Numerical Computation of Solitary Waves on Infinite Cylinders

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Abstract

The numerical computation of solitary waves to semilinear elliptic equations in infinite cylinders is investigated. Rather than solving on the infinite cylinder, the equation is approximated by a boundary-value problem on a finite cylinder. Convergence and stability results for this algorithm are given. In addition, it is shown that Galerkin approximations can be used to calculate solitary waves for the elliptic problem on the finite cylinder. The theoretical predictions are compared with numerical computations. In particular, post buckling of an infinitely long cylindrical shell under axial compression is considered; it is shown numerically that, for a fixed spatial truncation, the error in the truncation on the length of the cylinder scales in accordance with the theoretical predictions.

Keywords: solitary wave, boundary-value problem, elliptic equation

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1 Introduction

The numerical computation of solitary-wave solutions to elliptic systems

$$u_{xx} + \Delta_y u + g(y, u, u_x, \nabla_y u) = 0, \quad (x, y) \in \mathrm{R} \times \Omega, \tag{1.1}$$

in infinite cylinders $\mathrm{R} \times \Omega$ is investigated. Here, $u \in \mathrm{R}^m$, and Ω is an open and bounded subset of R^n with Lipschitz boundary. Appropriate boundary conditions

$$R((u, u_x, \nabla_y u)|_{\mathrm{R}\times\partial\Omega}) = 0 \tag{1.2}$$

on $\mathrm{R} \times \partial\Omega$ should be added. Solitary waves are solutions $h(x, y)$ satisfying

$$\lim_{x\to\pm\infty} h(x, y) = p_\pm(y)$$

uniformly for $y \in \Omega$. In applications, they frequently arise as travelling waves $h(x - ct, y)$ for parabolic equations

$$u_t = u_{xx} + \Delta_y u + g(y, u, u_x, \nabla_y u), \quad (x, y) \in \mathrm{R} \times \Omega. \tag{1.3}$$

These applications include problems in structural mechanics such as shells and struts, chemical kinetics, combustion, and nerve impulses; see, for instance, [32] and the comprehensive bibliography there. Analytically, the existence of solitary-wave solutions exhibiting non-trivial structure in the direction of the cross-section is still a largely open problem. Existence has been proved in many cases for small solutions using center-manifold theory [19, 25]. In special cases, it may be possible to exploit maximum principles [1, 2, 17] and variational structure [27, 28]. Another approach establishing existence of front solutions uses topological methods [10, 12].

Suppose that $h(x, y)$ is a solitary wave which satisfies (1.1–1.2). In order to calculate h numerically, the problem on the infinite cylinder $\mathrm{R}\times\Omega$ has to be approximated by a suitable system

$$\begin{aligned} u_{xx} + \Delta_y u + g(y, u, u_x, \nabla_y u) &= 0, \quad (x, y) \in (T_-, T_+) \times \Omega \\ R((u, u_x, \nabla_y u)|_{[T_-,T_+]\times\partial\Omega}) &= 0, \end{aligned} \tag{1.4}$$

posed on a finite cylinder. Here, we have to specify appropriate conditions

$$\begin{aligned} R_-((u, u_x, \nabla_y u)|_{\{T_-\}\times\Omega}) &= 0, \\ R_+((u, u_x, \nabla_y u)|_{\{T_+\}\times\Omega}) &= 0, \end{aligned} \tag{1.5}$$

at the boundaries induced by the truncation of the cylinder axis. The issue is then to determine whether equation (1.4–1.5) has a unique solution close to the solitary wave h, and if it does, to derive estimates for the error caused by the truncation.

In this article, we give sufficient conditions on the equation and the boundary conditions (1.5) such that the aforementioned algorithm works. Boundary conditions which satisfy these conditions are called *admissible*. One implication of our assumptions is that the solitary wave $h(x,y)$ converges exponentially towards $p_\pm(y)$ as $|x| \to \infty$ uniformly in $y \in \Omega$. The difference of the solution h_T of (1.4–1.5) and the solitary wave h can then be estimated by

$$|h - h_T| \le C(|R_-(h|_{\{T_-\}\times\Omega})| + |R_+(h|_{\{T_+\}\times\Omega})|),$$

in appropriate norms, where the positive constant C does not depend on T_- and T_+. Here, the right-hand side converges to zero exponentially as $|T_-|, T_+ \to \infty$.

