into a linear condition on $y = 1$.

Lemma 3.1 *The map $\chi : U \to U$ defined by $\chi(\Phi, u, Z, \eta) = (\tilde{\Phi}, u, Z, \eta)$ where*

$$\tilde{\Phi} = \Phi + \int_0^y \big(f(\boldsymbol{w}) - f'(0)\boldsymbol{w}\big)\,\mathrm{d}y' - \frac{Z}{1+Z}\Phi(0)$$

is a C^1-diffeomorphism. Moreover, the restriction $\chi : V \to V$ is a C^1-diffeomorphism.

Proof. It is easy to check that χ is a smooth map from U into X. A direct calculation shows that

$$\tilde{\Phi} = \frac{1}{1+Z}\Phi + \frac{y^2\eta}{2b} - \frac{\eta}{\sqrt{b^2-\eta^2}} \int_0^y y'(1+u(y'))\,\mathrm{d}y',$$

so χ is invertible with inverse $\chi^{-1} : U \to U$ defined through $\chi^{-1}(\tilde{\Phi}, u, Z, \eta) = (\Phi, u, Z, \eta)$ with

$$\Phi = (1+Z)\left(\tilde{\Phi} - \frac{y^2\eta}{2b} + \frac{\eta}{\sqrt{b^2-\eta^2}} \int_0^y y'(1+u(y'))\,\mathrm{d}y'\right).$$

The fact that χ^{-1} is smooth proves the first part of the lemma. The second part follows from the fact that the restrictions to V, $\chi : V \to V$ and $\chi^{-1} : V \to V$, are well defined and smooth. ■

Set $\boldsymbol{w} = \chi^{-1}(\tilde{\boldsymbol{w}})$. Then (3.2)–(3.3) yields the following system for $\tilde{\boldsymbol{w}}$

$$\tilde{\boldsymbol{w}}_x = -\left[D\chi^{-1}(\tilde{\boldsymbol{w}})\right]^{-1} F(\chi^{-1}(\tilde{\boldsymbol{w}}); \lambda) =: G(\tilde{\boldsymbol{w}}; \lambda), \tag{3.4}$$

with boundary conditions

$$\tilde{\Phi}_y = 0, \quad \text{on } y = 0, \tag{3.5}$$

$$\tilde{\Phi}_y = \frac{\eta}{b}, \quad \text{on } y = 1, \tag{3.6}$$

since

$$\tilde{\Phi}_y = f(\boldsymbol{w}) + \frac{y\eta}{b}.$$

We treat this system as an infinite dimensional dynamical system on the phase space X. We write

$$\tilde{\boldsymbol{w}}_x = \tilde{A}(\lambda)\tilde{\boldsymbol{w}} + \tilde{G}(\tilde{\boldsymbol{w}}; \lambda), \tag{3.7}$$

where $\tilde{A}(\lambda) = D_{\tilde{\boldsymbol{w}}}G(0; \lambda)$ and $\tilde{G}(\tilde{\boldsymbol{w}}; \lambda) = G(\tilde{\boldsymbol{w}}; \lambda) - \tilde{A}(\lambda)\tilde{\boldsymbol{w}}$. The boundary conditions (3.5)–(3.6) are included in the domain of definition of the linear operator $\tilde{A}(\lambda)$ by taking

$$Y := \mathrm{Dom}(\tilde{A}(\lambda)) = \left\{(\tilde{\Phi}, u, Z, \eta) \in X^1 \mid \tilde{\Phi}_y(0) = 0,\ \tilde{\Phi}_y(1) = \frac{\eta}{b}\right\}.$$

Then $\tilde{A}(\lambda)$ is a closed linear operator in X with domain Y dense in X, and $\tilde{G}$ is a smooth map from $W = U \cap Y \times \mathbb{R}$ into X.

Note that $\chi(0) = 0$ and $D\chi(0) = I$, so $\tilde{A}(\lambda) = D_{\tilde{\boldsymbol{w}}}G(0; \lambda) = -D_{\boldsymbol{w}}F(0; \lambda)$. This means that the linear part of the system (3.2) is not changed by the transformation above. The same is true for the boundary conditions (3.3). A direct calculation shows that

$$\tilde{A}(\lambda)\tilde{\boldsymbol{w}} = \left(u, -\tilde{\Phi}_{yy}, \frac{\eta}{b}, \lambda Z + u_{|y=1}\right)^T.$$

Remark also that (3.7) is reversible with reverser R defined in Section 2, since $\chi(R\boldsymbol{w}) = R\chi(\boldsymbol{w})$.

We apply center manifold reduction directly to this system. We find a four-dimensional reduced system which describes the steady waves. Note that the reduced system obtained in [Ki88] is only two-dimensional. The two additional dimensions here are due to the invariance of (2.1)–(2.4) under translations in the fluid potential Φ and due to Galilean invariance. Both symmetries are inherited by the system (3.7) from the full Euler equations. In [Ki88], these invariances were factored out, already in the dynamical formulation of the problem, before the reduction procedure, such that the reduced equation did not possess these symmetries any more. Here, we only use them after the reduction, and show that it is possible to simplify the reduced system on the four-dimensional center-manifold to a two-dimensional differential equation with the help of reversibility and condition (3.1). The reason for this slightly different approach is that we cannot factor out these symmetries in the eigenvalue problem.

Theorem 1 *For any $b > 1/3$ and $k \geq 0$ there exist $\varepsilon^* > 0$ and $C > 0$ such that, for any $\varepsilon \in (0, \varepsilon^*)$ the system* (3.2)–(3.3) *with $\lambda = 1 + \varepsilon^2$ possesses a unique* solitary-wave *solution $\boldsymbol{w}_\varepsilon^* \in C_b^k(\mathbb{R}, X^1)$ with the following properties:*

(i) $\boldsymbol{w}_\varepsilon^* = \boldsymbol{w}_\varepsilon^{*0} + \tilde{\boldsymbol{w}}_\varepsilon^*$ *where* $\boldsymbol{w}_\varepsilon^{*0} = (U^0, u^0, -u^0, -bu_x^0)$ *with*

$$u^0(x) = \varepsilon^2 \mathrm{sech}^2\left(\frac{\sqrt{\beta}\varepsilon x}{2}\right), \quad U^0(x) = \int_0^x u^0(x')\,\mathrm{d}x', \quad \beta = \frac{3}{3b-1},$$

and $\|\tilde{\boldsymbol{w}}_\varepsilon^*(x)\|_{X^1} \leq C\varepsilon^3$ *for any* $x \in \mathbb{R}$*. Moreover,*

$$\|(I - P_\Phi)\,\tilde{\boldsymbol{w}}_\varepsilon^*(x)\|_{X^1} \leq C\varepsilon^4 \mathrm{e}^{-\sqrt{\beta}\varepsilon|x|}, \quad \|\partial_y P_\Phi \tilde{\boldsymbol{w}}_\varepsilon^*(x)\|_{X^1} \leq C\varepsilon^3 \mathrm{e}^{-\sqrt{\beta}\varepsilon|x|},$$

where P_Φ is the projection on the Φ–component of $\boldsymbol{w}$: $P_\Phi : X \to X$, $P_\Phi = \mathrm{diag}\,(1,0,0,0)$.

(ii) $\boldsymbol{w}_\varepsilon^*$ *is reversible, i.e.* $R\boldsymbol{w}_\varepsilon^*(x) = \boldsymbol{w}_\varepsilon^*(-x)$*, and the components* $\Phi_\varepsilon^*, u_\varepsilon^*, Z_\varepsilon^*, \eta_\varepsilon^*$ *of* $\boldsymbol{w}_\varepsilon^*$ *satisfy*

$$Z_\varepsilon^*(x) + (1 + Z_\varepsilon^*(x)) \int_0^1 u_\varepsilon^*(x, y)\,\mathrm{d}y = 0.$$

(iii) $\boldsymbol{w}_\varepsilon^*$ *is a smooth function of* ε*.*

Proof. By Lemma 3.1 it is enough to show the existence of solitary waves for the system (3.7). As in [Ki88] one can show that $\tilde{A}(\lambda)$ has compact resolvent, so its spectrum consists only of isolated eigenvalues of finite multiplicities. The eigenvalue problem

$$\tilde{A}(\lambda)\tilde{\boldsymbol{w}} = \zeta\tilde{\boldsymbol{w}}, \quad \tilde{\boldsymbol{w}} \in Y$$

can be solved explicitly, and we find that ζ is an eigenvalue of $\tilde{A}(\lambda)$ if and only if it satisfies the equality

$$\zeta^2 \cos\zeta = (\lambda - b\zeta^2)\zeta \sin\zeta.$$

A direct calculation shows that 0 is always an eigenvalue of $\tilde{A}(\lambda)$ with generalized eigenvectors

$$\boldsymbol{w}_0 = (1, 0, 0, 0)^T, \quad \boldsymbol{w}_\lambda = (0, 1, -1/\lambda, 0)^T,$$

such that $\tilde{A}(\lambda)\boldsymbol{w}_0 = 0$, $\tilde{A}(\lambda)\boldsymbol{w}_1 = \boldsymbol{w}_0$. If $b > 1/3$ and $\lambda = 1$ this eigenvalue has algebraic multiplicity 4; the generalized eigenvectors

$$\boldsymbol{w}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{w}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad \boldsymbol{w}_2 = \begin{pmatrix} -\frac{y^2}{2} \\ 0 \\ 0 \\ -b \end{pmatrix}, \quad \boldsymbol{w}_3 = \begin{pmatrix} 0 \\ -\frac{y^2}{2} \\ \frac{1}{2} - b \\ 0 \end{pmatrix},$$

satisfy $\tilde{A}(1)\boldsymbol{w}_0 = 0$, $\tilde{A}(1)\boldsymbol{w}_i = \boldsymbol{w}_{i-1}$, $i = 1, 2, 3$, and form a basis for the generalized eigenspace associated to the eigenvalue 0.

We apply the reduction result in [Mi88] to system (3.7) with $b > 1/3$ and $\lambda = 1 + \varepsilon^2$ close to $\lambda_0 = 1$. By direct calculation one can prove that there exist positive constants $C(\lambda)$ and q_0 such that

$$\|(\mathrm{i}q - \tilde{A}(\lambda))^{-1}\|_{X\to X} \leq \frac{C(\lambda)}{|q|}, \tag{3.8}$$

for any $q \in \mathbb{R}$, $|q| > q_0$. Moreover, the map $\tilde{G}$ is smooth in $\tilde{\boldsymbol{w}}$ and ε^2 when considered as a map from the domain $Y = \mathrm{Dom}\,(\tilde{A})$ into X. With these preparations, the reduction theorem in [Mi88] shows that any small bounded solution $\tilde{\boldsymbol{w}} \in C_b^k(\mathbb{R}, Y)$ of (3.7) is of the form

$$\tilde{\boldsymbol{w}}(x) = a_0(x)\boldsymbol{w}_0 + a_1(x)\boldsymbol{w}_1 + a_2(x)\boldsymbol{w}_2 + a_3(x)\boldsymbol{w}_3 + \Psi(a_0, a_1, a_2, a_3; \varepsilon^2), \tag{3.9}$$

with $\Psi(a_0, a_1, a_2, a_3; \varepsilon^2) = \mathrm{O}(|a_j|(|a_j| + \varepsilon^2))$, and a_j satisfy the reduced system

$$\begin{aligned} a_{0,x} &= a_1 + f_0(a_0, a_1, a_2, a_3; \varepsilon^2), \\ a_{1,x} &= a_2 + f_1(a_0, a_1, a_2, a_3; \varepsilon^2), \\ a_{2,x} &= a_3 + f_2(a_0, a_1, a_2, a_3; \varepsilon^2), \\ a_{3,x} &= f_3(a_0, a_1, a_2, a_3; \varepsilon^2), \end{aligned} \tag{3.10}$$

in which $f_j(a_0, a_1, a_2, a_3; \varepsilon^2) = \mathrm{O}(|a_j|(|a_j| + \varepsilon^2))$.

By a careful choice of a cut-off function, necessary in the construction of the center-manifold, one can arrange to have the reduced flow inherit the symmetries of the full system (3.7). In particular, the invariance of (3.7) under translation in Φ implies that Ψ and (3.10) are invariant under transformations of the form $a_0 \to a_0 + \alpha$, for any $\alpha \in \mathbb{R}$, such that Ψ and the f_j, $j = 0, \ldots, 3$ do not depend upon a_0. The reduced equations (3.10) possess a skew-product structure and decouple into a system for a_1, a_2, a_3,

$$\begin{aligned} a_{1,x} &= a_2 + f_1(a_1, a_2, a_3; \varepsilon^2), \\ a_{2,x} &= a_3 + f_2(a_1, a_2, a_3; \varepsilon^2), \\ a_{3,x} &= f_3(a_1, a_2, a_3; \varepsilon^2), \end{aligned} \tag{3.11}$$

and one differential equation for a_0, which can be integrated. Reversibility can be used to uniquely determine a_0. The reduced system (3.10) is reversible with reverser R_0 acting through

$$R_0(a_0, a_1, a_2, a_3) = (-a_0, a_1, -a_2, a_3),$$

since $R\boldsymbol{w}_0 = -\boldsymbol{w}_0$, $R\boldsymbol{w}_1 = \boldsymbol{w}_1$, $R\boldsymbol{w}_2 = -\boldsymbol{w}_2$, $R\boldsymbol{w}_3 = \boldsymbol{w}_3$. Reversible solutions of (3.10) are those with a_0, a_2 odd and a_1, a_3 even functions in x. For such solutions a_0 is uniquely determined by the condition $a_0(0) = 0$, which leads to

$$a_0(x) = \int_0^x a_1 + f_0(a_1, a_2, a_3; \varepsilon^2)\,\mathrm{d}x'. \tag{3.12}$$

Next, we use the condition (3.1) to uniquely determine a_3 for solutions of (3.7) with mean flow one. For $\tilde{\boldsymbol{w}} = (\tilde{\Phi}, u, Z, \eta)$ this condition reads

$$Z(x) + (1 + Z(x)) \int_0^1 u(x, y)\,\mathrm{d}y = 0, \quad x \in \mathbb{R}.$$

Substitution of $\tilde{\boldsymbol{w}}$ from (3.9) yields an equality

$$\mathcal{F}(a_1, a_2, a_3; \varepsilon^2) = 0.$$

It is not difficult to see that $\mathcal{F}$ is smooth in its arguments, and a direct calculation shows that

$$D_{a_3}\mathcal{F}(0, 0, 0; \varepsilon^2) = \frac{1}{3} - b \neq 0.$$

Then by the implicit function theorem we obtain

$$a_3 = \psi(a_1, a_2; \varepsilon^2) = \mathrm{O}(|a_j|(|a_j| + \varepsilon^2)), \tag{3.13}$$

with ψ smooth function. Substituting (3.13) into (3.11) we obtain the two-dimensional system

$$\begin{aligned} a_{1,x} &= a_2 + g_1(a_1, a_2; \varepsilon^2), \\ a_{2,x} &= g_2(a_1, a_2; \varepsilon^2). \end{aligned} \tag{3.14}$$

This system is also reversible with reverser acting through $a_1 \to a_1$, $a_2 \to -a_2$. One can now argue as in [Ki88] and prove that (3.14) possesses a unique reversible homoclinic solution $(a_1^*(\varepsilon), a_2^*(\varepsilon))$, smooth function of ε, for sufficiently small $\varepsilon > 0$. Explicit calculation of the relevant quadratic terms shows that

$$a_1^*(x; \varepsilon) = \varepsilon^2 \mathrm{sech}^2\left(\frac{\sqrt{\beta}\varepsilon x}{2}\right) + O(\varepsilon^4).$$

The equalities (3.12), (3.13) give the reversible homoclinic solution of the reduced system (3.10), and from (3.9) we find the reversible solitary-wave solution of (3.7). This proves the theorem. ■

4 Spectral stability of solitary waves

In this section we formulate the stability problem in terms of the spectrum of a family of linear operators and state the main results.

