Transverse bifurcations of homoclinic cycles

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Abstract

Homoclinic cycles exist robustly in dynamical systems with symmetry, and may undergo various bifurcations, not all of which have an analogue in the absence of symmetry. We analyze such a bifurcation, the *transverse bifurcation*, and uncover a variety of phenomena that can be distinguished representation-theoretically. For example, exponentially flat branches of periodic solutions (a typical feature of bifurcation from homoclinic cycles) occur for some but not all representations of the symmetry group. Our study of transverse bifurcations is motivated by the problem of intermittent dynamos in rotating convection.

1 Introduction

Homoclinic cycles are structurally unstable for general dynamical systems but are known to exist robustly in dynamical systems with symmetry [19, 7, 14]. They are then associated with intermittent or bursting phenomena. A well-known example of a robust homoclinic cycle is given by Guckenheimer and Holmes [9] who consider a three-dimensional system of ordinary differential equations (ODEs) equivariant

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under the action of a certain 24 element group of symmetries. This example occurs in the context of rotating Rayleigh-Bénard convection [1, 2] and also in a population dynamics model [18].

The standard convection rolls that occur in Rayleigh-Bénard convection are unstable in rotating convection [17]. Busse and Clever [1] proposed a simple three-dimensional model involving three sets of rolls inclined at exactly 60°, based on an asymptotic expansion and truncation of the full Boussinesq equations for rotating convection. Busse and Heikes [2] pointed out that the presence of a homoclinic cycle in the Busse-Clever model corresponds to an intermittency phenomenon observed in fluid experiments where, locally, systems of rolls form but are then replaced as time goes on by new systems of rolls inclined at an angle of about 60° to the old rolls.

The motion of an electrically conducting fluid (mercury, say) is affected by the presence of a magnetic field. If the convection is strong enough, small magnetic field perturbations may grow and lead, as the two effects balance each other, to the existence of a self-sustained magnetic field, a mechanism known as the dynamo effect. Childress and Soward [5] proposed rotating convection as a suitable framework in which the dynamo effect could be realized. It is therefore a natural approach to apply methods of local bifurcation theory to the homoclinic cycle of rolls in the pure convection problem. Observe that any dynamo solution obtained in this way is an intermittent dynamo.

It transpires that the particular instability of the Busse-Clever-Heikes cycle corresponding to the instabilities considered by Childress and Soward [5] is a new kind of bifurcation which we call a *transverse bifurcation*. This motivates the general study of transverse bifurcations from robust homoclinic cycles. We consider the simplest possible context, namely in four dimensions, and obtain fairly complete results in this context. We expect that the corresponding results for cycles in higher dimensions are similar, but the proofs will be much more technical. In particular, our techniques should be applicable to the rotating magnetohydrodynamic problem.

In the remainder of this introduction, we describe how our results on transverse bifurcations fit together with previous results on bifurcation from homoclinic cycles both in the symmetric and nonsymmetric contexts.

A homoclinic orbit of an ODE is a trajectory which converges for both positive and negative time to the same equilibrium ξ. In the class of general vector fields, the appearance of such an orbit is a codimension one phenomenon. Dynamics and invariant sets near such homoclinic orbits have been studied extensively since the work of Shil'nikov in the 60's. In particular, many of the codimension one and codimension two phenomena associated with homoclinic orbits have been analyzed, see for example [24, 6, 12, 13, 20].

The dynamics near a homoclinic orbit is governed to some extent by the eigenvalues of the linearized vector field at the equilibrium. Particularly significant is the relative strengths of the weakest contracting and expanding eigenvalues. An important codimension two bifurcation, which we call a *resonant bifurcation*, occurs when the real parts of these eigenvalues are equal (in absolute value) and gives rise to an exponentially flat branch of periodic solutions [6].

Bifurcations of homoclinic orbits often possess analogues in the symmetric context, see for example [21, 23] and also in population dynamics [11]. The main differences are that the codimension is reduced and that the underlying homoclinic cycle, being robust, persists throughout the bifurcation. An important special case is when the cycle is initially asymptotically stable, but loses stability as a bifurcation parameter is varied. The equivariant analogue of the resonant bifurcation (now a codimension one bifurcation) is analyzed in [23] (see also [10]) and coincides with the loss of stability of the underlying homoclinic cycle. A cycle can lose stability also by undergoing a transverse bifurcation, as we now explain.

Robust homoclinic cycles arise because heteroclinic connections may lie in proper flow-invariant subspaces (forced by symmetry) and may be structurally stable within these subspaces. Eigenvalues with eigenvectors that do not lie in the relevant flow-invariant subspaces are called *transverse* eigenvalues. A necessary condition that the homoclinic cycle is asymptotically stable is that the transverse eigenvalues (if any) have negative real part. A *transverse bifurcation* occurs when the cycle loses asymptotic stability as the real part of a transverse eigenvalue passes through zero. [1] In the nonsymmetric context, such a transverse bifurcation is highly degenerate — at least of codimension three and therefore of

[1]Since the homoclinic cycle of Busse et al [1, 2] is purely convective, it is evident that a magnetic instability of the equilibrium rolls corresponds to a transverse instability of the cycle

less interest. Transverse bifurcations have been considered by [3] in a somewhat different context.

As evidenced by the results in this paper, transverse bifurcation leads to a variety of different phenomena. The lowest-dimensional instances of this bifurcation occur for vector fields on $\mathbb{R}^4$. Then there is a single real transverse eigenvalue. As the bifurcation takes place, the dimension of the unstable manifold of the equilibrium ξ is increased by one. Applying the center manifold theorem, we know that ξ undergoes a steady-state bifurcation. The presence of symmetry implies that this is a pitchfork bifurcation. To fix ideas, we assume that the pitchfork bifurcation is supercritical. Subcritically, there is an asymptotically stable homoclinic cycle. Supercritically, there is an unstable homoclinic cycle and a pair of saddle points. We investigate the further dynamics associated with this bifurcation.

Even within this restricted framework the possibilities for dynamics are quite rich. There is a trichotomy [16] underlying the asymptotic stability properties of heteroclinic cycles in the presence of symmetry. This trichotomy is particularly fundamental for cycles in $\mathbb{R}^4$. The cycles thus fall into three types which we label Type A, Type B and Type C and which are distinguished on purely representation-theoretic grounds, see Definition 2.3. The cycles in [4] are of Type A. If a cycle in $\mathbb{R}^3$, such as the example in [9], is embedded in a three-dimensional flow-invariant subspace in $\mathbb{R}^4$ then the resulting cycle is of Type B. There is a cycle of Type C in [8, 10].

We now describe our main results. Suppose that a homoclinic cycle in $\mathbb{R}^4$ undergoes a transverse bifurcation and that the associated pitchfork bifurcation at the equilibria is supercritical.

If the cycle is of Type A, there is a bifurcation of periodic solutions whose distance to the homoclinic cycle (or amplitude) is exponentially flat in the bifurcation parameter λ. Indeed the amplitude of the branch of periodic solutions is proportional at lowest order to $|d|^{1/\lambda}$ where d is a certain coefficient of the linearized flow near the homoclinic cycle but away from the equilibria. In particular, the direction of branching (and stability) is determined by d and thus is independent of the direction of the local pitchfork bifurcation. There is a supercritical bifurcation of asymptotically stable periodic solutions when $|d| < 1$ and a subcritical bifurcation of unstable periodic solutions when $|d| > 1$.