In order to prove this result, we interpret the variable x as time, and write (1.1) as a first-order system

$$\begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} v \\ -\Delta_y u - g(y,u,v,\nabla_y u) \end{pmatrix}. \tag{1.6}$$

Here, for each fixed $x \in \mathrm{R}$, $(u,v)(x)$ is a function of $y \in \Omega$ contained in some function space depending on the boundary conditions on $\partial\Omega$. A solitary wave of (1.1) corresponds to a homoclinic or heteroclinic solution of (1.6) which connects the equilibria $p_-(y)$ and $p_+(y)$, that is, we have

$$\lim_{x\to\pm\infty} (h(x), h_x(x)) \to (p_\pm, 0) \tag{1.7}$$

in the underlying function space. We then investigate the truncated boundary-value problem by replacing (1.7) by a condition of the form

$$R_-((u,v)(T_-)) = 0, \qquad R_+((u,v)(T_+)) = 0. \tag{1.8}$$

The key for solving the resulting boundary-value problem are exponential dichotomies for the linearization

$$\begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ -\Delta_y - D_u g - D_{\nabla_y u} g\, \nabla_y & D_{u_x} g \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{1.9}$$

of (1.6) about the solitary wave $(h(x), h_x(x))$. Here, derivatives of g are evaluated at $(y, h, h_x, \nabla_y h)$. Exponential dichotomies are projections onto x-dependent stable and unstable subspaces, say $E^s(x)$ and $E^u(x)$, such that solutions $(u,v)(x)$ of (1.9) associated with initial values $(u,v)(x_0)$ in the stable space $E^s(x_0)$ exist for $x > x_0$ and decay exponentially for $x \to \infty$. In contrast, solutions $(u,v)(x)$ associated with initial values $(u,v)(x_0)$ in the unstable space $E^u(x_0)$ satisfy (1.9) in backward x-direction $x < x_0$ and decay exponentially for decreasing x. In the context of elliptic equations, the stable and unstable spaces are both infinite-dimensional. Existence of exponential dichotomies for ordinary, parabolic or functional differential equations is well known. For elliptic equations, existence has recently been proved in [26] using a novel functional-analytic approach. The results in this latter article then allow us to solve the truncated boundary-value problem and to derive the aforementioned error estimate.

It remains to actually solve the truncated boundary-value problem. There are two different ways of accomplishing this task. Firstly, we concentrate on the elliptic formulation. Consider, for instance, equation (1.1) with Dirichlet boundary conditions, that is,

$$\begin{aligned} u_{xx} + \Delta_y u + g(y, u, u_x, \nabla_y u) &= 0, \qquad (x, y) \in \mathrm{R} \times \Omega \\ u|_{\mathrm{R}\times\partial\Omega} &= 0, \end{aligned}$$

and assume that the solitary wave converges to zero as $|x|$ tends to infinity. We may then want to take Dirichlet boundary conditions for the artificial conditions (1.5) which results in the truncated problem

$$\begin{aligned} u_{xx} + \Delta_y u + g(y, u, u_x, \nabla_y u) &= 0, \qquad (x, y) \in (T_-, T_+) \times \Omega \\ u|_{\partial((T_-,T_+)\times\Omega)} &= 0. \end{aligned}$$

This system can now be discretized using finite differences or finite elements. Of course, the same procedure works for Neumann or periodic boundary conditions provided they are admissible. Secondly, we could take a dynamical-systems point of view and consider the first-order system (1.6-1.8). We now discretize only in the cross-section Ω and obtain a large system of ODEs

$$\begin{aligned} \begin{pmatrix} u_x \\ v_x \end{pmatrix} &= Q_n \begin{pmatrix} v \\ -\Delta_y u - g(y, u, v, \nabla_y u) \end{pmatrix}, \\ 0 &= R_-((u, v)(T_-)), \\ 0 &= R_+((u, v)(T_+)), \end{aligned}$$

defined on $R(Q_n)$ where the Galerkin projection Q_n projects the function space in Ω onto a finite-dimensional subspace. ODE codes such as HOMCONT, see [7, 9], can then be used to solve the resulting boundary-value problem.