4.1 Linearized system

Consider the linearization of the problem (2.5)–(2.7) about the solitary wave $\boldsymbol{w}_\varepsilon^* \in C_b^k(\mathbb{R}, X^1)$ found in Theorem 1 for $\varepsilon \in (0, \varepsilon^*)$:

$$D\boldsymbol{W}_t = \boldsymbol{W}_x + D_{\boldsymbol{w}} F(\boldsymbol{w}_\varepsilon^*; 1+\varepsilon^2)\boldsymbol{W} \tag{4.1}$$

$$0 = f'(\boldsymbol{w}_\varepsilon^*)\boldsymbol{W}, \quad \text{on } y = 0 \tag{4.2}$$

$$B\boldsymbol{W}_t = f'(\boldsymbol{w}_\varepsilon^*)\boldsymbol{W}, \quad \text{on } y = 1. \tag{4.3}$$

We look for solutions of this system of the form

$$\boldsymbol{W}(t,x) = e^{\sigma t}\boldsymbol{W}_\sigma(x), \tag{4.4}$$

with $\boldsymbol{W}_\sigma$ bounded function from $\mathbb{R}$ into the complexification of X^1, for $\sigma \in \mathbb{C}$. For simplicity we denote the complexification of X^1, and later those of X and Y, also by X^1 (resp. X and Y).

Roughly speaking, the solitary wave $\boldsymbol{w}_\varepsilon^*$ is stable if (4.1)–(4.3) does not possess any solutions of the form (4.4) for any $\sigma \in \mathbb{C}$ with $\mathrm{Re}\,\sigma > 0$.

Substitution of (4.4) into (4.1)–(4.3) yields the following system for $\boldsymbol{W}_\sigma$:

$$\sigma D\boldsymbol{W} = \boldsymbol{W}_x + D_{\boldsymbol{w}} F(\boldsymbol{w}_\varepsilon^*; 1+\varepsilon^2)\boldsymbol{W} \tag{4.5}$$

$$0 = f'(\boldsymbol{w}_\varepsilon^*)\boldsymbol{W}, \quad \text{on } y = 0 \tag{4.6}$$

$$\sigma B\boldsymbol{W} = f'(\boldsymbol{w}_\varepsilon^*)\boldsymbol{W}, \quad \text{on } y = 1. \tag{4.7}$$

We write this system in abstract form

$$\mathcal{L}(\sigma,\varepsilon)\widetilde{\boldsymbol{W}} := \widetilde{\boldsymbol{W}}_x - L(\sigma,\varepsilon)\widetilde{\boldsymbol{W}} = 0,$$

with $L(\sigma,\varepsilon)$ some linear operator in X, and then formulate the stability problem for $\boldsymbol{w}_\varepsilon^*$ in terms of the spectrum of the family of operators $\mathcal{L}^\varepsilon = (\mathcal{L}(\sigma,\varepsilon))_{\sigma\in\mathbb{C}}$. We proceed as in the steady problem by constructing first a linear diffeomorphism χ_σ which transforms the non-autonomous boundary conditions (4.6)–(4.7) into autonomous boundary conditions.

Lemma 4.1 *Assume $\sigma \in \mathbb{C}$ and $\varepsilon \in (0,\varepsilon^*)$. The linear map $\chi_\sigma : X \to X$ defined by*

$$\chi_\sigma \boldsymbol{W} = D\chi(\boldsymbol{w}_\varepsilon^*)\boldsymbol{W} - \frac{\sigma}{2}\left(y^2 B\boldsymbol{W}, 0, 0, 0\right)$$

is bounded and has bounded inverse $\chi_\sigma^{-1} : X \to X$. Moreover, χ_σ and χ_σ^{-1} are analytic in σ, smooth in ε, and their restrictions to X^1 are well defined.

The proof is similar to the one of Lemma 3.1 so we omit it here. Note that χ_0 is the linearization about $\boldsymbol{w}_\varepsilon^*$ of the diffeomorphism χ in Lemma 3.1.

Set $\boldsymbol{W} = \chi_\sigma^{-1}\widetilde{\boldsymbol{W}}$. Then the system (4.5) becomes

$$\widetilde{\boldsymbol{W}}_x = \chi_\sigma \left[\sigma D - D_{\boldsymbol{w}} F(\boldsymbol{w}_\varepsilon^*; 1+\varepsilon^2)\right] \chi_\sigma^{-1}\widetilde{\boldsymbol{W}} + (\partial_x \chi_\sigma)\, \chi_\sigma^{-1}\widetilde{\boldsymbol{W}}, \tag{4.8}$$

with boundary conditions

$$\tilde{\Phi}_y = 0, \quad \text{on } y = 0, \tag{4.9}$$

$$\tilde{\Phi}_y = \frac{\eta}{b}, \quad \text{on } y = 1. \tag{4.10}$$

for $\widetilde{\boldsymbol{W}} = (\tilde{\Phi}, u, Z, \eta)$.

Explicit calculation of the equations in (4.8) show that it is of the form

$$\widetilde{\boldsymbol{W}}_x = D(\sigma,\varepsilon)\widetilde{\boldsymbol{W}} + A(\varepsilon)\widetilde{\boldsymbol{W}}, \tag{4.11}$$

with

$$D(\sigma,\varepsilon) = D_\infty(\sigma) + \varepsilon^2 D_1(x;\sigma,\varepsilon),$$

a bounded linear operator in X, and

$$A(\varepsilon) = A_\infty(\varepsilon^2) + \varepsilon^2 A_1(x;\varepsilon),$$

a closed linear operator in X. The parts A_∞ and D_∞ correspond to the linearization evaluated at the asymptotic state of the solitary wave, at $x = \infty$. The parts A_1 and D_1 correspond to the perturbation due to the solitary wave. These are operators with coefficients depending on x, and decaying to 0 at $x = \infty$ with the same rate as the decay rate of the solitary wave $\boldsymbol{w}_\varepsilon^*$. Since we do not need the explicit formulas of these operators in the following, we omit them here. However, note that $A_\infty(\varepsilon^2) = \tilde{A}(1+\varepsilon^2)$, and that $D_\infty(\sigma)$ and $D_1(x;\sigma,\varepsilon)$ depend upon σ in the following way

$$D_\infty(\sigma) = \sigma D_{\infty 1} + \sigma^2 D_{\infty 2}, \quad D_1(x;\sigma,\varepsilon) = \sigma D_{11}(x;\varepsilon) + \sigma^2 D_{12}(x;\varepsilon),$$

since $BD = 0$. As in the formulation of the steady problem (3.7), the boundary conditions (4.9)–(4.10) are included in the domain of definition of the operator $A(\varepsilon)$.

The properties of $D(\sigma,\varepsilon)$ and $A(\varepsilon)$ needed later are summarized in the next lemma. They follow from Lemma 2.1, the decay properties of $\boldsymbol{w}_\varepsilon^*$ in Theorem 1, and the definition of χ_σ in Lemma 4.1.

Lemma 4.2 *Assume $\sigma \in \mathbb{C}$, $\varepsilon \in (0,\varepsilon^*)$ and $x \in \mathbb{R}$.*

(i) $D_\infty(\sigma)$ *and* $D_1(x;\sigma,\varepsilon)$ *are bounded linear operators in* X *(resp.* X^1*), depending analytically upon* σ *and smoothly upon* ε*.*

(ii) $A_\infty(\varepsilon^2)$ *and* $A_1(x;\varepsilon)$ *are closed linear operators in* X *with dense domain* Y*, depend analytically upon* σ *and smoothly upon* ε*.*

Moreover, there exists a positive constant C *such that the following inequalities hold for any* $\sigma \in \mathbb{C}$*,* $\varepsilon \in (0, \varepsilon^*)$ *and* $x \in \mathbb{R}$*:*

$$\|D_\infty(\sigma)\|_{X(\text{resp. } X^1)\to X(\text{resp. } X^1)} \le C\,|\sigma|(1+|\sigma|),$$
$$\|D_1(x;\sigma,\varepsilon)\|_{X(\text{resp. } X^1)\to X(\text{resp. } X^1)} \le C\,|\sigma|(1+|\sigma|)e^{-\sqrt{\beta}\varepsilon|x|},$$
$$\|A_\infty(\varepsilon^2)\|_{Y\to X} \le C, \qquad \|A_1(x;\varepsilon)\|_{Y\to X} \le C\,e^{-\sqrt{\beta}\varepsilon|x|}.$$

4.2 Spectral stability

Set $L(\sigma,\varepsilon) = D(\sigma,\varepsilon) + A(\varepsilon)$, and consider the family of operators $\mathcal{L}^\varepsilon = (\mathcal{L}(\sigma,\varepsilon))_{\sigma\in\mathbb{C}}$ defined by

$$\mathcal{L}(\sigma,\varepsilon) = \frac{d}{dx} - L(\sigma,\varepsilon).$$

Equation (4.11) becomes $\mathcal{L}(\sigma,\varepsilon)\widetilde{\boldsymbol{W}} = 0$. Set $\mathscr{H} = L^2(\mathbb{R}, X)$ and $\mathscr{W} = H^1(\mathbb{R}, X) \cap L^2(\mathbb{R}, Y)$. Then $\mathcal{L}(\sigma,\varepsilon)$ is closed linear operator in $\mathscr{H}$ with dense domain $\mathscr{W}$.

Define the *resolvent* of the family of operators $\mathcal{L}^\varepsilon$ as the set

$$\rho(\mathcal{L}^\varepsilon) = \{\sigma \in \mathbb{C} \,:\, \mathcal{L}(\sigma,\varepsilon) \text{ invertible}\}.$$

The set $\Sigma(\mathcal{L}^\varepsilon) = \mathbb{C} \setminus \rho(\mathcal{L}^\varepsilon)$ is called the *spectrum* of $\mathcal{L}^\varepsilon$. We distinguish between *point spectrum*

$$\Sigma_p(\mathcal{L}^\varepsilon) = \Sigma(\mathcal{L}^\varepsilon) \cap \{\sigma \in \mathbb{C} \,:\, \mathcal{L}(\sigma,\varepsilon) \text{ Fredholm with index } 0\},$$

and *essential spectrum* $\Sigma_e(\mathcal{L}^\varepsilon) = \Sigma(\mathcal{L}^\varepsilon) \setminus \Sigma_p(\mathcal{L}^\varepsilon)$.

Definition 4.3 *The solitary wave* $\boldsymbol{w}^*_\varepsilon$ *is called* spectrally stable *if*

$$\Sigma(\mathcal{L}^\varepsilon) \subset \{\sigma \in \mathbb{C} \,:\, \operatorname{Re}\sigma \le 0\},$$

and spectrally unstable *otherwise.*

The main result in this paper is:

Theorem 2 *Fix* $b > 1/3$*, and choose any* $R > 0$ *large. Then there exists* $\varepsilon_b > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_b)$*, the spectrum of* $\mathcal{L}^\varepsilon$ *coincides with the imaginary axis in a ball of radius* R*:*

$$\Sigma(\mathcal{L}^\varepsilon) \cap \{\sigma \in \mathbb{C} \,:\, |\sigma| \le R\} = \mathrm{i}\mathbb{R} \cap \{\sigma \in \mathbb{C} \,:\, |\sigma| \le R\}$$

The proof consists of two parts summarized in the following two theorems.

Theorem 3 *There exists $\varepsilon_e > 0$ such that for any $\varepsilon \in (0, \varepsilon_e)$ the essential spectrum of $\mathcal{L}^\varepsilon$ coincides with the imaginary axis.*

Theorem 4 *Fix $b > 1/3$, and choose any $R > 0$ large. Then there exists $\varepsilon_p > 0$ such that for any $\varepsilon \in (0, \varepsilon_p)$ the point spectrum of $\mathcal{L}^\varepsilon$ is contained in $\mathrm{i}\mathbb{R} \cup \{|\sigma| \geq R\}$.*

Both theorems are proved in Sections 5 and 6. The result in Theorem 2 is a consequence of Theorems 3 and 4.