If the cycle is of Type B, there is a supercritical pitchfork bifurcation of asymptotically stable homoclinic cycles with amplitude proportional at lowest order to $\sqrt{\lambda}$.

If the cycle is of Type C, there is a supercritical pitchfork bifurcation of asymptotically stable homoclinic cycles as for Type B but the heteroclinic orbits connect every other equilibrium.

The remainder of this paper is organized as follows. In Section 2, we recall some terminology and results of [15, 16] and define cycles of Type A, B and C. The bifurcations for the cycles of Type B and C are analyzed in Section 3. The analysis for the cycles of Type A is given in Section 4.

2 Notation and results from [15, 16]

In this section we summarize the terminology and main results in [15, 16] that we shall need. In particular, cycles of Type A, B and C are defined.

Roughly speaking, a heteroclinic cycle for an ODE is a collection of equilibria $\xi_1, \dots, \xi_m$ together with trajectories $x_j(t)$, $j = 1, \dots, m$ that connect ξ_j to ξ_{j+1} (where $\xi_{m+1} = \xi_1$). More generally, if there is a group of symmetries Γ we relax the condition on $x_m(t)$ and demand only that $x_m(t)$ connects ξ_m to $\gamma\xi_1$ for some $\gamma \in \Gamma$. When $m = 1$ the cycle is called a homoclinic cycle.

The following simple construction leads to the class of robust homoclinic cycles in $\mathbb{R}^4$ that we shall study. This is a special case of the cycles in $\mathbb{R}^n$ that are considered in [15, 16]. Suppose that Γ is a finite group acting linearly on $\mathbb{R}^4$ and consider a Γ-equivariant vector field with the following properties. Let $\xi \neq 0$ be a hyperbolic saddle point with a one-dimensional unstable manifold $W^u(\xi)$ Suppose that $W^u(\xi) \subset P$ where $P = \mathrm{Fix}(\Sigma)$ is a two-dimensional fixed-point subspace corresponding to an isotropy subgroup $\Sigma \subset \Gamma$. Suppose further that there is an element $\gamma \in \Gamma$ such that $\gamma\xi$ is a sink in P and that there is a saddle-sink connection in P connecting ξ to $\gamma\xi$. Then the collection of saddle points $\gamma^j\xi$, $j \geq 1$, together with their unstable manifolds, forms a homoclinic cycle. Moreover, this cycle is robust: equivariance implies that P is flow-invariant and hence the saddle-sink connection persists under Γ-equivariant perturbation.

Since Γ is finite, the collection of saddle points $\gamma^j\xi$ is finite. Let $k \geq 1$ be the least positive integer such that $\gamma^k\xi = \xi$. Then the homoclinic cycle X is given by

$$X = \bigcup_{j=1}^{k} W^u(\gamma^j\xi).$$

Remark 2.1 The unstable manifold $W^u(\xi)$ consists of two branches, one of which is contained in $W^s(\gamma\xi)$. As is easily shown, the remaining branch of the unstable manifold is contained in $W^s(\gamma'\xi)$ for some $\gamma' \in \Gamma$.

The asymptotic stability of such cycles is considered in [15, 16]. Without loss of generality we may assume that Γ acts faithfully and orthogonally on $\mathbb{R}^4$, so Γ is a finite subgroup of $\mathbf{O}(4)$. Observe that ξ is a saddle point inside of P and is a sink inside of $\gamma^{-1}P$. There is an eigenvalue common to both subspaces (the radial eigenvalue), an additional eigenvalue in $\gamma^{-1}P$ (contracting) and in P (expanding), and one further eigenvalue (transverse). We denote the respective eigenvalues by $-r > 0$, $-c > 0$, $e > 0$ and $t \neq 0$. In particular, the eigenvalues are necessarily real.

Proposition 2.2 ([15]) *Suppose that $c > e$ and $t < 0$. Then generically the homoclinic cycle X is asymptotically stable. If $t > 0$, then X is unstable.*

The sufficient conditions for asymptotic stability in Proposition 2.2 are not always necessary for cycles in $\mathbb{R}^n$, $n > 3$. As shown in [16], there is an important representation-theoretic trichotomy that underlies the stability theory of homoclinic (and heteroclinic) cycles in dimensions four and higher. In our context, this trichotomy can be described very easily. Define the three-dimensional subspace $Q = P + \gamma^{-1}P$. There is the question as to whether or not Q is a fixed-point subspace, that is, whether or not $Q = \mathrm{Fix}(\tau)$ for some reflection $\tau \in \Gamma$.

Definition 2.3

The cycle X is of *Type A* if Q is not a fixed-point subspace.

The cycle X is of *Type B* if Q is a fixed-point subspace and $X \subset Q$.

The cycle X is of *Type C* if Q is a fixed-point subspace and $X \not\subset Q$.

As noted in the introduction, all three types of cycle are realized in examples in the literature.

Remark 2.4 The following representation-theoretic information from [16] is useful. For all three types of cycle, there is an involution $\rho \in \Sigma$ such that $P = \mathrm{Fix}(\rho)$. (When diagonalized, ρ has two $+1$'s and two -1's.) When X is of Type A, $\Sigma \cong \mathbb{Z}_2$, the unique nontrivial element being ρ. When X is of Type B or Type C, $\Sigma \cong \mathbb{Z}_2^2$, with nontrivial elements ρ, τ and $\rho\tau$. Finally, if X is of Type C, there is an even number of equilibria $\gamma^j\xi$ in the cycle, so k is even.

Proposition 2.5 *If X is of Type A or of Type B, then generically the cycle is asymptotically stable if and only if $c > e$ and $t < 0$. If X is of Type C, then generically the cycle is asymptotically stable if and only if $c - t > e$ and $t < 0$.*

Proof The stability of cycles of Types A and B is studied in [15]. The result for cycles of Type C is due to [8, 10]. ∎

3 Pitchfork bifurcations for cycles of Type B and Type C

In this section, we consider the bifurcations associated to a homoclinic cycle of Type B or C when the transverse eigenvalue passes through zero as a bifurcation parameter λ is varied. Specifically, the transverse eigenvalue $t(\lambda)$ is a function of λ and $t(0) = 0$. Generically, $t'(0) \neq 0$. So that the cycle X undergoes a loss of stability, we assume that $t'(0) > 0$. Reparametrizing, we may assume that $t(\lambda) \equiv \lambda$.

3.1 Pitchfork bifurcation of equilibria

First we consider the bifurcation associated with the equilibrium ξ. The analysis of this bifurcation holds also for cycles of Type A. When $\lambda = 0$ there is a one-dimensional center manifold tangent at ξ to the kernel of the linearized vector field. The dynamics on this center manifold is governed by a vector field of the form

$$\dot{z} = \lambda z + h(z, \lambda),$$

where $h(0, \lambda) = h_z(0, \lambda) = 0$. We may and shall assume that h is at least C^3.