Often, elliptic equations have an additional reflection symmetry. For instance, consider the fourth-order equation

$$u_{xxxx} + \Delta_y^2 u + g(y, u, u_{xx}, \Delta_y u) = 0, \qquad (x, y) \in \mathrm{R} \times \Omega, \tag{1.10}$$

which is included in our general set-up. The Z_2-symmetry $u(x, y) \mapsto u(-x, y)$ leaves (1.10) invariant. This symmetry manifests itself as a time-reversibility $S : (u, v_1, v_2, v_3) \mapsto (u, -v_1, v_2, -v_3)$ for the associated dynamical system

$$(u, v_1, v_2, v_3)_x = (v_1, v_2, v_3, -\Delta_y^2 u - g(y, u, v_2, \Delta_y u)).$$

It is shown that the algorithm given above can be adapted to this situation. We also remark that our results apply to parabolic equations

$$u_t = \Delta u + g(u, \nabla u), \qquad x \in \Omega,$$

with $\Omega \subset \mathrm{R}^n$ bounded and open. Here, we are interested in the computation of homoclinic or heteroclinic solutions $h(t,x)$ which satisfy $\lim_{t\to\pm\infty} h(t,x) = p_\pm(x)$.

For ordinary differential equations, well-posedness of the truncated problem has been investigated by Beyn [4], and Doedel and Friedman [11] for very general boundary conditions. In addition, error estimates have been derived in these articles. Hagstrom and Keller [15, 16] considered elliptic problems of the form (1.1) assuming that (1.9) has an exponential dichotomy. They investigated the so-called asymptotic boundary conditions. These conditions select precisely those solutions converging to $p_\pm(y)$ as $x \to \pm\infty$. In particular, the solution of the truncated problem coincides with the true wave h on the infinite cylinder. The actual calculation of the asymptotic boundary conditions, however, involves again certain approximations which were not investigated in [15, 16].

As a concrete application, we consider the post buckling of an infinitely long cylindrical shell under axial compression as modeled by the von Kármán–Donnell equations. In [22, 23] and [24] solitary-waves were computed as solutions representing localized buckling patterns and it was shown that these solutions provide a good approximation to the localized buckling pattern observed in experiments on long shells. The numerical procedure involved the reduction to a truncated boundary-value problem and its discretization using Galerkin approximation as discussed above. Here, we show numerically that, for a fixed spatial truncation, the error in the truncation on the length of the cylinder scales in accordance with our theoretical predictions.

This paper is organized as follows. Section 2 contains the general set-up and the main results. We summarize the results about exponential dichotomies from [26] in Section 3. The theorems on Galerkin approximations and the truncated boundary-value problem are proved in Sections 4–6. We show in Section 7 that our results apply to semilinear elliptic equations. Numerical simulations and a comparison of the numerical and theoretical error are presented in Section 8. Finally, the application to the von Kármán–Donnell equation is given in Section 9.

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2 Main Results

2.1 The Setting

Assume that A is a closed operator defined on a reflexive Banach space X with dense domain $D(A)$.

Let $B \in \mathcal{L}(X)$ be any bounded operator. We say that A and B commute if $Bu \in D(A)$ for any $u \in D(A)$ and $AB = BA$ on $D(A)$.

Hypothesis (A1) *Suppose that there is a constant C such that*

$$\|(A - \mathrm{i}\lambda)^{-1}\|_{\mathcal{L}(X)} \leq \frac{C}{1 + |\lambda|}$$

for all $\lambda \in \mathrm{R}$. Assume that there is a projection $\hat{P}_- \in \mathcal{L}(X)$ with the following properties: A^{-1} and $\hat{P}_-$ commute, and there exists a $\delta > 0$ such that $\operatorname{Re}\lambda < -\delta$ for any $\lambda \in \operatorname{spec}(A\hat{P}_-)$ and $\operatorname{Re}\lambda > \delta$ for any $\lambda \in \operatorname{spec}(A(\mathrm{id} - \hat{P}_-))$.

Throughout, C denotes various different constants all independent of T_- and T_+.

Sufficient conditions for the existence of the projection $\hat{P}_-$, which is sometimes referred to as the Calderon projector [6], have been given in [5, 13]. We also refer to the explicit construction of the projections for semilinear elliptic equations in [26, Section 6].