Remark 4.4 *In fact, we prove slightly more. We actually compute eigenvalues embedded into the essential spectrum $\Sigma_e(\mathcal{L}^\varepsilon) = \mathrm{i}\mathbb{R}$. We show that inside the essential spectrum, there is only the zero eigenvalue with geometric multiplicity two and algebraic multiplicity three. One eigenfunction is due to the invariance of the Euler equations under $\Phi \to \Phi + const.$, and the second eigenfunction is given by the x-derivative of the solitary wave. The generalized eigenvector to the second eigenfunction is given by the derivative of the solitary wave with respect to the wave speed.*

5 The essential spectrum of solitary waves

We prove Theorem 3. We study first the spectrum of the family of asymptotic operators $\mathcal{L}^\varepsilon_\infty = (\mathcal{L}_\infty(\sigma, \varepsilon))_{\sigma \in \mathbb{C}}$ where

$$\mathcal{L}_\infty(\sigma, \varepsilon) = \frac{d}{dx} - L_\infty(\sigma, \varepsilon), \quad L_\infty(\sigma, \varepsilon) = D_\infty(\sigma) + A_\infty(\varepsilon^2).$$

Lemma 5.1 *For any $\varepsilon \geq 0$, the essential spectrum of $\mathcal{L}^\varepsilon_\infty$ is equal to $\mathrm{i}\mathbb{R}$. The point spectrum of $\mathcal{L}^\varepsilon_\infty$ is empty.*

Proof. The asymptotic operators $D_\infty(\sigma)$ and $A_\infty(\varepsilon^2)$ are independent of x, so in order to determine the spectrum of $\mathcal{L}^\varepsilon_\infty$ we can use the Fourier transform in x. Let k denote the Fourier variable. Then the spectrum of $\mathcal{L}^\varepsilon_\infty$ in $\mathscr{H}$ coincides with the spectrum of $\widehat{\mathcal{L}}^\varepsilon_\infty = (\widehat{\mathcal{L}}_\infty(\sigma, \varepsilon))_{\sigma \in \mathbb{C}}$ where $\widehat{\mathcal{L}}_\infty(\sigma, \varepsilon) = \mathrm{i}k - L_\infty(\sigma, \varepsilon)$. The domain of $\widehat{\mathcal{L}}_\infty(\sigma, \varepsilon)$ is $\widehat{\mathscr{W}} = L^2(\mathbb{R}, Y) \cap \widehat{H^1}(\mathbb{R}, X)$, where

$$\widehat{H^1}(\mathbb{R}, X) = \{\widehat{f} \in L^2(\mathbb{R}, X) \ : \ (1 + |k|)\widehat{f} \in L^2(\mathbb{R}, X)\}.$$

The resolvent set of $\widehat{\mathcal{L}}^\varepsilon_\infty$ consists of the values $\sigma \in \mathbb{C}$ with the following two properties:

(i) $\Sigma(L_\infty(\sigma, \varepsilon)) \cap \mathrm{i}\mathbb{R} = \emptyset$, where $\Sigma(L_\infty(\sigma, \varepsilon))$ denotes the spectrum of $L_\infty(\sigma, \varepsilon)$ in X,

(ii) there exists a positive constant $C(\sigma,\varepsilon)$ such that the estimate

$$\|(\mathrm{i}k - L_\infty(\sigma,\varepsilon))^{-1}\|_{X\to X} \le \frac{C(\sigma,\varepsilon)}{1+|k|}, \tag{5.1}$$

holds for any $k \in \mathbb{R}$.

Indeed, assume that (i) and (ii) hold for some $\sigma \in \mathbb{C}$. Then, for any $\widehat{f} \in \mathscr{H}$ there exists $\widehat{g}(k) = (\mathrm{i}k - L_\infty(\sigma,\varepsilon))^{-1}\widehat{f}(k)$ with

$$\|(1+|k|)\widehat{g}\|^2_{\mathscr{H}} = \int_{\mathbb{R}} (1+|k|)^2 \|\widehat{g}(k)\|^2_X \,\mathrm{d}k \le C(\sigma,\varepsilon)^2 \int_{\mathbb{R}} \|\widehat{f}(k)\|^2_X \,\mathrm{d}k = C(\sigma,\varepsilon)^2 \|\widehat{f}\|^2_{\mathscr{H}}.$$

Hence $\widehat{g} \in \widehat{\mathscr{W}}$ and the map $\widehat{f} \to \widehat{g}$ is bounded from $\mathscr{H}$ into $\widehat{\mathscr{W}}$.

The operator $L_\infty(\sigma,\varepsilon)$ has compact resolvent, so its spectrum consists only of isolated eigenvalues of finite multiplicities. The eigenvalue problem

$$L_\infty(\sigma,\varepsilon)\tilde{\boldsymbol{w}} = \zeta\tilde{\boldsymbol{w}}, \quad \tilde{\boldsymbol{w}} \in Y,$$

can be solved explicitly. We find that ζ is an eigenvalue of $L_\infty(\sigma,\varepsilon)$ if

$$(\sigma+\zeta)^2\cos\zeta = (1+\varepsilon^2 - b\zeta^2)\zeta\sin\zeta. \tag{5.2}$$

Set $\sigma = \sigma_1 + i\sigma_2$ and $\zeta = \mathrm{i}k$. Then (5.2) yields

$$(\sigma_2+k)^2 - \sigma_1^2 = (1+\varepsilon^2+bk^2)k\,\tanh k, \tag{5.3}$$

$$2\sigma_1(\sigma_2+k) = 0. \tag{5.4}$$

If $\sigma_1 \neq 0$, i.e. $\sigma \notin \mathrm{i}\mathbb{R}$, the equality (5.4) implies $k = -\sigma_2$ which is clearly not a solution of (5.3). Hence (5.2) has no purely imaginary solutions, i.e. $\Sigma(L_\infty(\sigma,\varepsilon)) \cap \mathrm{i}\mathbb{R} = \emptyset$, for any $\sigma \notin \mathrm{i}\mathbb{R}$. If $\sigma_1 = 0$, i.e. $\sigma \in \mathrm{i}\mathbb{R}$, the last equality is always satisfied, and (5.3) has, for any $\sigma_2 \neq 0$, exactly two real solutions, one positive and one negative (recall that $b > 1/3$), so (5.2) has in this case two purely imaginary solutions, both simple and different from zero. For $\sigma = 0$, (5.2) has only one purely imaginary solution, $\zeta = 0$ which is a root of multiplicity two if $\varepsilon \neq 0$, and a root of multiplicity four if $\varepsilon = 0$. We conclude that (i) is satisfied for any $\sigma \notin \mathrm{i}\mathbb{R}$, and is not satisfied if $\sigma \in \mathrm{i}\mathbb{R}$.

We show that (ii) holds for any $\sigma \notin \mathrm{i}\mathbb{R}$. Recall that $A_\infty(\varepsilon^2) = \tilde{A}(1+\varepsilon^2)$ where $\tilde{A}(\lambda)$ is the linear operator in (3.7). Then (3.8) implies

$$\|(\mathrm{i}k - A_\infty(\varepsilon^2))^{-1}\|_{X\to X} \le \frac{C(\varepsilon)}{|k|},$$

for any $|k| \ge k_0$, for some positive k_0 and $C(\varepsilon)$. Since $D_\infty(\sigma)$ is a bounded operator in X,

$$\|(\mathrm{i}k - A_\infty(\varepsilon^2))^{-1}D_\infty(\sigma)\|_{X\to X} \le \|D_\infty(\sigma)\|\frac{C(\varepsilon)}{|k|} \le \frac{1}{2},$$

if $|k| \geq k_1(\sigma, \varepsilon) = \max\{k_0, 2\|D_\infty(\sigma)\|C(\varepsilon)\}$. Then

$$(\mathrm{i}k - L_\infty(\sigma, \varepsilon))^{-1} = (I + (\mathrm{i}k - A_\infty(\varepsilon^2))^{-1} D_\infty(\sigma))^{-1} (\mathrm{i}k - A_\infty(\varepsilon^2))^{-1},$$

so, for any $|k| \geq k_1(\sigma, \varepsilon)$,

$$\|(\mathrm{i}k - L_\infty(\sigma, \varepsilon))^{-1}\|_{X \to X} \leq \frac{2C(\varepsilon)}{|k|}.$$

Now (5.1) follows for $\sigma \notin \mathrm{i}\mathbb{R}$ from $\Sigma(L_\infty(\sigma, \varepsilon)) \cap \mathrm{i}\mathbb{R} = \emptyset$.

We conclude that any $\sigma \notin \mathrm{i}\mathbb{R}$ belongs to the resolvent of $\mathcal{L}^\varepsilon_\infty$. It remains to show that the entire imaginary axis belongs to the essential spectrum. We therefore exhibit an orthonormal sequence $\boldsymbol{w}_\ell \in X$, with $\mathcal{L}_\infty(\sigma, \varepsilon)\boldsymbol{w}_\ell \to 0$ and conclude that $\mathcal{L}_\infty(\sigma, \varepsilon)$ cannot be Fredholm of index zero, for $\sigma \in \mathrm{i}\mathbb{R}$.

From (5.3), (5.4), we find a $k_* = k_*(\sigma) \in \mathbb{R}$ and a vector $\boldsymbol{w}_0$ such that $(\mathrm{i}k_* - L_\infty(\sigma, \varepsilon))\boldsymbol{w}_0 = 0$. Let θ_R be a smooth, even cut-off function, with $\theta_R(x) = 1$ for $|x| \leq R$, $\theta_R(x) = 0$ for $|x| \geq R + 1$, and $\theta_R(x) = \theta_0(x - R)$ for $x \in [R, R + 1]$. Define $\tilde{\boldsymbol{w}}_\ell := \theta_\ell(x - 2\ell^2)\mathrm{e}^{\mathrm{i}k_* x}\boldsymbol{w}_0$ and renormalize $\boldsymbol{w}_\ell := \tilde{\boldsymbol{w}}_\ell / \|\tilde{\boldsymbol{w}}_\ell\|_{\mathscr{H}}$. Since the supports of all $\boldsymbol{w}_\ell$ are disjoint, the $\boldsymbol{w}_\ell$ form an orthonormal sequence. A straight forward computation shows that $\|\mathcal{L}_\infty(\sigma, \varepsilon)\boldsymbol{w}_\ell\|_{\mathscr{H}} = \mathrm{O}(\ell^{-1/2})$. This proves the Lemma. ■

We show now that the essential spectrum of $\mathcal{L}^\varepsilon$ is contained in $\mathrm{i}\mathbb{R}$.

Proposition 5.2 *There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$ and any $\sigma \notin \mathrm{i}\mathbb{R}$, the operator $\mathcal{L}(\sigma, \varepsilon)$ is Fredholm with zero index, so $\Sigma_e(\mathcal{L}^\varepsilon) \subset \mathrm{i}\mathbb{R}$.*

This proposition is proved in six steps contained in the following lemmas.

Lemma 5.3 *There exist positive constants ε_1, $c_1(\sigma, \varepsilon)$, $c_2(\sigma)$, such that the inequalities*

$$\|\boldsymbol{w}\|_{\mathscr{W}} \leq c_1(\sigma, \varepsilon)\|\mathcal{L}_\infty(\sigma, \varepsilon)\boldsymbol{w}\|_{\mathscr{H}}, \tag{5.5}$$

$$\|\boldsymbol{w}\|_{\mathscr{W}} \leq c_2(\sigma)\left(\|\boldsymbol{w}\|_{\mathscr{H}} + \|\mathcal{L}(\sigma, \varepsilon)\boldsymbol{w}\|_{\mathscr{H}}\right), \tag{5.6}$$

hold, for any $\varepsilon \in (0, \varepsilon_1)$, $\sigma \notin \mathrm{i}\mathbb{R}$ and $\boldsymbol{w} \in \mathscr{W}$.

Proof. From Lemma 5.1 follows

$$\|\mathcal{L}_\infty(\sigma, \varepsilon)^{-1}\boldsymbol{v}\|_{\mathscr{W}} \leq C(\sigma, \varepsilon)\|\boldsymbol{v}\|_{\mathscr{H}},$$

for any $\varepsilon > 0$, $\sigma \notin \mathrm{i}\mathbb{R}$, $\boldsymbol{v} \in \mathscr{H}$. For $\boldsymbol{w} \in \mathscr{W}$ set $\boldsymbol{v} = \mathcal{L}_\infty(\sigma, \varepsilon)\boldsymbol{w} \in \mathscr{H}$. Then

$$\|\boldsymbol{w}\|_{\mathscr{W}} = \|\mathcal{L}_\infty(\sigma, \varepsilon)^{-1}\boldsymbol{v}\|_{\mathscr{W}} \leq C(\sigma, \varepsilon)\|\boldsymbol{v}\|_{\mathscr{H}} = C(\sigma, \varepsilon)\|\mathcal{L}_\infty(\sigma, \varepsilon)\boldsymbol{w}\|_{\mathscr{H}}.$$

and (5.5) is proved.

Choose $\sigma_0 \notin \mathrm{i}\mathbb{R}$ and $\varepsilon_0 \in (0, \varepsilon^*)$. Then

$$\begin{aligned}\|\boldsymbol{w}\|_{\mathscr{W}} \;\le\; & c_1(\sigma_0,\varepsilon_0)\|\mathcal{L}_\infty(\sigma_0,\varepsilon_0)\boldsymbol{w}\|_{\mathscr{H}} \le c_1(\sigma_0,\varepsilon_0)\Big[\|\mathcal{L}(\sigma,\varepsilon)\boldsymbol{w}\|_{\mathscr{H}} \\ & + \|(D_\infty(\sigma) - D_\infty(\sigma_0))\boldsymbol{w}\|_{\mathscr{H}} + \|(A_\infty(\varepsilon^2) - A_\infty(\varepsilon_0^2))\boldsymbol{w}\|_{\mathscr{H}} \\ & + \varepsilon^2\|D_1(x;\sigma,\varepsilon)\boldsymbol{w}\|_{\mathscr{H}} + \varepsilon^2\|A_1(x;\varepsilon)\boldsymbol{w}\|_{\mathscr{H}}\Big].\end{aligned}$$

From the explicit formula for $A_\infty(\varepsilon^2) = \tilde{A}(1+\varepsilon^2)$ we deduce that $A_\infty(\varepsilon^2) - A_\infty(\varepsilon_0^2)$ is a bounded operator in X, and

$$\|(A_\infty(\varepsilon^2) - A_\infty(\varepsilon_0^2))\boldsymbol{w}\|_{\mathscr{H}} \le |\varepsilon^2 - \varepsilon_0^2|\|\boldsymbol{w}\|_{\mathscr{H}}.$$

Furthermore, Lemma 4.2 implies

$$\|D_1(x;\sigma,\varepsilon)\boldsymbol{w}\|_{\mathscr{H}} \le C|\sigma|(1+|\sigma|)\|\boldsymbol{w}\|_{\mathscr{H}}, \quad \|A_1(x;\varepsilon)\boldsymbol{w}\|_{\mathscr{H}} \le C\|\boldsymbol{w}\|_{\mathscr{W}},$$

for any $\boldsymbol{w} \in \mathscr{W}$. The constant C is independent of ε and σ. Finally, recall that $D_\infty(\sigma)$ is bounded in X and conclude

$$\|\boldsymbol{w}\|_{\mathscr{W}} \;\le\; c_1(\sigma_0,\varepsilon_0)\Big[\|\mathcal{L}(\sigma,\varepsilon)\boldsymbol{w}\|_{\mathscr{H}} + C(\sigma)\|\boldsymbol{w}\|_{\mathscr{H}} + |\varepsilon^2 - \varepsilon_0^2|\|\boldsymbol{w}\|_{\mathscr{H}} + \varepsilon^2 C\|\boldsymbol{w}\|_{\mathscr{W}}\Big].$$

Choose ε_1 such that $\varepsilon_1^2 c_1(\sigma_0,\varepsilon_0)C \le 1/2$ and (5.6) is proved. ∎

For the next two lemmas we follow [RS95]. For each $T > 0$ define the Hilbert spaces

$$\mathscr{H}_T = L^2([-T,T],X),$$
$$\mathscr{W}_T = L^2([-T,T],Y) \cap H^1([-T,T],X).$$

The embedding $\mathscr{W}_T \subset \mathscr{H}_T$ is compact (*cf.* [RS95], Lemma 3.8).