Recall that ξ lies in the flow-invariant subspace $P = \mathrm{Fix}(\rho)$ where $\rho \in \Gamma$ is an involution. The isotropy subgroup of ξ, and in particular the involution ρ, acts on a suitably chosen neighborhood of ξ and we can arrange that the dynamics on the center manifold commutes with ρ. Since the eigenspace associated with the transverse eigenvalue is orthogonal to P, ρ acts nontrivially in the central direction. Specifically, the action of ρ is given by $z \mapsto -z$ and $h(z, \lambda)$ is an odd function of z. We assume that $h_{zzz}(0, 0) \neq 0$ and for definiteness that $h_{zzz}(0, 0) < 0$. Then ξ undergoes a supercritical pitchfork bifurcation to a pair of saddle points as λ passes through zero. Moreover, these saddle points are interchanged by ρ.

The eigenvalue data associated with the cycle X varies with the bifurcation parameter λ. For λ near 0, X has eigenvalues $-r(\lambda)$, $-c(\lambda)$, $e(\lambda)$ and $t(\lambda) \equiv \lambda$ where $r(0), c(0), e(0) > 0$. We assume that $c(0) \neq e(0)$. Then it follows from Proposition 2.5 that the stability of the cycle (all three types) is determined for λ small by the sign of $c(0) - e(0)$. To ensure that the cycle undergoes a loss of stability, we assume that $c(0) > e(0)$.

Remark 3.1 We have made the assumptions $t'(0) \neq 0$, $h_{zzz}(0, 0) \neq 0$ and $c(0) \neq e(0)$ which are satisfied generically. Then we have concentrated attention on the case

$$t'(0) > 0, \quad h_{zzz}(0, 0) < 0, \quad c(0) > e(0). \tag{3.1}$$

This is the case that is the most interesting from the point of view of asymptotically stable dynamics. However, the analysis of the other cases is completely analogous.

3.2 Cycles of Type B

Next we specialize to cycles X of Type B with saddle point ξ connected to $\gamma\xi$ where $\gamma \in \Gamma$. Thus $X \subset Q = \mathrm{Fix}(\tau)$ where $\tau \in \Gamma$ is a reflection. We can replace

ρ by τ in the analysis of the pitchfork bifurcation at ξ and so we may denote the bifurcating equilibria by ξ' and $\tau\xi'$. The element γ preserves Q and hence acts on the two components of $\mathbb{R}^4 - Q$. Replacing γ by $\gamma\tau$ if necessary, we may assume that γ preserves each of these components.

Theorem 3.2 *Suppose that Γ is a finite group acting linearly on $\mathbb{R}^4$, and that X is a homoclinic cycle of Type B with equilibria $\gamma^j\xi$, $j = 1, \ldots, k$. Assume that X undergoes a transverse bifurcation and that the nondegeneracy conditions (3.1) are satisfied. Then there is a supercritical pitchfork bifurcation to two asymptotically stable homoclinic cycles, X' and $\tau X'$, one lying in each component of $\mathbb{R}^4 - Q$. The cycle X' consists of heteroclinic connections from $\gamma^{j-1}\xi'$ to $\gamma^j\xi'$ for $j = 1, \ldots, k$.*

Proof There is a second three-dimensional flow-invariant subspace $Q' = \mathrm{Fix}(\rho\tau)$ containing P together with the transverse directions corresponding to ξ and $\gamma\xi$. Thus the pitchfork bifurcations of equilibria at ξ and $\gamma\xi$ each take place in Q'. In particular, the four equilibria ξ', $\tau\xi'$, $\gamma\xi'$ and $\tau\gamma\xi'$ lie in Q'. The first two equilibria are saddles (with one-dimensional unstable manifolds) and the remaining equilibria are sinks. These stability assignments follow from the fact that the pitchfork bifurcation is supercritical.

We claim that there is a saddle-sink connection in Q' between ξ' and $\gamma\xi'$ (and hence between $\tau\xi'$ and $\tau\gamma\xi'$). It then follows that there is a supercritical pitchfork bifurcation of robust homoclinic cycles. Moreover, it follows from [15] that the bifurcating cycles are asymptotically stable: ξ' lies in the flow-invariant subspace $Q' \cap \gamma^{-1}Q'$ and there are two radial eigenvalues, one close to $-r$ and one close to 0 but negative. There is also an expanding eigenvalue in Q' close to e and a contracting eigenvalue in $\gamma^{-1}Q'$ close to $-c$. Since $c > e$, the contracting eigenvalue dominates the expanding eigenvalue for λ small and the cycle is asymptotically stable. (There are no transverse eigenvalues for the new bifurcating cycles.)

It remains to establish the existence of the heteroclinic connection between ξ' and $\gamma\xi'$. We can choose a neighborhood $V \subset Q'$ of the equilibrium ξ so that for λ close enough to zero, trajectories that begin in V and remain in V for all time are attracted to the center manifold of ξ. (Note that the expanding direction associated with ξ does not lie in Q'.) On the center manifold, $\gamma\xi$ is repelling and all other trajectories are asymptotic to the new equilibria $\gamma\xi'$ and $\tau\gamma\xi'$. Hence, all trajectories in V are attracted to one of three equilibria $\gamma\xi$, $\gamma\xi'$ and $\tau\gamma\xi'$.

By continuity, the unstable manifolds of the equilibria near ξ have to intersect V for $\lambda > 0$ small and so there are connections to the equilibria near $\gamma\xi$. Since P divides Q' into two flow-invariant regions, one containing ξ' and $\gamma\xi'$ and the other containing $\tau\xi'$ and $\tau\gamma\xi'$, the required connections are forced. ∎

3.3 Cycles of Type C

In this subsection, we consider cycles of Type C. The analysis is very similar to that for cycles of Type B. Again $Q = P + \gamma^{-1}P = \mathrm{Fix}(\tau)$, but this time $X \not\subset Q$. Just as before, we may choose γ so that γ preserves each of the components of $\mathbb{R}^4 - Q$. We continue to denote the equilibria bifurcating from ξ by ξ' and $\tau\xi'$.

Theorem 3.3 *Suppose that Γ is a finite group acting linearly on $\mathbb{R}^4$, and that X is a homoclinic cycle of Type C with equilibria $\gamma^j\xi$, $j = 1, \dots, k$ (where k is even). Assume that X undergoes a transverse bifurcation and that the nondegeneracy conditions (3.1) are satisfied. Then there is a supercritical pitchfork bifurcation to four asymptotically stable homoclinic cycles: X' and $\gamma X'$ lying in one component of $\mathbb{R}^4 - Q$, $\tau X'$ and $\gamma\tau X'$ lying in the other component. The cycle X' consists of heteroclinic connections from $\gamma^{j-2}\xi'$ to $\gamma^j\xi'$ for $j = 2, 4, \dots, k$.*