Define $\hat{P}_+ = \mathrm{id} - \hat{P}_-$ and $A_- = -\hat{P}_- A$, $A_+ = \hat{P}_+ A$, and let $X_- = R(\hat{P}_-)$ and $X_+ = R(\hat{P}_+)$. Here, range and kernel of an operator L are denoted by $R(L)$ and $N(L)$, respectively. By Hypothesis (A1), the operators A_- and A_+ are sectorial with their spectrum contained in the right half-plane. Therefore, for $\alpha \geq 0$, we can define the interpolation spaces $X_+^\alpha = D(A_+^\alpha)$ and $X_-^\alpha = D(A_-^\alpha)$, see [18]. Finally, we set $X^\alpha = X_+^\alpha \times X_-^\alpha$. We denote the norm in X^α by $|\cdot|_\alpha$ and the operator norm in $\mathcal{L}(X^\alpha)$ by $\|\cdot\|_\alpha$. The projection $\hat{P}_-$ is then in $\mathcal{L}(X^\alpha)$ for any $\alpha < 1$.

In addition, we assume that A has compact resolvent.

Hypothesis (A2) *The operator $A^{-1} \in \mathcal{L}(X)$ is compact.*

In the following, we consider the abstract evolution equation

$$\dot{u} = Au + f(u, \mu), \quad (u, \mu) \in X^\alpha \times \mathrm{R} \tag{2.1}$$

for some fixed $\alpha \in [0, 1)$ and for $f \in C^2(X^\alpha \times \mathrm{R}, X)$. We say that $u(t)$ is a solution of (2.1) on the interval $[0, T)$ if

$$u \in C^1((0, T), X) \cap C^0((0, T), D(A)) \cap C^0([0, T), X^\alpha)$$

and $u(t)$ satisfies (2.1) in X for $t \in (0,T)$. We assume the existence of a hyperbolic equilibrium and of a homoclinic orbit for $\mu = 0$.

Hypothesis (H1) *Equation (2.1) has a hyperbolic equilibrium $p_0 \in D(A)$ for $\mu = 0$. In particular, $A + D_u f(p_0, 0)$ meets Hypothesis (A1).*

Hypothesis (H2) *Let $h(t) \in C^1(\mathrm{R}, X^\alpha) \cap C^0(\mathrm{R}, X^1)$ be a homoclinic solution of (2.1) for $\mu = 0$ with $h(t) \to p_0$ as $|t| \to \infty$. We assume that $h(t)$ is non-degenerate, that is, $\dot{h}(t)$ is the only bounded solution, up to constant multiples, of the variational equation*

$$\dot{v} = Av + D_u f(h(t), 0)v \tag{2.2}$$

about $h(t)$.

Next, we introduce the adjoint variational equation

$$\dot{v} = -(A^* + D_u f(h(t), 0)^*)v, \tag{2.3}$$

about the homoclinic solution $h(t)$. To describe the asymptotic behavior of solutions of (2.2) and (2.3), we have to assume forward and backward uniqueness.

Hypothesis (A3) *The only bounded solution of (2.2) and (2.3) on R^+ or R^- with $v(0) = 0$ is the trivial solution $v(t) = 0$.*

Hypotheses (H2) and (A3) imply that the adjoint equation (2.3) has a unique, up to scalar multiples, bounded solution $\psi(t)$ on R. Finally, we assume that the Melnikov integral associated with $h(t)$ does not vanish.

Hypothesis (H3) $M := \int_{-\infty}^{\infty} \langle \psi(t), D_\mu f(h(t), 0) \rangle \, dt \neq 0.$

2.2 The Galerkin Approximation

First, we show persistence of the homoclinic orbit $h(t)$ under finite-dimensional Galerkin approximations of equation (2.1). We may think of a Galerkin approximation as a family of projections denoted by $Q_\rho \in \mathcal{L}(X)$ for $\rho > 0$. Here, $Q_0 = \mathrm{id}$, while Q_ρ will have finite-dimensional range for $\rho > 0$ and approximates the identity in a weak sense.

Hypothesis (Q)

(i) A commutes with Q_ρ.

(ii) The norms $\|Q_\rho\|_{\mathcal{L}(X)} \leq C$ *are bounded uniformly in* ρ.

(iii) For any $u \in X$, *we have* $|Q_\rho u - u|_0 \to 0$ *as* $\rho \to 0$.