Lemma 5.4 *Assume $\varepsilon \in (0,\varepsilon_1)$ and $\sigma \notin \mathrm{i}\mathbb{R}$. There exist $T = T(\sigma,\varepsilon) > 0$ and $c_3(\sigma,\varepsilon) > 0$, such that the inequality*

$$\|\boldsymbol{w}\|_{\mathscr{W}} \le c_3(\sigma,\varepsilon)\left(\|\boldsymbol{w}\|_{\mathscr{H}_T} + \|\mathcal{L}(\sigma,\varepsilon)\boldsymbol{w}\|_{\mathscr{H}}\right), \tag{5.7}$$

holds, for any $\boldsymbol{w} \in \mathscr{W}$.

Proof. Assume $\boldsymbol{w} \in \mathscr{W}$ is such that $\boldsymbol{w}(x) = 0$, for $|x| \le T$, for some $T > 0$. Then (5.5) and the inequalities in Lemma 4.2 imply

$$\begin{aligned}\|\boldsymbol{w}\|_{\mathscr{W}} \;&\le\; c_1(\sigma,\varepsilon)\|\mathcal{L}_\infty(\sigma,\varepsilon)\boldsymbol{w}\|_{\mathscr{H}} \\ &\le\; c_1(\sigma,\varepsilon)\left[\|\mathcal{L}(\sigma,\varepsilon)\boldsymbol{w}\|_{\mathscr{H}} + \varepsilon^2\|D_1(x;\sigma,\varepsilon)\boldsymbol{w}\|_{\mathscr{H}} + \varepsilon^2\|A_1(x;\varepsilon)\boldsymbol{w}\|_{\mathscr{H}}\right] \\ &\le\; c_1(\sigma,\varepsilon)\|\mathcal{L}(\sigma,\varepsilon)\boldsymbol{w}\|_{\mathscr{H}} + C_0(\sigma,\varepsilon)\varepsilon^2\mathrm{e}^{-\sqrt{\beta}\varepsilon T}\|\boldsymbol{w}\|_{\mathscr{W}}.\end{aligned}$$

Then, there exist $T = T(\sigma,\varepsilon) > 0$ and $C_1(\sigma,\varepsilon) > 0$ such that for any $\boldsymbol{w} \in \mathscr{W}$, with $\boldsymbol{w}(x) = 0$, for $|x| \le T-1$, we have

$$\|\boldsymbol{w}\|_{\mathscr{W}} \le C_1(\sigma,\varepsilon)\|\mathcal{L}(\sigma,\varepsilon)\boldsymbol{w}\|_{\mathscr{H}}. \tag{5.8}$$

Take a smooth cutoff function $\phi : \mathbb{R} \to [0,1]$ such that $\phi(x) = 0$ for $|x| \ge T$, $\phi(x) = 1$ for $|x| \le T-1$, and $|\phi'(x)| \le m$. Using (5.6) and (5.8) we obtain

$$\begin{aligned}\|\boldsymbol{w}\|_{\mathscr{W}} &\le \|\phi\boldsymbol{w}\|_{\mathscr{W}} + \|(1-\phi)\boldsymbol{w}\|_{\mathscr{W}} \le c_2(\sigma)\left(\|\phi\boldsymbol{w}\|_{\mathscr{H}} + \|\mathcal{L}(\sigma,\varepsilon)\phi\boldsymbol{w}\|_{\mathscr{H}}\right)\\ &\quad + C_1(\sigma,\varepsilon)\|\mathcal{L}(\sigma,\varepsilon)(1-\phi)\boldsymbol{w}\|_{\mathscr{H}} \le c_3(\sigma,\varepsilon)\left(\|\boldsymbol{w}\|_{\mathscr{H}_T} + \|\mathcal{L}(\sigma,\varepsilon)\boldsymbol{w}\|_{\mathscr{H}}\right),\end{aligned}$$

since $\mathcal{L}(\sigma,\varepsilon)\phi\boldsymbol{w} = \phi\mathcal{L}(\sigma,\varepsilon)\boldsymbol{w} + \phi'\boldsymbol{w}$. ■

Lemma 5.5 *For any $\varepsilon \in (0,\varepsilon_1)$ and $\sigma \notin \mathrm{i}\mathbb{R}$, the operator $\mathcal{L}(\sigma,\varepsilon)$ has closed range and finite dimensional kernel.*

Proof. Since the restriction $\mathscr{W} \to \mathscr{H}_T$ is compact the conclusion follows from the Lemma 5.4 and the Abstract Closed Range Lemma (*cf.* [RS95]). ■

Lemma 5.6 *For any $\varepsilon \in (0,\varepsilon_1)$ and $\sigma \notin \mathrm{i}\mathbb{R}$, the adjoint operator $\mathcal{L}(\sigma,\varepsilon)^*$ has closed range and finite dimensional kernel.*

Proof. The proof is similar to the proof of Lemma 5.5 and we omit it. ■

Lemmas 5.5 and 5.6 imply:

Lemma 5.7 *For any $\varepsilon \in (0,\varepsilon_1)$ and $\sigma \notin \mathrm{i}\mathbb{R}$, the operator $\mathcal{L}(\sigma,\varepsilon)$ is Fredholm.*

Finally, we show

Lemma 5.8 *For any $\varepsilon \in (0,\varepsilon_1)$ and $\sigma \notin \mathrm{i}\mathbb{R}$, the Fredholm index of $\mathcal{L}(\sigma,\varepsilon)$ is zero.*

Proof. Since $\mathcal{L}(\sigma,\varepsilon) - \mathcal{L}_\infty(\sigma,\varepsilon)$ is a small perturbation of $\mathcal{L}_\infty(\sigma,\varepsilon)$, and since this operator has a bounded inverse from $\mathscr{H}$ into $\mathscr{W}$, for any $\sigma \notin \mathrm{i}\mathbb{R}$, a perturbation argument shows that $\mathcal{L}(\sigma,\varepsilon)$ is invertible for σ in an open set in the right half plane $\mathrm{Re}\,\sigma > 0$, and for σ in an open set in the left half plane $\mathrm{Re}\,\sigma < 0$. Hence, for σ in these open subsets the Fredholm index of $\mathcal{L}(\sigma,\varepsilon)$ is zero. Since the Fredholm index of $\mathcal{L}(\sigma,\varepsilon)$ is constant on connected subsets of $\mathbb{C} \setminus \mathrm{i}\mathbb{R}$, we conclude that its Fredholm index is zero, for any $\varepsilon \in (0,\varepsilon_1)$ and $\sigma \notin \mathrm{i}\mathbb{R}$. ■

Proposition 5.9 *For any $\varepsilon \in (0,\varepsilon_1)$, the entire imaginary axis $\sigma \in \mathrm{i}\mathbb{R}$ belongs to the essential spectrum of $\mathcal{L}^\varepsilon$.*

Proof. The proof is identical to the proof for $\mathcal{L}^\varepsilon_\infty$ from Lemma 5.1. The orthonormal sequence $\boldsymbol{w}_\ell$, which was constructed there, satisfies $\mathcal{L}(\sigma,\varepsilon)\boldsymbol{w}_\ell \to 0$ for $\ell \to \infty$. ■

6 The point spectrum of solitary waves

The goal of this section is to prove Theorem 4. Equivalently, given the information on the essential spectrum from Theorem 3, we show that $\mathrm{Re}\,\sigma \neq 0$ belongs to the resolvent set for bounded $|\sigma|$ and small ε.

Proposition 6.1 *For any $R > 0$, there exists $\varepsilon_2 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_2)$ and any $\sigma \notin \mathrm{i}\mathbb{R}$, $|\sigma| \leq R$, the operator $\mathcal{L}(\sigma, \varepsilon)$ is invertible.*

The proposition is proved in several steps. Since we have bounds on the norm of $\mathcal{L}_\infty(\sigma, \varepsilon)^{-1}$, uniformly for values $|\mathrm{Re}\,\sigma| \geq \delta > 0$, $|\sigma| \leq R$, it is sufficient to consider a neighborhood of the imaginary axis $\sigma \in \mathrm{i}[-R, R]$. We therefore concentrate on a neighborhood of $\sigma = \mathrm{i}q$ for fixed q. There are then two different cases:

I) finite frequencies $q \neq 0$

II) small frequencies $q = 0$.

In both cases, we are interested in the kernel of the operator $\mathcal{L}(\sigma, \varepsilon)$, which is Fredholm index zero for $\mathrm{Re}\,\sigma \neq 0$. Elements of the kernel are bounded solutions of the abstract, non-autonomous, linear differential equation

$$\widetilde{\boldsymbol{W}}_x = D(\sigma, \varepsilon)\widetilde{\boldsymbol{W}} + A(\varepsilon)\widetilde{\boldsymbol{W}}. \tag{6.1}$$

It is sufficient to show that this ordinary differential equation does not possess any non-trivial, bounded solutions. We will see that, just as for the nonlinear steady equation, bounded solutions lie on a finite-dimensional, invariant manifold. To the abstract, quasilinear differential equation (6.1), we apply non-autonomous center-manifold reduction; see [Mi88]. The reduction is performed for σ close to $\mathrm{i}q \in \mathrm{i}\mathbb{R}$ and ε small. Note that for any q fixed, finite, and $\varepsilon = 0$, the linear equation is a relatively bounded perturbation of the principal part

$$\widetilde{\boldsymbol{W}}_x = L_\infty(\mathrm{i}q, 0)\widetilde{\boldsymbol{W}} = (D_\infty(\mathrm{i}q) + A_\infty(0))\widetilde{\boldsymbol{W}},$$

with small relative bound. In Lemma 5.1 we proved the resolvent estimate

$$|(\mathrm{i}k - L_\infty(\mathrm{i}q, 0))^{-1}|_{X \to X} \leq \frac{C}{|k|},$$

for all $|k| \geq k_0(q)$, and we may apply the reduction theorem in [Mi88] in a neighborhood of any fixed point q, uniformly for bounded q.

6.1 The case of non-zero frequency

6.1.1 The reduction

We exclude point spectrum in a neighborhood of $\mathrm{i}q \neq 0$, case (I). Set $\sigma = \mathrm{i}q + \delta$ and rewrite (6.1) as

$$\widetilde{\boldsymbol{W}}_x = L_\infty(\mathrm{i}q, 0)\widetilde{\boldsymbol{W}} + \delta B_\infty(\delta)\widetilde{\boldsymbol{W}} + \varepsilon^2(B_0 + B_1(x;\delta,\varepsilon))\widetilde{\boldsymbol{W}}, \tag{6.2}$$

where $L_\infty(\mathrm{i}q, 0) = D_\infty(\mathrm{i}q) + A_\infty(0)$, and

$$\delta B_\infty(\delta) = D_\infty(\mathrm{i}q + \delta) - D_\infty(\mathrm{i}q), \quad \varepsilon^2 B_0 = A_\infty(\varepsilon^2) - A_\infty(0),$$

$$B_1(x;\delta,\varepsilon) = D_1(x;\mathrm{i}q + \delta, \varepsilon) + A_1(x;\varepsilon).$$

We view equation (6.2) as a small perturbation of the eigenvalue problem for $\delta = 0$ and $\varepsilon = 0$. This is justified by the following inequalities

$$\|B_\infty(\delta)\|_{Y(\text{resp. } X)\to Y(\text{resp. } X)} \le C(1 + |q|), \quad \|B_0\|_{Y(\text{resp. } X)\to Y(\text{resp. } X)} \le C$$

$$\|B_1(x;\delta,\varepsilon)\|_{Y\to X} \le C(1 + |q|^2)\mathrm{e}^{-\sqrt{\beta}\varepsilon|x|}$$

for $\varepsilon \in (0, \varepsilon_3)$ and any $q \neq 0$.

The reduction procedure is performed for small ε and δ. We have to find the center eigenspace of the linear operator $L_\infty(\mathrm{i}q, 0)$. The linear operator $L_\infty(\mathrm{i}q, 0)$ is closed in X with dense domain Y. Moreover, it has compact resolvent, so its spectrum consists only of isolated eigenvalues of finite multiplicities. As shown in the proof of Lemma 5.1, ζ is an eigenvalue of $L_\infty(\mathrm{i}q, 0)$ if

$$(\mathrm{i}q + \zeta)^2 \cos\zeta = (1 - b\zeta^2)\zeta \sin\zeta.$$

Imaginary solutions $\zeta = \mathrm{i}k$ of this equation satisfy

$$(q + k)^2 = (1 + bk^2)k \tanh k.$$

We find exactly two simple roots $\mathrm{i}k_1$ and $\mathrm{i}k_2$ with $k_2 < 0 < k_1$ (since $b > 1/3$). Hence, $L_\infty(\mathrm{i}q, 0)$ has two simple, purely imaginary eigenvalues $\mathrm{i}k_1, \mathrm{i}k_2$. The corresponding eigenvectors are

$$\boldsymbol{w}_{1,2} = \begin{pmatrix} \cosh(k_{1,2}y) + \frac{1}{2}qy^2\tilde{z}_{1,2} \\ \mathrm{i}k_{1,2}\cosh(k_{1,2}y) \\ \mathrm{i}\tilde{z}_{1,2} \\ -bk_{1,2}\tilde{z}_{1,2} \end{pmatrix}, \quad \tilde{z}_{1,2} = -\frac{k_{1,2}\sinh k_{1,2}}{k_{1,2} + q} = -\frac{(k_{1,2} + q)\cosh k_{1,2}}{\lambda + bk_{1,2}^2}.$$

The center manifold reduction implies that small bounded solutions of (6.2) are of the form

$$\widetilde{\boldsymbol{W}}(x) = a_1(x)\boldsymbol{w}_1 + a_2(x)\boldsymbol{w}_2 + \mathrm{O}((|\delta| + \varepsilon^2)|a_j|). \tag{6.3}$$

For the amplitudes $\boldsymbol{a} = (a_1, a_2)$, we find a linear, non-autonomous system of ordinary differential equations, depending on the eigenvalue parameter σ and the bifurcation parameter ε

$$\boldsymbol{a}_x = A(x; \delta, \varepsilon)\boldsymbol{a}. \tag{6.4}$$

In $\varepsilon = 0$, the 2×2-matrix A does not depend on x any more and possesses two distinct purely imaginary eigenvalues. In the remainder of this section, we set up a perturbation argument, which shows that for ε small and $\mathrm{Re}\,\delta > 0$, there are no bounded solutions to (6.4).