Proof This is very similar to the proof of Theorem 3.2. The only point to verify, is that there is a heteroclinic connection from ξ' to $\gamma^2\xi'$. Again we consider the flow-invariant subspace $Q' = \mathrm{Fix}(\rho\tau)$. Although the transverse direction for ξ is contained in Q', this is not the case for the transverse eigenvalue at $\gamma\xi$. Thus the bifurcating equilibria ξ' and $\tau\xi'$ lie in Q' but $\gamma\xi', \gamma\tau\xi' \notin Q'$. At the same time, the expanding direction at $\gamma\xi$ does lie in Q' and hence the connection from $\gamma\xi$ to $\gamma^2\xi$ is contained in Q'. (Another way of saying this is that $\gamma Q' = Q$.) In particular, $\gamma^2\xi \in Q'$. Moreover, the transverse direction at $\gamma^2\xi$ lies in Q'. Arguing similarly to before, we deduce that inside of Q' the unstable manifold of ξ' passes nearby to the saddle point $\gamma\xi$ and is asymptotic to the sink $\gamma^2\xi'$. ∎

3.4 Completion of analysis for cycles of Type B

We end this section by refining our results for cycles of Type B. Indeed, we prove that for a cycle of Type B undergoing a transverse bifurcation and satisfying conditions (3.1), generically there is no local asymptotic dynamics except for the homoclinic cycles described in Theorem 3.2 — locally, every trajectory that is bounded in forward or backward time is asymptotic to (the group orbit of) the unstable homoclinic cycle X or the stable homoclinic cycle X'.

To prove this result, it is necessary to construct a Poincaré map around the homoclinic cycle. Assuming certain nondegeneracy conditions (that we shall not write down) on the eigenvalues at the saddle point ξ, we can locally linearize the flow except on the center manifold [25]. More precisely, for any positive integer k, there is a C^k-change of coordinates so that the equations in a neighborhood of ξ take the form

$$
\begin{aligned}
\dot{u} &= -r(z,\lambda)u && (3.2a)\\
\dot{v} &= -c(z,\lambda)v && (3.2b)\\
\dot{w} &= e(z,\lambda)w && (3.2c)\\
\dot{z} &= \lambda z + h(z,\lambda) && (3.2d)
\end{aligned}
$$

where r, c, e and h are C^k functions of z and λ. Here, h is the same function that appeared at the beginning of this section. When $z = 0$, r, c and e recapture their original meaning. In this subsection, we shall require only that $k \geq 1$.

The cases $c(0) > r(0)$ and $c(0) < r(0)$ are similar and we carry out the proof under the assumption that $c(0) < r(0)$. Then, generically, the unstable manifold of ξ is tangent to the direction of v. Hence we can define the ingoing cross-section

$$H(in) = \{(u, 1, w, z);\ |u|, |w|, |z| \leq 1\}.$$

We also introduce the outgoing cross-section

$$H(out) = \{(u, v, 1, z);\ |u|, |v|, |z| \leq 1\}.$$

The heteroclinic connections emanating from ξ intersect $H(in)$ and $H(out)$ at the points $(u_0, 1, 0, 0)$ and $(0, 0, 1, 0)$. We consider the first hit map $g : H(in) \to \gamma H(in)$

defined for sufficiently small $(u,1,w,z) \in H(in)$, $w > 0$, as the composition $g = \psi\circ\phi$ of a first hit map $\phi : H(in) \to H(out)$ and a connecting diffeomorphism $\psi : H(out) \to \gamma H(in)$.

The diffeomorphism $\gamma^{-1}\psi : H(out) \times \mathbb{R} \to H(in)$ has the form

$$\gamma^{-1}\psi(u,v,1,z,\lambda) = (\eta\,,\,1\,,\,\alpha_1 v + \alpha_2 z\,,\,\alpha_3 v + \alpha_4 z),$$

where η and $\alpha_1,\ldots,\alpha_4$ are smooth functions of u,v,z,λ. Since the z-directions are transverse to the invariant subspace Q, it is evident that $\alpha_3 = 0$. Similarly, invariance of the subspace Q' implies that $\alpha_2 = 0$. Hence, the w-component of $\gamma^{-1}\psi$ is dominated by $2|\alpha_1(0)||v|$ for all u,v,z,λ small enough.

Rescaling time, we can assume that $e(z,\lambda) \equiv 1$ (and so $c(0) > 1$). Equation (3.2c) becomes $\dot{w} = w$ and hence the time of flight between $H(in)$ and $H(out)$ for a point $(u,1,w,z) \in H(in)$ is given by $T = -\ln w$. We deduce from equation (3.2b) that for any $\epsilon > 0$, the v-component of ϕ is dominated by $|w|^{c(0)-\epsilon}$.

Combining these two estimates we have that the iterate of w under the Poincaré map $\gamma^{-1}g = \gamma^{-1}\psi\circ\phi : H(in) \to H(in)$ is dominated by $2|\alpha_1(0)||w|^{c(0)-\epsilon}$. Choosing ϵ so that $c(0) - \epsilon > 1$ we deduce that provided u and z remain small, the w-coordinate contracts to zero under the Poincaré map. (In fact, u is forced to remain small; it is only z that we cannot easily control.) Hence, we have shown that the local asymptotic dynamics is confined to the invariant subspace Q' and its symmetric images as required.

The corresponding analysis for cycles of Type C is considerably more complicated.

4 Flat bifurcations for a Type A homoclinic cycle

In this section we state and prove our result on transverse bifurcations of a Type A homoclinic cycle. The result requires various genericity assumptions:

(i) The assumption that the transverse eigenvalue passes through zero with nonzero speed, that is, $t'(0) \neq 0$.

(ii) Nondegeneracy conditions on the eigenvalues of the linearized vector field at the saddle point ξ as in Subsection 3.4. In particular, we again require that $c(0) \neq e(0)$.

(iii) Nondegeneracy conditions on global coefficients such as the constant d that appears in the statement of the theorem.

As in Section 3, we shall concentrate attention on certain cases and assume the nondegeneracy conditions

$$t'(0) > 0, \quad c(0) > e(0). \tag{4.1}$$

This time, the analysis of the cases $c(0) > e(0)$ and $c(0) < e(0)$ is slightly different. In particular, the coefficient d is different for the two cases. We will indicate at the end how to extend the analysis to the case $c(0) < e(0)$. Finally we note that the conditions in (iii) are indeed satisfied generically, see for example [15].

Theorem 4.1 *Suppose that Γ is a finite group acting linearly on $\mathbb{R}^4$, and that X is a homoclinic cycle of Type A undergoing a transverse bifurcation. Assume that the nondegeneracy conditions (4.1) are satisfied. Generically, there exists a unique (up to conjugacy) branch of limit cycles bifurcating from the homoclinic cycle as λ passes through* 0*. Moreover, there is a positive constant $d \neq 1$ depending only on the global part of the flow, such that the distance of the limit cycle to the homoclinic cycle is at leading order proportional to $d^{\frac{1}{\lambda}}$. If $d < 1$, the bifurcation is supercritical and the periodic solutions are asymptotically stable. If $d > 1$, the bifurcation is subcritical and the periodic solutions are unstable.*

Remark 4.2 In bifurcations of this type in the nonsymmetric context, periodic solutions typically come in two forms: 1-periodic and 2-periodic. Roughly speaking, this means that the bifurcating periodic solutions wind around either once or twice in a neighborhood of the homoclinic trajectory. Moreover, the 2-periodic case is considerably harder to analyze and leads to nonuniqueness, see [6]. In our context, as a consequence of the symmetry, everything reduces to the 1-periodic case.