It is a consequence of Hypothesis (Q)(i) that $A^\alpha Q_\rho = Q_\rho A^\alpha$. Therefore, we have $Q_\rho \in \mathcal{L}(X^\alpha)$ and $\|Q_\rho\|_{\mathcal{L}(X^\alpha)} \leq C$ independently of $\rho > 0$. Furthermore, $|Q_\rho u - u|_\alpha \to 0$ as $\rho \to 0$ for any $u \in X^\alpha$. In order to obtain uniform convergence of the Galerkin approximation, we assume compactness of the nonlinearity f.

Hypothesis (C) *If* $Q_\rho \neq \mathrm{id}$ *for some* $\rho > 0$, *we assume that* $f : X^\alpha \times \mathrm{R} \to X$ *is a compact map.*

The next theorem shows the persistence of the equilibrium and the homoclinic orbit under the Galerkin approximation

$$\dot{u} = Au + Q_\rho f(u, \mu), \quad (u, \mu) \in X^\alpha \times \mathrm{R}, \tag{2.4}$$

of equation (2.1).

We emphasize that the subspaces $R(Q_\rho)$ and $N(Q_\rho)$ are both invariant under equation (2.4). For initial data $u_0 \in (\mathrm{id} - Q_\rho)X^\alpha$, equation (2.4) reduces to the linear equation $\dot{u} = Au$, which has no bounded solution on R except $u = 0$.

We also remark that the norms on $Q_\rho X^\alpha$ and $Q_\rho X$ are equivalent, but the equivalence constants tend to infinity as $\rho \to 0$. Therefore, estimates which are uniform with respect to ρ can only be expected in the X^α-norm.

Theorem 1 *Assume that Hypotheses (A1)–(A3), (H1)–(H3), (C), and (Q) are satisfied. There are then positive numbers* ρ_0, δ_0, *and* C *such that the following is true for any* $0 \leq \rho < \rho_0$ *and* $|\mu| < \delta_0$.

(i) Equation (2.4) has a hyperbolic equilibrium $p_\rho(\mu) \in R(Q_\rho)$ *with* $p_0(0) = p_0$ *and*

$$|p_\rho(\mu) - p_0|_\alpha \leq C(|(\mathrm{id} - Q_\rho)p_0|_\alpha + |\mu|).$$

(ii) For every ρ, *there exists a* μ_ρ *such that equation (2.4) has a non-degenerate homoclinic orbit* $h_\rho(t) \in Q_\rho X^\alpha$ *with* $h_\rho(t) \to p_\rho(\mu_\rho)$ *as* $|t| \to \infty$, *and*

$$|\mu_\rho| + \sup_{t \in \mathrm{R}} |h_\rho(t) - h(t)|_\alpha \leq C \sup_{t \in \mathrm{R}} |(\mathrm{id} - Q_\rho)h(t)|_\alpha.$$

(iii) Besides $p_\rho(\mu)$ *and* $h_\rho(t)$, *there are no other equilibria or homoclinic solutions of (2.4) in the open set* $\{(u, \mu) \in X^\alpha \times \mathrm{R};\ |\mu| + \inf_{t \in \mathrm{R}} |u - h(t)|_\alpha < \delta_0\}$.

We denote the spectral projections associated with $A + D_u f(p_0, 0)$ and $A + Q_\rho D_u f(p_\rho(\mu), \mu)$ in X^α by $P_\pm$ and $P_{\pm,\rho}(\mu)$, respectively. In particular, P_- and $P_{-,\rho}(\mu)$ project onto the stable eigenspaces corresponding to eigenvalues with negative real part of $A + D_u f(p_0, 0)$ and $A + Q_\rho D_u f(p_\rho(\mu), \mu)$, respectively.

2.3 The Truncated Boundary-Value Problem

Here, we investigate the numerical computation of the homoclinic orbits h_ρ of the Galerkin approximation. The approach most commonly used consists of truncating the infinite interval R to a finite interval $[T_-, T_+]$ for some $T_- < 0 < T_+$ and imposing boundary conditions at the end points $t = T_-$ and $t = T_+$. The truncated boundary-value problem is given by

$$\begin{pmatrix} \dot{u} - Au - Q_\rho f(u, \mu) \\ R_\rho(u(T_+), u(T_-), \mu) \\ J_{T,\rho}(u, \mu) \end{pmatrix} = 0 \tag{2.5}$$

for $t \in T = (T_-, T_+)$. Here, $J_{T,\rho}$ denotes a phase condition and R_ρ encodes the boundary conditions. They have to satisfy the following conditions.