6.1.2 Exponential dichotomies

In $\varepsilon = 0$, the equation (6.4) is autonomous. At $\mathrm{Re}\,\sigma = 0$, the spectrum of the matrix A consists precisely of the eigenvalues $\zeta_1 = \mathrm{i}k_1$, $k_1 > 0$ and $\zeta_2 = \mathrm{i}k_2$, $k_2 < 0$. Depending on $\delta = \sigma - \mathrm{i}q$, the eigenvalues may move off the imaginary axis. A direct computation shows that $\mathrm{d}\zeta_1/\mathrm{d}\delta > 0$ and $\mathrm{d}\zeta_2/\mathrm{d}\delta < 0$, such that the eigenvalues leave the axis, with non-vanishing speed, in opposite directions. In particular, for $\mathrm{Re}\,\sigma > 0$ small and $\varepsilon = 0$, we find that (6.4) is a hyperbolic, linear ordinary differential equation. The eigenspaces are analytic in $\delta = \sigma - \mathrm{i}q$.

For $\varepsilon > 0$, the eigenvalues ζ_1 and ζ_2 still describe the dynamics at $x = \pm\infty$, since the solitary wave and therefore the coefficients of the matrix $A(x; \delta, \varepsilon)$ converge to zero, with rate $\mathrm{e}^{-\sqrt{\beta\varepsilon}|x|}$.

Therefore, when $\mathrm{Re}\,\delta > 0$, the dynamics for $|x| \to \infty$ are hyperbolic, with stable eigenvalue ζ_2 and unstable eigenvalue ζ_1. The following lemma on exponential dichotomies shows in which sense the hyperbolic structure can be continued to finite x. We therefore consider a general non-autonomous, linear differential equation

$$\boldsymbol{a}_x = A(x; \delta, \mu), \ \boldsymbol{a} \in \mathbb{R}^n, \tag{6.5}$$

depending on a real parameter μ and a complex spectral parameter δ. In our example, μ represents the (small) parameter ε.

Lemma 6.2 *Consider (6.5) with fundamental solution $\varphi(x, y)$. Assume asymptotically constant coefficients $A(x; \delta, \mu) \to A_\pm(\delta, \mu)$ as $x \to \infty$, and smoothness: A and $A_\pm$ are C^k in the parameter $\mu \in U_\mu \subseteq \mathbb{R}$, $k \geq 0$, and analytic in the spectral parameter $\delta \in U_\delta \subseteq \mathbb{C}$, and A is continuous in x. Furthermore assume that $A_\pm$ are hyperbolic, that is, they do not possess eigenvalues on the imaginary axis, for all $\mu \in U_\mu$ and all $\delta \in U_\delta$.*

Then there exists a unique decomposition of the phase space $\mathbb{R}^n$ into linear, stable and unstable subspaces $E^{\mathrm{s}}_+(x; \delta, \mu)$ and $E^{\mathrm{u}}_-(x; \delta, \mu)$, which are as smooth as A. The subspaces are invariant under the linear evolution $\varphi(x, y)$:

$$\varphi(x, y)E^{\mathrm{s}}_+(y) = E^{\mathrm{s}}_+(x), \quad \varphi(x, y)E^{\mathrm{u}}_-(y) = E^{\mathrm{u}}_-(x).$$

Moreover, any initial value to a bounded solution on $[0,\infty)$ *is contained in* $E^{\mathrm{s}}_{+}(0)$ *and initial values to bounded solutions on* $(-\infty,0]$ *are contained in* $E^{\mathrm{u}}_{-}(0)$.

On the other hand, there are positive constants C, $\eta_+ > 0$, *and* $\eta_- > 0$ *such that we have uniform exponential decay for solutions in forward time,*

$$|\varphi(x,y)\boldsymbol{a}| \leq C\mathrm{e}^{-\eta_+|x-y|}|\boldsymbol{a}|$$

for all $\boldsymbol{a} \in E^{\mathrm{s}}_{+}(y)$, $x \geq y \geq 0$, *and in backward time*

$$|\varphi(x,y)\boldsymbol{a}| \leq C\mathrm{e}^{-\eta_-|x-y|}|\boldsymbol{a}|$$

for all $\boldsymbol{a} \in E^{\mathrm{u}}_{-}(y)$, $x \leq y \leq 0$. *The constants* C *and* $\eta_\pm$ *can be chosen independently of* μ, δ *in compact subsets of* $U_\mu \times U_\delta$.

For the proof, see [Co78], for example.

By the above lemma, we find nontrivial, bounded solutions, if and only if stable and unstable subspaces intersect nontrivially

$$E^{\mathrm{u}}_{-}(0) \cap E^{\mathrm{s}}_{+}(0) \neq \{0\}.$$

We may choose bases $\boldsymbol{a}^j_\pm$, analytic in δ and continuous in μ in the two subspaces and compute the determinant

$$\mathcal{E}(\delta;\mu) = \det(\boldsymbol{a}^j_\pm). \tag{6.6}$$

A variant of this analytic function is usually referred to as the Evans function [Ev72, AGJ90]. Clearly, zeroes of $\mathcal{E}$ detect precisely the nontrivial bounded solutions to (6.4), and therefore the point spectrum coincides with the zeroes of $\mathcal{E}$. The algebraic multiplicity of eigenvalues coincides with the order of the zeroes of $\mathcal{E}$; see [AGJ90]. By analyticity in δ and continuous dependence on μ, the number of zeroes counted with multiplicity varies continuously with μ. We are going to exploit this fact in Section 6.2.

In our setting, both subspaces are well-defined and complex one-dimensional for $\mathrm{Re}\,\delta > 0$. They are spanned by the complex vectors $a^{\mathrm{s}}_{+}(0)$ and $a^{\mathrm{u}}_{-}(0)$, which lead to solutions $a^{\mathrm{s}}_{+}(x)$ and $a^{\mathrm{u}}_{-}(x)$. It is our goal to show, that both solutions can be extended, analytically in δ and continuously in ε in an open neighborhood of $\delta = 0$, in particular, across the imaginary axis where hyperbolicity at $x = \pm\infty$ is lost, into the left half plane. We show that in the limit $\varepsilon = 0$ and $\mathrm{Re}\,\delta = 0$, the initial values $a^{\mathrm{s}}_{+}(0)$ and $a^{\mathrm{u}}_{-}(0)$ converge to eigenvectors e_2 and e_1 to the eigenvalues ζ_2 and ζ_1, respectively.

In particular, $\mathcal{E}(0;0) \neq 0$, and by continuity, we can exclude unstable eigenvalues in a neighborhood of $\sigma = \mathrm{i}q$.

6.1.3 A gap lemma

The goal here is, to continue the Evans function across the essential spectrum. The idea is to exploit rapid convergence of the coefficients of the non-autonomous differential equation $A(x)$, compensating for the loss of hyperbolicity in the asymptotic equation at $x = \pm\infty$.

The main idea was already used in [GZ98, Theorem 2.3] and [KS98, Lemma 2.2].

We recall the results stated there.

Theorem 5 *[GZ98, KS98] Consider a non-autonomous, linear differential equation*

$$\boldsymbol{a}_x = A(x;\delta,\mu)\boldsymbol{a} \in \mathbb{R}^n$$

with fundamental solution $\varphi(x,y)$, with paramters $\delta \in U_\delta(0) \subset \mathbb{C}$ and $\mu \in U_\mu(0) \subset \mathbb{R}$ close to the origin. Assume exponential convergence to asymptotically constant coefficients

$$|A(x;\delta,\mu) - A_\infty(\delta,\mu)| \leq C\mathrm{e}^{-\eta|x|}$$

with positive constants $C, \eta > 0$. Assume furthermore that A and A_∞ are C^k in μ, $k \geq 0$, and analytic in δ, and A is continuous in x. At $\mu = 0$, $\delta = 0$, we require the existence of a spectral projection P to A_∞ such that $\mathrm{Re}\,\mathrm{spec}\,PA_\infty \leq 0$ *and* $\mathrm{Re}\,\mathrm{spec}\,(\mathrm{id}\,-P)A_\infty \geq 0$.

Then there exists a unique decomposition of the phase space $\mathbb{R}^n$ into linear, stable and unstable subspaces $E^{\mathrm{s}}_+(x;\delta,\mu)$ and $E^{\mathrm{u}}_-(x;\delta,\mu)$, which are as smooth as A. The subspaces are invariant under the linear evolution $\varphi(x,y)$:

$$\varphi(x,y)E^{\mathrm{s}}_+(y) = E^{\mathrm{s}}_+(x), \quad \varphi(x,y)E^{\mathrm{u}}_-(y) = E^{\mathrm{u}}_-(x).$$

Solutions to initial values in $E^{\mathrm{s}}_+(0)$ converge to $E^{\mathrm{s}}(\delta,\mu)$ as $x \to \infty$, where the eigenspace $E^{\mathrm{s}}(\delta,\mu)$ smoothly depends on δ and μ and coincides with the range $\mathrm{Im}\,P$ *for $\mu = 0$, $\delta = 0$.*

Also, solutions to initial values in $E^{\mathrm{u}}_-(0)$ converge to $E^{\mathrm{u}}(\delta,\mu)$ as $x \to \infty$, where the eigenspace $E^{\mathrm{u}}(\delta,\mu)$ smoothly depends on δ and μ and coincides with the kernel $\mathrm{Ker}\,P$ *for $\mu = 0$, $\delta = 0$.*

In particular, for parameter values δ, μ where the eigenspaces $E^{\mathrm{s/u}}(\delta,\mu)$ are actually the stable and unstable eigenspaces, respectively, the subspaces $E^{\mathrm{s}}_+(x;\delta,\mu)$ and $E^{\mathrm{u}}_-(x;\delta,\mu)$ coincide with the eigenspaces from Lemma 6.2.

In our problem, one additional difficulty arises. The convergence rate η of the non-autonomous perturbation depends on $\mu = \varepsilon$. The rate, $\sqrt{\beta}\varepsilon$, although fast compared to the eigenvalues of the asymptotic matrix $\mathrm{O}(\varepsilon^2)$, is not bounded away from zero, as required in the above theorem.

We therefore restate a parameter-dependent version of these results, taking into account the different orders of convergence of the solitary wave and possible eigenfunctions.

Proposition 6.3 *Consider a non-autonomous, linear differential equation $\boldsymbol{a}_x = A(x;\delta,\mu)\boldsymbol{a}$, depending on a parameter $\mu \in \mathbb{R}^p$ and an eigenvalue parameter $\delta \in \mathbb{C}$, in a neighborhood of the origin in $\mathbb{R}^p \times \mathbb{C}$. Assume that the coefficients A are C^k, $k \geq 0$, in μ and analytic in δ, and that A is continuous in x. Furthermore assume that the coefficients $A(x;\delta,\mu)$ converge to constant matrices, as $|x| \to \infty$*

$$|A(x;\delta,\mu) - A_\infty(\delta,\mu)| \leq C|\mu|\mathrm{e}^{-\eta(\mu)|x|},$$

and, as $\mu \to 0$,

$$|A(x;\delta,\mu) - A_0(\delta)| \leq C|\mu|.$$

Assume $\operatorname{spec} A_0(0) \subset \mathrm{i}\mathbb{R}$, *and $A_\infty(\delta,\mu)$ is hyperbolic for* $\operatorname{Re}\delta \neq 0$, *with*

$$|(\mathrm{i}k - A_\infty(\delta,\mu))^{-1}| \leq \frac{C}{|\operatorname{Re}\delta|}, \tag{6.7}$$

for all $k \in \mathbb{R}$ and with $C > 0$ independent of $\mu \geq 0$.

Suppose that spatial convergence of the coefficients is fast compared to the rate of hyperbolicity:

$$\mu/\eta(\mu) \to 0 \text{ as } \mu \to 0.$$

Then, the Evans function $\mathcal{E}(\delta;\mu)$ defined for $\operatorname{Re}\delta > 0$, *can be extended continuously in μ and analytically in δ, in a sector $\{(\delta,\mu); -\operatorname{Re}\delta \leq M|\mu|\}$, for any fixed constant $M > 0$. In the limit $\mu \to 0$, we find $\mathcal{E}(\delta;0) \neq 0$ for δ close to zero.*

Proof. For any $\mu \neq 0$ small, the conclusions of the proposition directly follow from the gap lemma, Theorem 5. We have to show that the limit $\mu \to 0$ of $\mathcal{E}(\delta;\mu)$ exists, and is nonzero.