4.1 The Poincaré map and periodic solutions

Again we assume nondegeneracy conditions on the eigenvalues at the saddle point ξ so that under a C^k change of coordinates, the equations in a neighborhood of ξ take the form (3.2). Note that $h = \mathcal{O}(z^3)$. We shall require that $k \geq 3$. We believe that Theorem 4.1 holds for $k \geq 1$, but are not able to prove it in such generality. In fact $k \geq 2$ suffices for the existence proof and we use $k \geq 3$ in proving uniqueness. As before, we assume that $c(0) > r(0)$ and rescale time so that $e(0) \equiv 1$ and $c(0) > 1$. The cross-sections $H(in)$ and $H(out)$ and the Poincaré map $g = \psi \circ \phi$ are defined just as in Subsection 3.4.

Proposition 4.3 *Suppose $\delta \in \Gamma$. Then $\delta H(in) = \gamma H(in)$ if and only if $\delta = \gamma$ or $\delta = \rho\gamma$.*

Proof Suppose that $\delta H(in) = \gamma H(in)$. Then $\gamma^{-1}\delta$ fixes $H(in)$ or equivalently fixes points in the subspace $\gamma^{-1}P$. Hence, $\gamma^{-1}\delta$ lies in the isotropy subgroup $\gamma^{-1}\Sigma\gamma$ and it follows from Remark 2.4 that $\delta = \gamma$ or $\delta = \rho\gamma$ as required. ∎

Suppose $x(t)$ is a periodic solution sufficiently close to X. Then the group orbit $\Gamma x(t)$ intersects $H(in)$. Relabeling, we can arrange that $x_0 = x(0) \in H(in)$ and $g(x_0) \in \gamma H(in)$. We say that $x(t)$ is 1-*periodic* if $g(x_0) \in \Gamma x_0$. Proposition 4.3 implies the following corollary.

Corollary 4.4 *Suppose $x_0 \in H(in)$ and $g(x_0) \in \gamma H(in)$. Then x_0 lies on a 1-periodic solution if and only if x_0 is a fixed point of $\delta^{-1}g$ where $\delta = \gamma$ or $\delta = \rho\gamma$.*

Remark 4.5 Suppose x_0 is a fixed point of $\delta^{-1}g$, where $\delta = \gamma$ or $\delta = \rho\gamma$, and let $x(t)$ be the corresponding 1-periodic solution. Then $x(t)$ may wind once or twice around X before closing up and this property is determined by the action of the element δ. Let ℓ be the least positive integer such that $\delta^\ell H(in) = H(in)$. Then δ^ℓ acts on $H(in)$ either as identity or as minus identity. In the former case $x(t)$ winds once around X and in the latter case twice.

In the sequel we will find fixed points of $\gamma^{-1}g$ and $\gamma^{-1}\rho g$ and show that all periodic solutions bifurcating from X are 1-periodic.

As noted in Subsection 3.4, the diffeomorphism $\gamma^{-1}\psi : H(out) \times \mathbb{R} \to H(in)$ has the form

$$\gamma^{-1}\psi(u, v, 1, z, \lambda) = (\eta\,,\,1\,,\,\alpha_1 v + \alpha_2 z\,,\,\alpha_3 v + \alpha_4 z),$$

where η and $\alpha_1, \ldots, \alpha_4$ are smooth functions of u, v, z, λ. Let $b = \alpha_2(0)$, $d = \alpha_4(0)$. When X is of Type A, we require that $b \neq 0$ and $d \neq 0, \pm 1$. By [15] these conditions are satisfied generically. (This is certainly not the case for cycles of Type B and Type C: we have already seem that if X is of Type B, $\alpha_2 = \alpha_3 = 0$, whereas if X is of Type C, $\alpha_1 = \alpha_4 = 0$.)

We introduce new variables

$$\tau = w^{\lambda}, \quad Z = z/\lambda.$$

Lemma 4.6 *The first hit map $\phi : H(in) \times \mathbb{R} \to H(out)$ has the form*

$$\phi(u, 1, w, \lambda Z, \lambda) = (u\tau^{r/\lambda}\phi_r\,,\,w\tau^{(c-1)/\lambda}\phi_c\,,\,1\,,\,\lambda Z/\tau + \mathcal{O}(\lambda Z^2)),$$

where r, c, ϕ_r, ϕ_c are smooth functions of τ, Z, λ for $\tau \neq 0$.

The proof of this lemma is somewhat technical and is postponed to Subsection 4.3.

Remark 4.7 Recall that $r(0) > 0$ and $c(0) > 1$. It follows that if $\tau \in (0, 1)$ and $\lambda \geq 0$, then all components of ϕ are smooth and are $\mathcal{O}(\lambda)$. The same is true if $\tau > 1$ and $\lambda \leq 0$.

4.2 Analysis of the bifurcation equations

The Poincaré map $\gamma^{-1}g : H(in) \times \mathbb{R} \to H(in)$ is given by $\gamma^{-1}g = \gamma^{-1}\psi \circ \phi$ and we search for solutions of the equation

$$\gamma^{-1}g(u, 1, w, \lambda Z, \lambda) = (u, 1, w, \lambda Z).$$

Applying the implicit function theorem, we can solve the u-component of this equation for u as a smooth function of w, Z, λ and $\tau \neq 0$. Moreover, $u = \eta(0) +$

$\mathcal{O}(\lambda)$. Now we substitute u into the remaining components of $\gamma^{-1}g$ to obtain the bifurcation equations

$$w = \alpha_1 w \tau^{(c-1)/\lambda} \phi_c(\tau, Z, \lambda) + \alpha_2 \lambda Z/\tau + \mathcal{O}(\lambda Z^2),$$
$$\lambda Z = \alpha_3 w \tau^{(c-1)/\lambda} \phi_c(\tau, Z, \lambda) + \alpha_4 \lambda Z/\tau + \mathcal{O}(\lambda Z^2).$$

Abusing notation slightly, we can regard $\alpha_1, \dots, \alpha_4$ as smooth functions of w, Z, λ, τ. Moreover, up to a constant these four functions are $\mathcal{O}(\lambda)$.

Next, we make the substitution $\lambda Z = \rho w$, multiply by τ, and divide out by the solution $w = 0$. We obtain the equations for τ and ρ

$$\tau = \alpha_1 \tau^{(c-1)/\lambda+1} \phi_c + \alpha_2 \rho + \mathcal{O}(\tfrac{\tau^{\frac{1}{\lambda}}}{\lambda}),$$
$$\rho\tau = \alpha_3 \tau^{(c-1)/\lambda+1} \phi_c + \alpha_4 \rho + \mathcal{O}(\tfrac{\tau^{\frac{1}{\lambda}}}{\lambda}).$$

Write $\alpha_2 = b + \mathcal{O}(\lambda)$, $\alpha_4 = d + \mathcal{O}(\lambda)$ where b and d are constants. Generically, $b \neq 0$, $d \neq 0, \pm 1$. We consider first the case $d \in (0,1)$. The function $\tau^{\frac{1}{\lambda}}$ defined for $\lambda > 0$ extends uniquely to a smooth function of λ and τ defined on a neighborhood of $(0,d)$ and identically equal to 0 for $\lambda \leq 0$. This extended function, which we also denote by $\tau^{1/\lambda}$, satisfies the property $\mathcal{O}(\frac{\tau^{\frac{1}{\lambda}}}{\lambda}) = \mathcal{O}(\lambda)$.