Hypothesis (T1)

(i) $J_{T,\rho} : C^0(T, X^\alpha) \times \mathrm{R} \to \mathrm{R}$ *is of class* C^2*, and* $J_{T,\rho}(h_\rho, \mu_\rho) \to 0$ *as* $|T_-|$*,* $T_+ \to \infty$*. Furthermore, there is a constant* $d_0 > 0$ *independent of* T_-*,* T_+ *and* ρ *such that* $D_u J_{T,\rho}(h_\rho, \mu_\rho)\dot{h}_\rho \geq d_0 > 0$ *for all* $|T_-|$*,* T_+ *sufficiently large. Finally,* $D_u J_{T,\rho}(u, \mu)$ *and* $D_u^2 J_{T,\rho}(u, \mu)$ *are bounded in a ball in* $C^0(T, X^\alpha) \times \mathrm{R}$ *of fixed radius centered at* (h_ρ, μ_ρ) *uniformly in* T_-*,* T_+*, and* ρ*.*

(ii) We have $R_\rho \in C^2(X^\alpha \times X^\alpha \times \mathrm{R}, X^\alpha)$ *such that* DR_ρ *and* $D^2 R_\rho$ *are bounded in a small ball centered at* $(p_\rho(\mu_\rho), p_\rho(\mu_\rho), \mu_\rho)$ *in* $X^\alpha \times X^\alpha \times \mathrm{R}$ *uniformly in* ρ*. Furthermore,*

$$D_{u_+,u_-} R_\rho(p_\rho(\mu_\rho), p_\rho(\mu_\rho), \mu_\rho)|_{R(P_{+,\rho}(\mu_\rho)) \times R(P_{-,\rho}(\mu_\rho))}$$

is invertible, and the inverse is bounded uniformly in ρ*.*

Note that $\dot{h}(\cdot)$ and $\dot{h}_\rho(\cdot)$ are contained in $C^0(T, X^\alpha)$. Therefore, the condition on J in (T1)(i) makes sense.

Remark 2.1 *The boundary conditions are often separated, that is, given by*

$$R_\rho(u_+, u_-, \mu) = (R_{+,\rho}(u_+, \mu), R_{-,\rho}(u_-, \mu)) \in R(P_{+,\rho}(\mu_\rho)) \times R(P_{-,\rho}(\mu_\rho)) = X^\alpha.$$

If the operators $D_u R_{\pm,\rho}(p_\rho(\mu_\rho),\mu_\rho)|_{R(P_{\pm,\rho}(\mu_\rho))}$ *are invertible, and the inverses are bounded uniformly in* ρ*, then the invertibility condition in Hypothesis (T1)(ii) is also satisfied.*

We have the following theorem.

Theorem 2 *Assume that (A1)–(A3), (H1)–(H3), (C), (Q) and (T1) are met. There exist positive numbers* ρ_0*,* η *and* C *such that for all sufficiently large intervals* T *the following is true. For any* $\rho \in [0,\rho_0)$*, the boundary-value problem (2.5) has a unique solution* $(\bar{h}_\rho(t),\bar{\mu}_\rho)$ *in the tube*

$$\{(u,\mu) \in C^0([T_-,T_+],X^\alpha) \times \mathrm{R};\ |\mu| + \sup_{t\in[T_-,T_+]} |u(t)-h(t)|_\alpha \le \eta\},$$

and

$$|\bar{\mu}_\rho - \mu_\rho| + \sup_{t\in[T_-,T_+]} |\bar{h}_\rho(t) - h_\rho(t+\gamma_{T,\rho})|_\alpha \quad \le \quad C|R_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho)|_\alpha$$

for an appropriate small number $\gamma_{T,\rho}$*.*

Combining Theorems 1 and 2, and exploiting Hypothesis (T1), we obtain the following corollary.

Corollary 1 *Under the assumptions of Theorem 2, we have the estimate*

$$|\bar{\mu}_\rho| + \sup_{t\in[T_-,T_+]} |\bar{h}_\rho(t) - h(t)|_\alpha| \quad \le \quad C\Big(|R_\rho(h(T_+),h(T_-),0)|_\alpha + \sup_{t\in\mathrm{R}} |(\mathrm{id} - Q_\rho)h(t)|_\alpha\Big)$$

for the difference of the true homoclinic orbit h *and the numerical approximation* $\bar{h}$ *obtained by solving the boundary-value problem on a finite interval for the Galerkin approximation of (2.1).*

The error estimate can be made more explicit.