For $\operatorname{Re}\delta > 0$, $\mu \geq 0$, the equation possesses exponential dichotomies, as stated in Lemma 6.2. The subspaces can actually be constructed from a fixed point argument. We focus on E^{s}_+, first. From the resolvent estimate (6.7), we conclude that the subspaces corresponding to stable and unstable eigenvalues $E^{\mathrm{s/u}}_+(\delta)$ for the equation with $\mu = 0$ continue analytically in a neighborhood of $\delta = 0$. We write P_+ for the projection on $E^{\mathrm{s}}_+(0)$ along $E^{\mathrm{u}}_+(0)$, and $B(y) := A(y) - A_\infty$, suppressing the dependence on δ and μ. For $\operatorname{Re}\delta > 0$, solutions $\boldsymbol{a}(x)$ which are bounded on $x \geq 0$ then solve the integral equation

$$\boldsymbol{a}(x) = \mathrm{e}^{A_\infty x}\boldsymbol{a}_0 + \int_0^x \mathrm{e}^{A_\infty(x-y)}P_+B(y)\boldsymbol{a}(y)\mathrm{d}y + \int_\infty^x \mathrm{e}^{A_\infty(x-y)}(\mathrm{id} - P_+)B(y)\boldsymbol{a}(y)\mathrm{d}y$$

with $\boldsymbol{a}_0 = P_+\boldsymbol{a}(0)$. We substitute $\hat{\boldsymbol{a}}(x) = \mathrm{e}^{-A_\infty x}\boldsymbol{a}(x)$ and arrive at

$$\hat{\boldsymbol{a}}(x) = \boldsymbol{a}_0 + \int_0^x \mathrm{e}^{-A_\infty y}P_+B(y)\mathrm{e}^{A_\infty y}\hat{\boldsymbol{a}}(y)\mathrm{d}y + \int_\infty^x \mathrm{e}^{-A_\infty y}(\mathrm{id} - P_+)B(y)\mathrm{e}^{A_\infty y}\hat{\boldsymbol{a}}(y)\mathrm{d}y.$$

We view the right side as an affine operator on the space of bounded, continuous functions on $[0, \infty)$, equipped with the supremum norm. Since $B(y) \le C|\mu|\mathrm{e}^{-\eta(\mu)|y|}$, and $|\mathrm{e}^{A_\infty y}| \le C\mathrm{e}^{C|\mu|y}$ for $-\operatorname{Re}\delta \le C|\mu|$, we find that the norm of the linear part of the right side is $C'\mu/\eta(\mu)$, which converges to zero for $\mu \to 0$ by assumption. We therefore find a unique solution $\hat{\boldsymbol{a}}(x)$ in the sector, which converges to the constant solution as $\mu \to 0$. We find the stable subspace as $\hat{\boldsymbol{a}}(0)$, parameterized over $\boldsymbol{a}_0$. The construction of the unstable subspace is similar. In the limit, $\mu = 0$, we find the Evans function for the constant coefficient equation, which is nonzero, since we have a spectral decomposition on the imaginary axis corresponding to the limits of stable and unstable subspaces. ∎

Together with the considerations in Section 6.1.2, this proves absence of point spectrum in a neighborhood of the imaginary axis, outside a given small neighborhood of the origin, which we consider next.

6.2 The case of small frequency

We exclude point spectrum in a neighborhood of the origin $\sigma = 0$, off the imaginary axis. As a first step, we reduce the eigenvalue problem to finding non-trivial solutions to a four-dimensional non-autonomous ordinary differential equation; Section 6.2.1. We then introduce and justify a long-wave scaling corresponding to the Korteweg-de Vries limit; Section 6.2.2. We then recall from [PW92] the structure of the spectrum in the scaling limit, where we find the spectrum of the Korteweg-de Vries soliton; Section 6.2.3. The last part of this chapter, Section 6.2.4, is devoted to the central perturbation arguments. We show that the spectrum of the capillary-gravity waves coincides with the point spectrum of the Korteweg-de Vries soliton in a neighborhood of the imaginary axis.

6.2.1 The reduction

Rewrite (6.1) for $\sigma = \delta$ small as

$$\widetilde{\boldsymbol{W}}_x = L_\infty(0,0)\widetilde{\boldsymbol{W}} + \delta B_\infty(\delta)\widetilde{\boldsymbol{W}} + \varepsilon^2(B_0 + B_1(x;\delta,\varepsilon))\widetilde{\boldsymbol{W}}, \tag{6.8}$$

where $L_\infty(0,0) = A_\infty(0)$, and

$$\delta B_\infty(\delta) = D_\infty(\delta), \quad \varepsilon^2 B_0 = A_\infty(\varepsilon^2) - A_\infty(0), \quad B_1 = D_1 + A_1.$$

Recall that $A_\infty(0) = \tilde{A}(1)$, so it is exactly the linear operator used for the analysis of the steady problem in Theorem 1. From those results we find that $A_\infty(0)$ has only one purely imaginary eigenvalue $\zeta = 0$, with algebraic multiplicity four. The corresponding (generalized) eigenvectors are $\boldsymbol{w}_0$, $\boldsymbol{w}_1$, $\boldsymbol{w}_2$, $\boldsymbol{w}_3$ found in the proof of Theorem 1.

The center manifold reduction implies that the bounded solutions of (6.8) are of the form

$$\widetilde{\boldsymbol{W}}(x) = a_0(x)\boldsymbol{w}_0 + a_1(x)\boldsymbol{w}_1 + a_2(x)\boldsymbol{w}_2 + a_3(x)\boldsymbol{w}_3 + \mathrm{O}((|\delta| + \varepsilon^2)|a_j|),$$

and the amplitudes a_j satisfy a non-autonomous, linear, reduced system of the form

$$\begin{aligned}
a_{0,x} &= a_1 + \delta(c_{00}a_0 + c_{02}a_2) + \varepsilon^2(c_{01}a_1 + c_{03}a_3) + \varepsilon^2 f_0(x; a_1, a_2, a_3) + \mathrm{O}((|\delta| + \varepsilon^2)^2|a_j|)\\
a_{1,x} &= a_2 + \delta(c_{11}a_1 + c_{13}a_3) + \varepsilon^2 f_1(x; a_1, a_2, a_3) + \mathrm{O}((|\delta| + \varepsilon^2)^2|a_j|)\\
a_{2,x} &= a_3 + \delta(c_{20}a_0 + c_{22}a_2) + \varepsilon^2(c_{21}a_1 + c_{23}a_3) + \varepsilon^2 f_2(x; a_1, a_2, a_3) + \mathrm{O}((|\delta| + \varepsilon^2)^2|a_j|)\\
a_{3,x} &= \delta(c_{31}a_1 + c_{33}a_3) + \varepsilon^2 f_3(x; a_1, a_2, a_3) + \mathrm{O}((|\delta| + \varepsilon^2)^2|a_j|)
\end{aligned} \tag{6.9}$$

The constants c_{ij} are $\mathrm{O}(1)$ and can be determined explicitly. In particular, we have $c_{20} = c_{31} = -\beta$ and $c_{21} = \beta$. Note that the functions f_j are independent of a_0. This is due to the invariance of (2.1)–(2.4) under $\Phi \to \Phi + const.$ which implies the invariance of the reduced system under $a_0 \to a_0 + const.$ if $\delta = 0$. A direct calculation of the relevant terms gives

$$\begin{aligned}
\varepsilon^2 f_2(x; a_1, a_2, a_3) &= -\beta u^0 a_1 + \varepsilon^2 f_{22}(x; a_2, a_3)\\
\varepsilon^2 f_3(x; a_1, a_2, a_3) &= -2\beta u^0_x a_1 - 2\beta u^0 a_2 + \varepsilon^2 f_{33}(x; a_3)
\end{aligned}$$

with u^0 from Theorem 1 (i).

6.2.2 Justifying the Korteweg-de Vries scaling

As a first step, we prove that any eigenvalue δ, $\mathrm{Re}\,\delta \neq 0$ is necessarily located in an $\mathrm{O}(\varepsilon^3)$-neighborhood of the origin. Suppose therefore $\varepsilon = \nu|\delta|^{1/3}$ with ν small. We shall prove, that the system (6.9) does not possess non-trivial, bounded solutions, provided $\mathrm{Re}\,\delta \neq 0$. We may scale the system (6.9) according to

$$\xi = |\delta|^{1/3}x, \quad a_j(x) = |\delta|^{j/3}A_j(\xi), \quad j = \overline{0,3},$$

and obtain

$$\begin{aligned}
A_{0,\xi} &= A_1 + \mathrm{O}(\delta^{2/3} + \nu^2\delta^{2/3})\\
A_{1,\xi} &= A_2 + \mathrm{O}(\delta^{2/3} + \nu^2\delta^{1/3})\\
A_{2,\xi} &= A_3 - \beta\arg(\delta)A_0 + \mathrm{O}(\delta^{2/3} + \nu^2)\\
A_{3,\xi} &= -\beta\arg(\delta)A_1 + \mathrm{O}(\delta^{2/3} + \nu^2).
\end{aligned} \tag{6.10}$$

At $\nu = \delta = 0$, we have an autonomous linear ODE with eigenvalues $\zeta_0 = 0$, $\zeta^3_{1,2,3} = -2\beta\arg(\delta)$, and corresponding eigenvectors $A^0_1 = A^0_2 = 0$, $A^0_3 = -\beta\arg(\delta)$, $A^0_0 = 1$, and $A^k_j = (-\zeta_k)^j$, $k = 1, 2, 3$ and $j = 0, \ldots, 3$. Now suppose first $\mathrm{Re}\,\delta \neq 0$. Then ζ_j, $j = 1, 2, 3$ are hyperbolic.

Therefore the eigenspace to the eigenvalue ζ_0 forms a normally hyperbolic center-manifold for the linear flow. This center-manifold persists under small, non-autonomous perturbations and contains all bounded solutions (we may construct the center-manifold as the robust intersection of center-stable manifold at $x = \infty$ and center-unstable manifold at $x = -\infty$). On the other hand, the eigenvalue $\zeta_0 = 0$ is easily seen from (5.2) to move off the imaginary axis whenever ζ moves off the axis. But this eigenvalue determines the asymptotic behavior of solutions in the center-manifold at $x = +\infty$ *and* $x = -\infty$. If now δ approaches the imaginary axis, we have to refine the arguments as in Case 1, above. Using the gap lemma, Proposition 6.3, we continue the center-stable manifold at $x = +\infty$ and the center-unstable manifold at $x = -\infty$ smoothly across the imaginary axis, exploiting fast convergence of the non-autonomous terms on the scale, $\mathrm{O}(\varepsilon)$ compared to the order of the perturbation $\mathrm{O}(\varepsilon^2)$. We omit the details which are similar to the case of non-zero frequency, Section 6.1.

6.2.3 The Korteweg-de Vries limit

We may now assume that the eigenvalue δ is necessarily of the order ε^3 and therefore scale $\delta = \varepsilon^3\Lambda$. We obtain in the KdV-scaling

$$\xi = \varepsilon x, \quad a_j(x) = \varepsilon^j A_j(\xi), \quad j = \overline{0,3},$$

the scaled reduced system

$$\begin{aligned} A_{0,\xi} &= A_1 + \mathrm{O}(\varepsilon^2) \\ A_{1,\xi} &= A_2 + \mathrm{O}(\varepsilon) \\ A_{2,\xi} &= A_3 - \beta\Lambda A_0 + \beta A_1 - \beta A_1^* A_1 + \mathrm{O}(\varepsilon^2) \\ A_{3,\xi} &= -\beta\Lambda A_1 - 2\beta A_{1,\xi}^* A_1 - 2\beta A_1^* A_2 + \mathrm{O}(\varepsilon). \end{aligned} \tag{6.11}$$

Here A_1^* is the steady solitary wave solution of the KdV-equation

$$2\beta A_{1,\tau} + A_{1,\xi\xi\xi} - \beta A_{1,\xi} + 3\beta A_1 A_{1,\xi} = 0, \tag{6.12}$$

$$A_1^*(\xi) = \operatorname{sech}^2\left(\frac{\sqrt{\beta}\xi}{2}\right).$$

We consider the case $\varepsilon = 0$ first. We transform variables

$$B_0 = A_0,\ B_1 = A_1,\ B_2 = A_2,\ B_3 = A_3 - \beta\Lambda A_0 + \beta A_1 - \beta A_1^* A_1$$

and obtain at $\varepsilon = 0$

$$\begin{aligned} B_{0,\xi} &= B_1 \\ B_{1,\xi} &= B_2 \\ B_{2,\xi} &= B_3 \\ B_{3,\xi} &= -2\beta\Lambda B_1 + \beta B_2 - 3\beta A_{1,\xi}^* B_1 - 3\beta A_1^* B_2 \end{aligned} \tag{6.13}$$

which is the KdV-equation, linearized in the soliton solution A_1^*, for $B_1 = B_{0,\xi}$. The equation at $|\xi| = \infty$ reduces to

$$B_{0,\xi} = B_1, \quad B_{1,\xi\xi\xi} + 2\beta\Lambda B_1 - \beta B_{1,\xi} = 0$$

with characteristic polynomial $\zeta^4 + 2\beta\Lambda\zeta - \beta\zeta^2$ for the ζ-eigenvalues, determining exponential spatial decay or growth of possible eigenfunctions. Besides $\zeta = 0$ with eigenvector $(1,0,0,0)^T$, we have precisely the spectrum of the linearization about the KdV-soliton. In particular, dynamics in the space $(1,0,0,0)^\perp$ are precisely the (linear) dynamics around the KdV-soliton. This strongly suggests, that eigenfunctions will appear wherever the KdV-soliton possesses eigenfunctions — and nowhere else. Given the stability of the KdV-soliton [PW92], this would then prove stability of the solitary wave in the Euler-equations!

We construct in the sequel a more refined picture of the spectrum in $\varepsilon = 0$, which will, in particular, be persistent for $\varepsilon > 0$.

First of all, we note that the trivial zero-eigenvalue moves out of zero as soon as ε becomes positive and Λ non-zero. This can be readily seen from (5.2), by substituting the KdV-scaling $\delta = \varepsilon^3\Lambda$ and $\zeta = \varepsilon Z$. From the dispersion relation (5.2) we then obtain a new equation for Z, Λ and ε. The Taylor expansion of this equation in ε^2 is, up to third order

$$\varepsilon^4[(b - \frac{1}{3})Z^4 - Z^2 + 2\Lambda Z + \varepsilon^2(-\frac{1}{6}(b - \frac{1}{5})Z^6 + \frac{1}{6}Z^4 - \Lambda Z^3 + \Lambda^2) + \mathrm{O}(\varepsilon^4)] = 0. \quad (6.14)$$

To second order in ε^2, there is still one eigenvalue $\zeta_0 = 0$ which can be seen to be perturbed to $\zeta_0 = -\frac{1}{2}\varepsilon^2\Lambda + \mathrm{O}(\varepsilon^4\Lambda)$ by the third order terms in ε^2.

We emphasize here, that *all* eigenvalues are, for $\varepsilon \geq 0$ small, *smooth* functions in Λ and ε.

6.2.4 Perturbing the Korteweg-de Vries spectrum

We consider the scaled eigenvalue-problem for the water-waves (6.11) as a small perturbation of the eigenvalue problem for the KdV-equation (6.13).

We distinguish three cases, with increasing difficulty. First we consider Λ bounded away from the imaginary axis. We then continue the arguments for Λ close to the imaginary axis, but bounded away from the origin. Finally, we study the eigenvalue problem for Λ in a neighborhood of the origin.