We can now rewrite the equations in the form

$$\tau - b\rho + \mathcal{O}(\lambda) = 0, \quad d\rho - \tau\rho + \mathcal{O}(\lambda) = 0.$$

There is a trivial solution $(\tau, \rho, Z, \lambda) = (d, d/b, 0, 0)$. Applying the implicit function at the point $(d, d/b, 0, 0)$, we can solve smoothly for τ and ρ. Tracing back through the substitutions, we obtain the branch of solutions

$$z = \lambda Z = \rho w = \rho \tau^{1/\lambda} \approx (d/b) d^{1/\lambda}, \quad \lambda \geq 0.$$

Thus we have a flat supercritical bifurcation of limit cycles bifurcating from the homoclinic cycle.

In the case $d > 1$ we consider $\lambda \leq 0$ and obtain a subcritical bifurcation of limit cycles. When $d < 0$, the above procedure yields no fixed point for $\gamma^{-1}g$. Note however that d is of opposite sign for $\gamma^{-1}g$ and $\gamma^{-1}\rho g$. Hence the bifurcating periodic solution exists if $|d| \neq 1$ and joins either z_0 to γz_0 or z_0 to $\rho\gamma z_0$.

4.3 Proof of Lemma 4.6

Under the substitution $Z = z/\lambda$, equation (3.2d) becomes

$$\dot{Z} = \lambda(Z + \tilde{h}(Z, \lambda)), \tag{4.2}$$

where $\tilde{h}(Z, \lambda) = \frac{1}{\lambda^2} h(\lambda Z, \lambda) = \mathcal{O}(Z^2)$. Let $Z(t, Z_0, \lambda)$ be the solution of (4.2) with initial condition Z_0. The time of flight between $H(in)$ and $H(out)$ for a point $(u, 1, w, z) \in H(in)$ is given by $T = -\ln w = -\frac{1}{\lambda} \ln \tau$. Define $\hat{Z}(\tau, Z_0, \lambda) = Z(-\ln(\tau)/\lambda, Z_0, \lambda)$.

Lemma 4.8 *The function $\hat{Z}$ is smooth in τ, Z_0 and λ for $\tau \neq 0$. Moreover $\hat{Z}(\tau, Z_0, \lambda) = Z_0/\tau + \mathcal{O}(Z_0^2)$.*

Proof By the chain rule,

$$\frac{dZ}{d\tau} = -\frac{1}{\lambda\tau}\frac{dZ}{dt}.$$

Substituting for Z in terms of $\hat{Z}$ in equation (4.2), we find that $\hat{Z}$ is a solution of the differential equation

$$\frac{d\hat{Z}}{d\tau} = -\frac{1}{\tau}(\hat{Z} + \tilde{h}(\hat{Z}, \lambda)). \tag{4.3}$$

The right-hand-side of the equation is smooth for $\tau \neq 0$ and the first statement of the lemma follows. In order to obtain an expansion in Z near $Z = 0$ we consider the linearization of (4.3), namely

$$\frac{d\hat{Z}^{\text{lin}}}{d\tau} = -\tau^{-1}\hat{Z}^{\text{lin}}.$$

This equation has the solution $\hat{Z}^{\text{lin}}(\tau, Z_0, \lambda) = \tau^{-1} Z_0$. ∎

It follows from this lemma that the z-component of ϕ in Lemma 4.6 is as stated. We now give the proof for the v-component. The proof for the u-component is almost identical. The v-component of the solution to equation (3.2) with initial condition $(u, 1, w, z_0)$ can be written in the form

$$v(t) = e^{-tc(0,\lambda)} e^{\int_0^t c_1(z(s,z_0,\lambda),\lambda)ds},$$

where $z(t, z_0, \lambda)$ is the solution of (3.2d) with initial condition z_0. Recalling that $T = -\ln w$, we compute that

$$v(T) = w^{c(z,0)} e^{\int_0^T c_1(z(t,z_0,\lambda),\lambda)dt}. \tag{4.4}$$

The exponential appearing in equation (4.4) can be expressed in terms of the variables (τ, Z):

$$\int_0^T c_1(\lambda Z(t, z_0, \lambda), \lambda)dt = \int_{Z_0}^{\hat{Z}(\tau,Z_0,\lambda)} c_1(\lambda Z, \lambda)(\dot{Z})^{-1}dZ$$
$$= \int_{Z_0}^{\hat{Z}(\tau,Z_0,\lambda)} c_1(\lambda Z, \lambda)\frac{1}{\lambda(Z + \tilde{h}(Z, \lambda))}dZ.$$

Since $\tilde{h}(Z, \lambda) = \mathcal{O}(Z^2)$ and $c_1(\lambda Z, \lambda) = \mathcal{O}(\lambda Z)$, the integrand is a smooth function of λ and Z for Z small. It follows from Lemma 4.8 that the integral and hence the exponentiated integral is smooth for Z_0 small and $\tau \neq 0$. Thus we can write the v-component of ϕ in the form $w^{c(0,\lambda)}\phi_c(\tau, Z, \lambda)$ where ϕ_c is smooth. Finally, we write $w^{c(0,\lambda)} = ww^{c(0,\lambda)-1}$ and make the substitution $w = \tau^{1/\lambda}$. ∎

4.4 Uniqueness

In this section we prove the following theorem.

Theorem 4.9 *The branch of periodic solutions found in Section 4.2 is unique.*

Consider the map $G = \gamma^{-1}g$ or $\gamma^{-1}\rho g$, chosen so that $d > 0$. The domain of G is $H_+(in) = \{(u, 1, w, z) \in H(in) \ : \ w > 0\}$. We look for a sequence $\{(u_i, 1, w_i, z_i)\}_{i=0,\ldots,k}$ such that $(u_k, 1, w_k, z_k) = \pm(u_0, 1, w_0, z_0)$ and

$$u_{j+1} = \eta(u_j, w_j, z_j) \tag{4.5a}$$
$$w_{j+1} = \pm(\alpha_1 w_j^c + \alpha_2 z(T_j, z_j)) \tag{4.5b}$$
$$z_{j+1} = \pm(\alpha_3 w_j^c + \alpha_4 z(T_j, z_j)), \tag{4.5c}$$

where $j = 0, \ldots, k-1$, $T_j = -\ln w_j$ and $z(t, z_j)$ is the solution of

$$\dot{z} = \lambda z + h(z) \tag{4.6}$$

with initial condition z_j. The sign in equations (4.5b,c) is + if $G(u_j, 1, w_j, z_j) \in H_+(in)$ and $-$ if $G(u_j, 1, w_j, z_j) \notin H_+(in)$. The above sequence gives rise to a k-periodic solution, and every periodic solution has a symmetry conjugate which can be obtained from such a sequence. The 1-periodic solution found in Section 4.2 provides a solution to (4.5) for every integer $k \geq 1$. Hence, to prove uniqueness it suffices to show that (4.5) has a unique solution for every $k \geq 1$.