(I) Eigenvalues far from the imaginary axis Suppose first that $\mathrm{Re}\,\Lambda \geq \nu_* > 0$ for some $\nu_* > 0$. We have to exclude bounded solutions to (6.11) for $\varepsilon > 0$, small. As in Section 6.1, we exploit the fact that the ξ-dependent coefficients in (6.11) converge exponentially as $|\xi| \to \infty$, uniformly in $\varepsilon \geq 0$. In order to construct stable and unstable subspaces as in Section 6.1, we discuss the spatial eigenvalues ζ_j of (6.11) at $|\xi| = \infty$. From the scaled dispersion relation (6.14), we find two eigenvalues with positive real part, ζ_1 and ζ_3, one eigenvalue with negative real part, which we call ζ_2 and the eigenvalue ζ_0, which for $\varepsilon = 0$ remains in the origin, and moves into the left half plane for $\varepsilon > 0$:

$$\operatorname{Re}\zeta_1,\ \operatorname{Re}\zeta_3 > 0 \qquad \operatorname{Re}\zeta_0 \leq 0,\ \operatorname{Re}\zeta_2 < 0.$$

With Lemma 6.2, we can construct linear subspaces $E^{\mathrm{s}}(0)$ and $E^{\mathrm{u}}(0)$, such that all initial values at $\xi = 0$ of the linear equation (6.11) leading to bounded solutions on $\mathbb{R}^+$ or $\mathbb{R}^-$ are contained in $E^{\mathrm{s}}(0)$ or $E^{\mathrm{u}}(0)$, respectively. Both subspaces depend analytically on Λ, $\operatorname{Re}\Lambda \geq \nu_* > 0$, and smoothly on $\varepsilon \geq 0$. Choosing analytic bases $B^{1/2}_{\mathrm{s/u}}$ in $E^{\mathrm{s/u}}(0)$, we can compute the Evans function

$$\mathcal{E}(\Lambda;\varepsilon) = \det\left(B^1_{\mathrm{s}}, B^2_{\mathrm{s}}, B^1_{\mathrm{u}}, B^2_{\mathrm{u}}\right).$$

We show that $\mathcal{E}(\Lambda;0)$ is nonzero for $\operatorname{Re}\Lambda \geq 0$. By continuity in ε and the previous considerations for large Λ, this excludes eigenvalues in $\operatorname{Re}\Lambda \geq \nu_* > 0$.

The Evans function $\mathcal{E}(\Lambda;0)$ can be computed almost explicitly from (6.13). Recall, that the equation for (B_1, B_2, B_3) does not depend on B_0 and is precisely the linearization about the KdV-soliton. We therefore define the subspace $(0,*,*,*) = (1,0,0,0)^\perp$ as the *KdV-subspace*. This subspace is *not* flow-invariant, but the dynamics in this subspace are independent of the value of B_0 in the first component if $\varepsilon = 0$. This gives the equations a skew-product structure. We may first solve the equation in the KdV-subspace and then solve the equation for B_0. Within the KdV-subspace, we find the eigenvalues ζ_1, ζ_2, and ζ_3. We find the stable and unstable subspaces $E^{\mathrm{s}}_{\mathrm{KdV}}(0)$ and $E^{\mathrm{u}}_{\mathrm{KdV}}(0)$ by intersecting the subspaces $E^{\mathrm{s}}(0)$ and $E^{\mathrm{u}}(0)$ with the KdV-subspace. In particular, $E^{\mathrm{s}}_{\mathrm{KdV}}(0)$ is one-dimensional and $E^{\mathrm{u}}_{\mathrm{KdV}}(0)$ is two-dimensional. Choosing analytic bases in these two subspaces, we can compute an analytic function $\mathcal{E}_{\mathrm{KdV}}(\Lambda)$, the Evans function of the KdV-soliton. We are now going to use information from [PW92] on the zeroes of $\mathcal{E}_{\mathrm{KdV}}(\Lambda)$.

Theorem 6 *[PW92] The Evans function $\mathcal{E}_{\mathrm{KdV}}(\Lambda)$ for the KdV-soliton can be extended analytically into $\operatorname{Re}\Lambda > -4/3$. It vanishes precisely in the origin, where we have*

$$\mathcal{E}_{\mathrm{KdV}}(0) = 0,\ \mathcal{E}'_{\mathrm{KdV}}(0) = 0,\ \mathcal{E}''_{\mathrm{KdV}}(0) \neq 0.$$

From this information, we can infer absence of zeroes for $\mathcal{E}(\Lambda;0)$ in $\operatorname{Re}\Lambda \geq \nu_*$.

Lemma 6.4 *The reduced, scaled Evans function of the water-wave problem, $\mathcal{E}(\Lambda;0)$, and the Evans function for the KdV-soliton, $\mathcal{E}_{\mathrm{KdV}}(\Lambda)$, differ by a non-vanishing analytic function $\mathcal{S}(\Lambda)$:*

$$\mathcal{E}(\Lambda;0) = \mathcal{S}(\Lambda)\mathcal{E}_{\mathrm{KdV}}(\Lambda); \quad \mathcal{S}(\Lambda) \neq 0$$

for $\operatorname{Re}\Lambda > 0$.

Proof. [of Lemma 6.4] We compute $\mathcal{E}(\Lambda)$ choosing a particular analytic basis in $E^{\mathrm{s}}(0)$ and $E^{\mathrm{u}}(0)$. Note first that $B^1_{\mathrm{s}} := \mathcal{B}_0 = (1,0,0,0)^T \in E^{\mathrm{s}}(0)$ since this vector is constant under time-ξ-evolution. Next, let $B^2_{\mathrm{s,\,KDV}}(\Lambda)$, $B^1_{\mathrm{u,\,KdV}}(\Lambda)$, $B^1_{\mathrm{u,\,KdV}}(\Lambda) \in \mathbb{C}^3$ denote the basis vectors for stable and unstable KdV-subspaces $E^{\mathrm{s/u}}_{\mathrm{KdV}}(0)$. Solving $B_{0,\xi} = B_1$, with B_1 given from the KdV-subspace, with initial condition $B^j_{\mathrm{s/u,\,KDV}}(\Lambda)$, we find particular bases $B^j_{\mathrm{s/u}}$ of $E^{\mathrm{s/u}}(0)$, which coincide with $B^j_{\mathrm{s/u,\,KdV}}$ in the KdV-subspace. Since $B^1_{\mathrm{s}} = (1,0,0,0)^T$, we find that in these coordinates the determinant $\det(B^1_{\mathrm{s}}, B^2_{\mathrm{s}}, B^1_{\mathrm{u}}, B^2_{\mathrm{u}})$ is of the form

$$
\begin{aligned}
\mathcal{E}(\Lambda;0) &= \det \begin{pmatrix} 1 & * & * & * \\ 0 & (B^1_{\mathrm{s,\,KdV}})_1(\Lambda) & (B^1_{\mathrm{u,\,KdV}})_1(\Lambda) & (B^2_{\mathrm{u,\,KdV}})_1(\Lambda) \\ 0 & (B^1_{\mathrm{s,\,KdV}})_2(\Lambda) & (B^1_{\mathrm{u,\,KdV}})_2(\Lambda) & (B^2_{\mathrm{u,\,KdV}})_2(\Lambda) \\ 0 & (B^1_{\mathrm{s,\,KdV}})_3(\Lambda) & (B^1_{\mathrm{u,\,KdV}})_3(\Lambda) & (B^2_{\mathrm{u,\,KdV}})_3(\Lambda) \end{pmatrix} \\
&= \det(B^1_{\mathrm{s,\,KdV}}(\Lambda), B^1_{\mathrm{u,\,KdV}}(\Lambda), B^2_{\mathrm{u,\,KdV}}(\Lambda)) \\
&= \mathcal{E}_{\mathrm{KdV}}(\Lambda).
\end{aligned}
$$

Choosing different analytic bases, the determinant only differs by a nonzero, analytic factor, which proves the lemma. ■

Corollary 6.5 *The scaled Evans function of the water-wave problem $\mathcal{E}(\Lambda;0)$ does not vanish in the right half plane. In particular, for $0 < \varepsilon \le \varepsilon_*(\nu)$, there are no unstable eigenvalues of the solitary wave in $\operatorname{Re}\delta \ge \nu\varepsilon^{3/2}$.*

(II) Eigenvalues close to the imaginary axis We show that we may continue the construction from Lemma 6.4 across the imaginary axis, outside a neighborhood of the origin.

Lemma 6.6 *The reduced Evans function $\mathcal{E}(\Lambda;\varepsilon)$ can be continued analytically in Λ and continuously in ε in a region $\{\operatorname{Re}\Lambda \ge -\nu,\ |\Lambda| \ge \nu\} \subset \mathbb{C}$.*

Proof. We have to show that the stable and unstable subspaces $E^{\mathrm{s}}(0)$ and $E^{\mathrm{u}}(0)$ continue analytically in Λ and continuously in ε across the imaginary axis. This in turn is an immediate consequence of the gap lemma, Theorem 5. ■

Corollary 6.7 *The scaled Evans function of the water-wave problem $\mathcal{E}(\Lambda;\varepsilon)$ does not vanish in a region $\{\operatorname{Re}\Lambda \ge -\nu,\ |\Lambda| \ge \nu\} \subset \mathbb{C}$.*

In order to finish the proof, it remains to exclude eigenvalues for the perturbed, scaled eigenvalue problem (6.14) in a neighborhood of the origin.

(III) Eigenvalues close to the origin Finally, we address the crucial neighborhood of the origin. We may already suspect that transversality as above might not hold, since already the KdV-equation possesses an eigenvalue $\Lambda = 0$ of algebraic multiplicity two, embedded in the essential spectrum. Again, the strategy consists of first continuing the Evans function $\mathcal{E}(\Lambda;\varepsilon)$ for the water-wave problem analytically in Λ and continuously in ε in a neighborhood of the origin, first. As a second step, we show how this Evans function is related to the Evans function of the Korteweg-de Vries equation, $\mathcal{E}_{\mathrm{KdV}}(\Lambda)$. The goal of this step to conclude that for all $\varepsilon \geq 0$ sufficiently small, $\mathcal{E}$ possesses at most three zeroes in a neighborhood of the origin — exploiting that the number of zeroes of an analytic functions is invariant under small perturbations. We then conclude the stability proof exhibiting two explicit eigenvectors in the kernel and an explicit principal vector in the generalized kernel.

We start with some notational preliminaries for the asymptotic equation at $|\xi| = \infty$. The eigenvalues of the linear equation on the right side of (6.13) at $\Lambda = 0$, $|\xi| = \infty$ are $\zeta_{\mathrm{s}} = \zeta_{\mathrm{u}} = 0$, a double zero eigenvalue, and $\zeta_{\mathrm{ss}} = -\sqrt{\beta}$ and $\zeta_{\mathrm{uu}} = \sqrt{\beta}$. The zero eigenvalue is geometrically simple with eigenvector $(1, 0, 0, 0)^T$.

The central observation now is that for $\Lambda, \varepsilon \neq 0$ the zero eigenvalues unfold smoothly:

$$\zeta_{\mathrm{s}} = -\frac{1}{2}\varepsilon^2\Lambda + \mathrm{O}(\varepsilon^2\Lambda^2), \quad \zeta_{\mathrm{u}} = 2\Lambda + \mathrm{O}(\Lambda^2 + \varepsilon^2\Lambda).$$

These expansions are readily computed from the Newton polygon to (6.14), with leading order contribution $-Z^2 + 2\Lambda Z + \varepsilon^2\Lambda^2$. Eigenvectors are smooth as well and given by $e_j = (-1, \zeta_j, \zeta_j^2, \zeta_j^3)$ for $j = \mathrm{s}, \mathrm{u}, \mathrm{ss}, \mathrm{uu}$. For $\varepsilon > 0, \mathrm{Re}\,\Lambda > 0$, the stable eigenspace is spanned by $E^{\mathrm{s}} = \mathrm{span}\,\{e_{\mathrm{s}}, e_{\mathrm{ss}}\}$ and the unstable eigenspace by $E^{\mathrm{u}} = \mathrm{span}\,\{e_{\mathrm{u}}, e_{\mathrm{uu}}\}$. At $\Lambda = 0$, we find a nontrivial intersection of stable and unstable subspaces $E^{\mathrm{s}} \cap E^{\mathrm{u}} = \mathrm{span}\,\{e_{\mathrm{s}}\} = \mathrm{span}\,\{e_{\mathrm{u}}\}$.

We emphasize that this smooth unfolding is non-generic: in a typical unfolding of the Jordan block with a parameter Λ, the eigenvalues are smooth functions of $\sqrt{\Lambda}$! The smooth unfolding, here, is due to reversibility: in the scaled dispersion relation (6.14), there is no linear term Λ, which would make the leading order contribution in the Z-Λ Newton polygon for (6.14) to be $-Z^2 + \Lambda = 0$, with $Z \sim \sqrt{\Lambda}$. Reversibility implies invariance of the dispersion relation under $Z \mapsto -Z$ and $\Lambda \mapsto -\Lambda$, for all ε! It is this symmetry which excludes linear terms in Λ.

We next show, that the subspaces $E^{\mathrm{s}}(\xi)$ and $E^{\mathrm{u}}(\xi)$, constructed for Λ outside a neighborhood of zero above, can be continued analytically in Λ and smoothly in ε across this neighborhood.

Lemma 6.8 *The Evans function $\mathcal{E}(\Lambda;\varepsilon)$ to the scaled linearization about the solitary wave in the water-wave problem* (6.14) *possesses an analytic extension into an open neighborhood of the origin $|\Lambda| \leq \nu_0$, which depends continuously on $\varepsilon \geq 0$ sufficiently small. The neighborhood is uniform in ε, that is, ν_0 does not depend on $\varepsilon \geq 0$.*

Proof. The construction very much relies, in the spirit of the gap lemma, Theorem 5, on a stable manifold theorem. However, we cannot apply the gap lemma directly, since additional hyperbolic eigenvalues are present, which actually are in resonance with spatial convergence of the coefficients at $\Lambda = 0$.