Initially we assume that $b > 0$ and $z_0 > 0$. In the this case the signs in (4.5) is always +.

Lemma 4.10 *Consider $w > 0$, $z > 0$ such that $w \geq Cz^k$ for some constants k, $C > 0$. Let $z(t)$ be the solution of (4.6) with initial condition z and let $T = -\ln w$. Then*

$$z(T) = w^{-\lambda}\left(z + \mathcal{O}\left(z^{\frac{3}{2}}\right)\right). \tag{4.7}$$

Proof Write

$$z(t) = e^{\lambda t}\left(z + \int_0^t e^{-\lambda s} h(z(s))ds\right).$$

In particular

$$z(T) = w^{-\lambda}\left(z + \int_0^T e^{-\lambda s} h(z(s))ds\right).$$

We show that $\left|\int_0^T h(z(s))ds\right| = \mathcal{O}(z^{\frac{3}{2}})$. To this end we estimate $z(t)$, $0 \leq t \leq T$. Fix $\epsilon > 0$ small and consider the linear equation

$$\dot{z} = \epsilon z. \tag{4.8}$$

Let $\tilde{z}(t)$ be a solution of (4.8) and $z(t)$ a solution of (4.6), both with initial condition $\tilde{z}(0) = z(0) = z$. Suppose $\lambda < \frac{\epsilon}{2}$. If ϵ is small enough then $\lambda + h(z)/z \leq \epsilon$ for all $0 \leq z \leq \epsilon$. It follows that if $\tilde{z}(t) \leq \epsilon$ then $z(s) \leq \tilde{z}(t)$ for all $0 \leq s \leq t$. Since $\tilde{z}(t) = e^{\epsilon t} z$ it follows that, for small enough ϵ,

$$\tilde{z}(T) = w^{-\epsilon} z \leq C^{-\epsilon} z^{1-\epsilon k} \leq 4z^{1-\epsilon k}.$$

We require that $\epsilon k < \frac{1}{2}$. It follows that if $z \leq \frac{1}{4}\epsilon^2$ then $\tilde{z}(T) \leq \epsilon$. Hence, for $0 \leq t \leq T$,

$$z(t) \leq \tilde{z}(T) \leq 4z^{1-\epsilon k}.$$

We now estimate the expression $\left|\int_0^T h(z(s))ds\right|$:

$$\left|\int_0^T h(z(s))ds\right| \le |-\ln C - k\ln z| \max_{s\in[0,T]} |h(z(s)| \\ \le |-\ln C - k\ln z|\mathcal{O}(z^{3(1-\epsilon k)}) \le z^{\frac{3}{2}}.$$

The lemma follows. ∎

In the remainder of this section we deal with the simplified bifurcation equation

$$\begin{aligned} w_{j+1} &= \pm\alpha_1 w_j^c + \alpha_2 z(T_j, z_j) \\ z_{j+1} &= \alpha_3 w_j^c + \alpha_4 z(T_j, z_j), \end{aligned} \tag{4.9}$$

$i = 0, \ldots, k-1$, $w_k \equiv w_0$, $z_k \equiv z_0$. The results for (4.9) easily extend to the general case.

Lemma 4.11 *Suppose that* $\{(u_i, 1, w_i, z_i)\}_{i=0,\ldots,k}$ *are solutions of (4.9). Suppose there exists a* $C > 1$ *such that*

$$\frac{1}{C} \le \frac{z_i}{w_i} \le C \tag{4.10}$$

for some i *and all* λ *sufficiently small. Then* $z_i/w_i \to d/b$ *as* $\lambda \to 0$ *for all* $i = 0, \ldots, k-1$.

Proof Without loss of generality we can assume that $i = 0$. The inequality (4.10) implies that

$$w_0^{c/e} \le w_0^{-\lambda}(Cz_0)^{c/e+\lambda} = w_0^{-\lambda} o(z_0).$$

This estimate, combined with Lemma 4.10, implies that

$$w_1 = \alpha_2 w_0^{-\lambda}(z_0 + o(z_0)), \quad z_1 = \alpha_4 w_0^{-\lambda}(z_0 + o(z_0)).$$

Hence $z_1/w_1 \to d/b$ as $\lambda \to 0$. Using the same argument for $i = 1, \ldots, k-1$, we prove the assertion of the lemma. ∎

Lemma 4.12 *If* $\{(u_i, 1, w_i, z_i)\}_{i=0,\ldots,k}$ *are solutions of (4.9) then there exists* $i \in \{0, \ldots, k-1\}$ *and* $C > 1$ *such that* $1/C \le z_i/w_i \le C$ *for all* λ *small enough.*

Proof Suppose no such C exists. Then we must have $z(T_i, z_i) = \mathcal{O}(w_i^{c/e})$ for $i = 0, \dots, k-1$. This would imply that $w_{i+1} = \mathcal{O}(w_i^{c/e})$, $i = 0, \dots, k-1$, and finally that $w_0 = w_k = \mathcal{O}(w_0^{(c/e)^k})$. Dividing out w_0 we get $1 = \mathcal{O}(w_0^{(c/e)^k - 1})$, which is impossible for $w_0 \approx 0$. ∎

Proof of Theorem 4.9 We assume that $\lambda > 0$. The case $\lambda < 0$ is similar. The Lemmas 4.10, 4.11 and 4.12 imply that we can always use the expansion (4.7). We introduce a new variable $w = \rho z$ and look for (z, ρ) near $(0, b/d)$. Other solutions are impossible by Lemma 4.11. Let $\kappa = \min\{\frac{1}{2}, c/e - 1\}$. We obtain

$$\begin{aligned} \rho_{i+1} z_{i+1} &= \alpha_2 \rho_i^{-\lambda} z_i^{-\lambda}(z_i + \mathcal{O}(z_1^{1+\kappa})) = \Omega_i z_i^{1-\lambda} \\ z_{i+1} &= \alpha_4 \rho_i^{-\lambda} z_i^{-\lambda}(z_i + \mathcal{O}(z_1^{1+\kappa})) = \Delta_i z_i^{1-\lambda}, \end{aligned} \tag{4.11}$$

where Δ_i, Ω_i are continuous functions of (λ, ρ_i, z_i) such that $\Delta_i \to d$ as $\lambda \to 0$ and $z \to 0$. Composing the equation k times we get

$$z_0 = \Delta z_0^{(1-\lambda)^k} = \Delta z_0^{1-k\lambda+\mathcal{O}(\lambda^2)},$$

where Δ is a continuous function of (λ, z_0, ρ_0) such that $\Delta \to d^k$ as $(\lambda, z_0) \to (0, 0)$. Dividing out z_0 and taking logarithms we get $\lim_{\lambda \to 0} z_0^{\lambda} = d$, $i = 0, \dots, k-1$. This is clearly impossible when $d > 1$. We introduce a new coordinate $\tau = z^{\lambda}$ and look for solutions near $(\rho_0, \tau_0) = (b/d, d)$. Equations (4.11) transform to