We compactify time $2\beta\xi = \log(\frac{1+\tau}{1-\tau})$, $\tau \in [-1, 1]$ and obtain a smooth (C^1 in τ and analytic in Λ) differential equation, suspended with the equation $\tau_\xi = \beta(1 - \tau^2)$. The fibers $\tau = +1$ and $\tau = -1$ are invariant and describe the limiting situation at $\xi = \pm\infty$. In these fibers the dynamics possesses invariant subspaces which are the linear eigenspaces to the eigenvalues ζ_j, $j = \mathrm{s}, \mathrm{u}, \mathrm{ss}, \mathrm{uu}$. In the τ-direction, the asymptotic $\tau = \pm 1$-subspaces are linearly stable ($\tau = +1$) and linearly unstable ($\tau = -1$), respectively, with exponential rate $\pm 2\beta$.

The flows inside $\tau = 1$ and $\tau = -1$ are linear and coincide. Subspaces corresponding to eigenspaces and generalized eigenspaces are flow-invariant subspaces. For example, the two-dimensional subspace in $\tau = \pm 1$ corresponding to the generalized kernel for $\Lambda = 0$, can be viewed as a smooth, normally hyperbolic, local center-manifold. Inside this center-manifold, we find the particularly important flow-invariant subspaces $\mathrm{span}\,\{e_\mathrm{s}\}$ in $\tau = +1$ and $\mathrm{span}\,\{e_\mathrm{u}\}$ in $\tau = -1$. The subspaces are analytic in Λ and continuous in ε. They possess strong unstable and strong stable foliations, which are as smooth as the vector field. Indeed, we may smoothly transform variables, $B_j \mapsto B_j \mathrm{e}^{-\zeta_{\mathrm{s/u}}\xi}$ to trivialize the flow inside the eigenspace, which consists of a line of equilibria after the rescaling. The foliations are then given as the strong stable manifolds of the equilibria in the eigenspaces. Analyticity follows from differentiability and the Cauchy-Riemann differential equations. We denote by $W^{\mathrm{ss}}(\mathrm{span}\,\{e_\mathrm{s}\})$ the three-dimensional stable manifold of the subspace $\mathrm{span}\,\{e_\mathrm{s}\}$ in the extended phase-space (τ, B). Analogously, let $W^{\mathrm{uu}}(\mathrm{span}\,\{e_\mathrm{u}\})$ denote the three-dimensional unstable manifold of the subspace $\mathrm{span}\,\{e_\mathrm{u}\}$. By construction, these manifolds are the smooth continuations of $E^\mathrm{s}(\xi)$ and $E^\mathrm{u}(\xi)$, that we already constructed in the region $Re\Lambda > 0$: $W^{\mathrm{ss}}(\mathrm{span}\,\{e_\mathrm{s}\}) \cap \{\tau = 0\} = E^\mathrm{s}(0)$ and $W^{\mathrm{uu}}(\mathrm{span}\,\{e_\mathrm{u}\}) \cap \{\tau = 0\} = E^\mathrm{u}(0)$. Choosing analytic bases in these subspaces, and evaluating the determinant, we have continued the Evans-function E into a neighborhood of the origin $\Lambda = 0$ smoothly, analytically in Λ and continuously in ε. ■

Remark 6.9 *The above construction does not show, that we can smoothly single out a particular one-dimensional subspace of initial conditions which converges to* $\mathrm{span}\,\{e_\mathrm{u}\}$ *or* $\mathrm{span}\,\{e_\mathrm{u}\}$ *faster than the other solutions — which is part of the proof of the gap lemma; see the proof of Proposition 6.3. In fact, we believe that this is in general impossible, since precisely at the origin,* $\Lambda = 0$, *the contracting and expanding eigenvalues* ζ_{ss} *and* ζ_{uu} *are equal to the rate of exponential approach in the* ξ*-direction, which makes it impossible to single out a strong stable or unstable direction.*

The next step provides an expansion for $\mathcal{E}(\Lambda; 0)$ near $\Lambda = 0$.

Lemma 6.10 *There exists a nonzero coefficient $\mathcal{E}_3 \neq 0$ such that*

$$\mathcal{E}(\Lambda; 0) = \mathcal{E}_3 \Lambda^3 + \mathrm{O}(\Lambda^4).$$

Proof. For $\varepsilon = 0$, the linear equation (6.13) possesses a skew-product structure, already exploited in the previous paragraphs (I) and (II).

In the KdV-subspace, the dynamics are independent of B_0. Stable and unstable subspaces $E^{\mathrm{s}}_{\mathrm{KdV}}(0)$ and $E^{\mathrm{u}}_{\mathrm{KdV}}(0)$ are well-defined. We may choose particular bases

$$E^{\mathrm{s}}_{\mathrm{KdV}}(0) = \mathrm{span}\,\{B^{\mathrm{ss}}_{\mathrm{KdV}}(0)\}, \quad E^{\mathrm{u}}_{\mathrm{KdV}}(0) = \mathrm{span}\,\{B^{\mathrm{uu}}_{\mathrm{KdV}}(0), B^{\mathrm{u}}_{\mathrm{KdV}}(0)\}$$

such that solutions in the KdV-subspace with these initial conditions satisfy

$$\begin{aligned} \mathrm{e}^{-\zeta^{\mathrm{ss}}\xi} B^{\mathrm{ss}}_{\mathrm{KdV}}(\xi) &\rightarrow b^{\mathrm{ss}}_{\mathrm{KdV}} \text{ for } \xi \to \infty \\ \mathrm{e}^{-\zeta^{\mathrm{uu}}\xi} B^{\mathrm{uu}}_{\mathrm{KdV}}(\xi) &\rightarrow b^{\mathrm{uu}}_{\mathrm{KdV}} \text{ for } \xi \to -\infty \\ \mathrm{e}^{-\zeta^{\mathrm{u}}\xi} B^{\mathrm{u}}_{\mathrm{KdV}}(\xi) &\rightarrow b^{\mathrm{u}}_{\mathrm{KdV}} \text{ for } \xi \to -\infty. \end{aligned}$$

From these solutions, we are going to construct a basis of stable and unstable subspaces for the full water-wave problem (6.14), $E^{\mathrm{s}}(0)$ and $E^{\mathrm{u}}(0)$. We start with $E^{\mathrm{s}}(0)$. First, $B^{\mathrm{s}}(1,0,0,0)^T$ is a ξ-independent, bounded solution and belongs to $E^{\mathrm{s}}(0)$. The second basis vector is readily computed from $B^{\mathrm{ss}}_{\mathrm{KdV}}(\xi)$. Define

$$B^{\mathrm{ss}}_0(\xi) = \int_{\infty}^{\xi} (B^{\mathrm{ss}}_{\mathrm{KdV}})_1(s)\mathrm{d}s$$

and

$$(B^{\mathrm{ss}}_1(\xi), B^{\mathrm{ss}}_2(\xi), B^{\mathrm{ss}}_3(\xi))^T = B^{\mathrm{ss}}_{\mathrm{KdV}}(\xi).$$

Then $B^{\mathrm{ss}}(\xi) = (B^{\mathrm{ss}}_0(\xi), B^{\mathrm{ss}}_1(\xi), B^{\mathrm{ss}}_2(\xi), B^{\mathrm{ss}}_3(\xi))^T$ is exponentially decaying for $\xi \to \infty$ and $B^{\mathrm{ss}}(0)$ is the desired second basis vector in $E^{\mathrm{s}}(0)$.

Similarly, we define

$$B^{\mathrm{uu}}_0(\xi) = \int_{-\infty}^{\xi} (B^{\mathrm{uu}}_{\mathrm{KdV}})_1(s)\mathrm{d}s$$

and

$$(B^{\mathrm{uu}}_1(\xi), B^{\mathrm{uu}}_2(\xi), B^{\mathrm{uu}}_3(\xi))^T = B^{\mathrm{uu}}_{\mathrm{KdV}}(\xi).$$

Then $B^{\mathrm{uu}}(\xi) = (B^{\mathrm{uu}}_0(\xi), B^{\mathrm{uu}}_1(\xi), B^{\mathrm{uu}}_2(\xi), B^{\mathrm{uu}}_3(\xi))^T$ is exponentially decaying for $\xi \to \infty$ and $B^{\mathrm{uu}}(0) \in E^{\mathrm{u}}(0)$.

The same construction for $B^{\mathrm{u}}_{\mathrm{KdV}}$ would give a pole in $\Lambda = 0$ since the integral diverges due to slow exponential decay, $\zeta^{\mathrm{u}} = 2\Lambda + \mathrm{O}(\Lambda^2 + \varepsilon^2\Lambda)$,

$$B^{\mathrm{u}}_{\mathrm{KdV}}(\xi) = b^{\mathrm{u}}_{\mathrm{KdV}} \mathrm{e}^{\zeta^{\mathrm{u}}\xi} + r(\xi)$$

with $r(\xi) = \mathrm{O}(\mathrm{e}^{(\zeta^{\mathrm{u}}+\nu)\xi})$ for $\xi \to -\infty$ with some $\nu > 0$, uniformly in Λ close to zero. We therefore rescale the KdV-eigenvector with Λ and set

$$\tilde{B}^{\mathrm{u}}_{\mathrm{KdV}}(\xi) = \Lambda B^{\mathrm{u}}_{\mathrm{KdV}}(\xi).$$

We then proceed as for B^{uu} and define

$$B^{\mathrm{u}}_0(\xi) = \int_0^{\xi} (\tilde{B}^{\mathrm{u}}_{\mathrm{KdV}})_1(s)\mathrm{d}s + B^{\mathrm{u}}_0(0).$$

with

$$B^{\mathrm{u}}_0(0) = \frac{\Lambda b^{\mathrm{u}}}{\zeta^{\mathrm{u}}} + \int_{-\infty}^{0} r(s)\mathrm{d}s.$$

With this choice of $B^{\mathrm{u}}_0(0)$, $B^{\mathrm{u}}_0(\xi)$ decays to zero exponentially for $\mathrm{Re}\,\Lambda > 0$. Note that $B^{\mathrm{u}}_0(0)$ is analytic in a neighborhood of $\Lambda = 0$ and that, with a suitable choice of b^{u} we can arrange to have $B^{\mathrm{u}}_0(0) = 1 + \mathrm{O}(\Lambda)$.

$$(B^{\mathrm{u}}_1(\xi), B^{\mathrm{u}}_2(\xi), B^{\mathrm{u}}_3(\xi))^T = B^{\mathrm{u}}_{\mathrm{KdV}}(\xi).$$

Then $B^{\mathrm{u}}(\xi) = (B^{\mathrm{u}}_0(\xi), B^{\mathrm{u}}_1(\xi), B^{\mathrm{u}}_2(\xi), B^{\mathrm{u}}_3(\xi))^T$ is exponentially decaying for $\xi \to -\infty$ and $\mathrm{Re}\,\Lambda > 0$ and $B^{\mathrm{u}}(0) \in E^{\mathrm{u}}(0)$.

The Evans function for the water-wave problem is then given by the determinant

$$\mathcal{E}(\Lambda, 0) = \det(B^{\mathrm{uu}}, B^{\mathrm{u}}, B^{\mathrm{s}}, B^{\mathrm{ss}}).$$

Exploiting that $B^{\mathrm{s}}(0) = 1$, we find that

$$\mathcal{E}(\Lambda, 0) = \det(B^{\mathrm{uu}}_{\mathrm{KdV}}, \tilde{B}^{\mathrm{u}}_{\mathrm{KdV}}, B^{\mathrm{ss}}_{\mathrm{KdV}}) = \Lambda \mathcal{E}_{\mathrm{KdV}}(\Lambda).$$

Together with Theorem 6 for the Evans function of the KdV equation, this proves the lemma. ■

Geometrically, the unfolding of the subspaces is as follows, roughly speaking. For $\Lambda = 0$, B^{u} and B^{s} coincide and B^{ss} and B^{uu} can be assumed to coincide as well. The weak directions, B^{s} and B^{u} cross transversely in $\Lambda = 0$, contributing a factor Λ to $\mathcal{E}$. The strong directions B^{ss} and B^{uu} unfold with quadratic tangency, just as in the KdV-equation, contributing a factor Λ^2 to $\mathcal{E}$.

By continuity in ε and analyticity in Λ, Lemma 6.8, we conclude using Rouché's theorem that for $\varepsilon > 0$ small, $\mathcal{E}(\Lambda; \varepsilon)$ possesses precisely three roots close to the origin, counted with multiplicity. The following lemma therefore shows that there are indeed no unstable eigenvalues in a small enough neighborhood of the origin.

Lemma 6.11 *The Evans function for the water-wave problem $\mathcal{E}(\Lambda; \varepsilon)$ possesses a triple root in the origin for all $\varepsilon \geq 0$ sufficiently small.*

Proof. Let $\Lambda = 0$. We find for $\varepsilon \geq 0$ a two-dimensional intersection of $\mathrm{E}^s(0)$ and $\mathrm{E}^u(0)$, generated by the derivative of the solitary wave and the translation of the potential $(1, 0, 0, 0)^T$. Indeed, by construction, Lemma 6.8, any bounded solution necessarily lies in the intersection, since solutions which at $\Lambda = 0$ do not belong to the intersection grow at least linearly. From Galilean invariance, we find the exponentially localized derivative of the solitary wave with respect to the wave speed as a principal vector to the derivative of the solitary wave. Following [PW92], we conclude that $\mathcal{E}(\Lambda; \varepsilon)$ possesses at least a triple zero in $\Lambda = 0$. On the other hand, Lemma 6.10 shows that the multiplicity is at most three. This proves the lemma. ∎

6.3 Proof of Proposition 6.1

We conclude the proof of absence of point spectrum in the right half plane. First, we showed in Section 6.1 that there are no unstable eigenvalues in a neighborhood of the imaginary axis, up to possible eigenvalues with large imaginary part or in a neighborhood of the imaginary axis. We then showed in Section 6.2.2 that eigenvalues in a neighborhood of the imaginary axis necessarily scale with ε^3, justifying the Korteweg-de Vries scaling. Finally, we showed in Section 6.2.4 that in the Korteweg-de Vries scaling, there are no unstable eigenvalues. The main part was a perturbation argument, based on the construction of an analytic Evans function. We showed that any eigenvalue is a root of an analytic function $\mathcal{E}(\Lambda; \varepsilon)$. We then continued $\mathcal{E}(\Lambda; \varepsilon)$ analytically in an open neighborhood of $\Lambda = 0$, for $\varepsilon \geq 0$. Lemma 6.10 showed that there are at most three eigenvalues in a neighborhood of zero, counting multiplicity, and Lemma 6.11 showed that all three eigenvalues are located in zero, for $\varepsilon \geq 0$ sufficiently small. This proves spectral stability up to possible eigenvalues with imaginary part tending to ∞ as $\varepsilon \to 0$, Proposition 6.1.

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