$$\rho_{i+1} \tau_{i+1}^{\frac{1}{\lambda}} = \tilde{\Omega}_i \tau_i^{\frac{1}{\lambda}-1}, \quad \tau_{i+1}^{\frac{1}{\lambda}} = \tilde{\Delta}_i \tau_i^{\frac{1}{\lambda}-1},$$

where the functions $\tilde{\Omega}_i$, $\tilde{\Delta}_i$ depend smoothly on ρ_i and τ_i. Moreover $\tilde{\Omega}_i = b/d + \mathcal{O}(\lambda)$, $\tilde{\Delta}_i = d + \mathcal{O}(\lambda)$. Composing k times we get

$$\rho_0 \tau_0^{\frac{1}{\lambda}} = \tilde{\Omega} \tau_0^{\frac{(1-\lambda)^k}{\lambda}}, \quad \tau_0^{\frac{1}{\lambda}} = \tilde{\Delta} \tau_0^{\frac{(1-\lambda)^k}{\lambda}},$$

where $\tilde{\Omega} = bd^{k-1} + \mathcal{O}(\lambda)$, $\tilde{\Delta} = d^k + \mathcal{O}(\lambda)$ and $\tilde{\Omega}$, $\tilde{\Delta}$ are smooth functions of τ_0, ρ_0. We substitute the second equation into the first equation and divide out $\tau_0^{\frac{1}{\lambda}}$ in the second equation to obtain

$$\rho_0 = \tilde{\Omega}/\tilde{\Delta}, \qquad 1 = \tilde{\Delta} \tau_0^{-k+\mathcal{O}(\lambda)}.$$

Applying the implicit function theorem we get a unique solution for (τ_0, ρ_0).

We now consider the cases $b > 0$, $z_0 < 0$ and $b < 0$. If $b > 0$ and $z_0 < 0$ then $G^w(u_0, 1, w_0, z_0) < 0$ and $G^z(u_0, 1, w_0, z_0) < 0$, so that we need to use the $-$ sign in (4.5). This way w_1, $z_1 > 0$ and w_j, $z_j > 0$, for $j = 2, \ldots, k-1$. The second equation in (4.11) has the form

$$z_1 = -\Delta_0 z_0; \qquad z_{j+1} = \Delta_j z_j, \quad j \geq 1.$$

Composing we get $z_k = z_0 = -\Delta z_0$, where $\Delta \to d^k$ as $(\lambda, z_0) \to (0, 0)$. This is clearly impossible for $z_0 \neq 0$.

If $b < 0$, $w_0 > 0$ and $z_0 < 0$ then $w_j > 0$, $z_j < 0$ for $j = 1 \geq 1$, so the sign taken in (4.5) is always $+$ and the arguments presented above apply. Suppose $b < 0$, $w_0 > 0$ and $z_0 > 0$. Then the sign in (4.5) is $-$ for $i = 0$ and $+$ for $i = 1, \ldots, k-1$. Proceeding as above we get $z_0 = -\Delta z_0$, where $\Delta \to d^k$ as $(\lambda, z_0) \to (0, 0)$, which is a contradiction. ∎

4.5 Stability

As in Section 4.2, let G be $\gamma^{-1} g$ or $\gamma^{-1} \rho g$, so that $d > 0$. Assume $d < 1$. The map G has the form:

$$\begin{aligned} \hat{u} &= \eta(w^r u, w^c, z(T)) \\ \hat{w} &= \alpha_1 w^c + \alpha_2 z(T) \\ \hat{z} &= \alpha_3 w^c + \alpha_4 z(T). \end{aligned} \tag{4.12}$$

Our periodic solution corresponds to a fixed point of (4.12). Let P denote this fixed point. Recall that the coordinates of P satisfy the following estimates:

$$\begin{aligned} z &\approx (d/b) d^{\frac{1}{\lambda}}, & w &\approx d^{\frac{1}{\lambda}}, \\ T &= -\ln w, & z(T) &= \frac{1}{b} w + o(w). \end{aligned}$$

Let $\tilde{b} = \alpha_2(\lambda; 0)$, $\tilde{d} = \alpha_4(\lambda; 0)$. The matrix of the linearization of (4.12) at P, up to a flat function of λ, has the form

$$\begin{pmatrix} 0 & 0 & * \\ 0 & & \\ 0 & & B \end{pmatrix}, \quad B = \begin{pmatrix} \tilde{b} \frac{dz(T)}{dw} & \tilde{b} \frac{dz(T)}{dz} \\ & \\ \tilde{d} \frac{dz(T)}{dw} & \tilde{d} \frac{dz(T)}{dz} \end{pmatrix}.$$

(Here $dz(T)/dw$ stands for $dz(T)/dw|_{(z_P,w_P)}$, and so on.) We compute that

$$\frac{dz(T)}{dw} = \frac{dz(T)}{dT} \cdot \frac{dT}{dw} = -\frac{1}{w}(\lambda z(T) + h(z(T))) = -\lambda z(T)/w + \text{ flat}(\lambda).$$

Note that $dz(t)/dz = dZ(t)/dZ$, where $\lambda Z = z$. Hence we can use Lemma 4.4 to conclude that

$$\frac{dz(T)}{dz} = \frac{1}{\tau} + \text{ flat}(\lambda).$$

Following the argument of the existence proof we see that

$$\tau = \tilde{d} + \text{ flat}(\lambda), \quad z(T)/w = 1/\tilde{b} + \text{ flat}(\lambda).$$

It follows that up to flat terms in λ the matrix B has the form

$$B = \begin{pmatrix} -\lambda & \tilde{b}/\tilde{d} \\ -\lambda\tilde{d}/\tilde{b} & 1 \end{pmatrix}.$$

Clearly the eigenvalues of B are 0 and $1-\lambda$. It follows that the homoclinic cycle is stable, if $\lambda > 0$. An analogous argument shows that the limit cycle is unstable for $d > 1$.

4.6 The case $c(0) < e(0)$

When $c(0) < e(0)$ we change variables letting $\omega = w^c$. The return map becomes

$$\begin{aligned} \hat{u} &= \eta(\omega^{r/c}u, \omega, z(T)) \\ \hat{\omega}^{1/c} &= \alpha_1\omega + \alpha_2 z(T) \\ \hat{z} &= \alpha_3\omega + \alpha_4 z(T). \end{aligned}$$

The remaining analysis is analogous to the case $c(0) > e(0)$. The coefficient determining the direction of branching (d in the $c(0) > e(0)$ case) is different and is given by

$$D = \frac{\alpha_1(0)\alpha_4(0) - \alpha_2(0)\alpha_3(0)}{\alpha_1(0)}.$$

In particular we must assume that $\alpha_1(0) \neq 0$ and $D \neq \pm 1$. Rather than repeating the already presented analysis we present a heuristic argument showing the existence of a periodic solution. Consider the simplified bifurcation equation

$$\omega^{1/c} = \alpha_1\omega + \alpha_2 z(T), \quad z = \alpha_3\omega + \alpha_4 z(T).$$

We can solve the first equation for ω obtaining $\omega \approx \frac{-\alpha_2(0)}{\alpha_1(0)} z(T)$. Substituting this expression into the second equation we obtain $z \approx Dz(T)$. Using Lemma 4.6, we obtain $z^\lambda \approx D$.

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