Corollary 2 *Under the assumptions of Theorem 2, we have the estimate*

$$|\bar{\mu}_\rho| + \sup_{t\in[T_-,T_+]} |\bar{h}_\rho(t) - h(t)|_\alpha \le C(e^{-\lambda^s T_+} + e^{\lambda^u T_-} + \sup_{t\in\mathrm{R}} |(\mathrm{id} - Q_\rho)h(t)|_\alpha),$$

where the constants λ^s *and* λ^u *are chosen such that* $\lambda \notin \mathrm{spec}(A + D_u f(p_0,0))$ *for any* $\lambda \in \mathrm{C}$ *with* $\mathrm{Re}\,\lambda \in [-\lambda^s,\lambda^u]$*.*

We point out that the case $Q_\rho = \mathrm{id}$ for all ρ is included in the analysis. It corresponds to truncating equation (2.1) directly without going to a finite-dimensional approximation. A more analytical consequence of Theorem 2 is the existence of periodic solutions with large period near the homoclinic orbit h.

Corollary 3 *Assume that (A1)–(A3) and (H1)–(H3) are met. There is then a constant* $\tau_* > 0$ *such that (2.1) has a periodic orbit* (u_τ, μ_τ) *with minimal period* τ *for any* $\tau > \tau_*$*. Furthermore,*

$$|\mu_\tau| + \sup_{t\in[-\frac{1}{2}\tau,\frac{1}{2}\tau]} |u_\tau(t) - h(t)|_\alpha \le C(e^{-\lambda^s\tau} + e^{\lambda^u\tau}),$$

where the constants λ^s *and* λ^u *are as in Corollary 2.*

Proof. Consider the phase condition $J_T(u) := \langle\varphi, u(0)\rangle$ where $\varphi \in (X^\alpha)^*$ is chosen such that $\langle\varphi, \dot h(0)\rangle = 1$. The boundary condition is $R(u_+, u_-, \mu) = u_+ - u_-$. Since $R(P_+) \times R(P_-) = X^\alpha$, Hypothesis (T1) is satisfied, and we can apply Theorem 2 with $Q_\rho = \mathrm{id}$ for all ρ. ■

This corollary has been proved for ordinary differential equations in [3] and [21]. The proof given in [21] also covers parabolic and functional-differential equations. Our contribution is the extension to elliptic equations.

2.4 The Algorithm in Practice

In practice, the Galerkin approximation is considered on the finite-dimensional space $R(Q_\rho)$, that is,

$$\dot q = Aq + Q_\rho f(q,\mu), \qquad (q,\mu) \in R(Q_\rho) \times \mathrm{R}. \tag{2.6}$$

If X is a Hilbert space, the phase condition may be chosen according to

$$J_{T,\rho}(q,\mu) = \int_{T_-}^{T_+} \langle \dot h_\rho(t), q(t) - h_\rho(t)\rangle_X \, dt. \tag{2.7}$$

For the boundary conditions, we may take, for instance, the projection boundary conditions which are defined by

$$\begin{aligned} R_{+,\rho}(q(T_+),\mu) &= Q_{+,\rho}(\mu)(q(T_+) - p_\rho(\mu)), \\ R_{-,\rho}(q(T_-),\mu) &= Q_{-,\rho}(\mu)(q(T_-) - p_\rho(\mu)), \end{aligned} \tag{2.8}$$

where $Q_{+,\rho}(\mu)$ and $Q_{+,\rho}(\mu)$ are the unstable and stable spectral projections in $R(Q_\rho)$ of the operator $(A + Q_\rho D_u f(p_\rho(\mu),\mu))|_{R(Q_\rho)}$. We have the following result for the finite-dimensional boundary-value problem on $R(Q_\rho)$ described above.

Theorem 3 *Assume that (A1)–(A3), (H1)–(H3) and (C) are met where* X *is a Hilbert space. There exist positive numbers* ρ_0*,* η*, and* C *such that for all sufficiently large intervals*

T the following is true. For any $\rho \in [0, \rho_0)$, the boundary-value problem (2.6–2.8) on $R(Q_\rho)$ has a unique solution $(\bar{h}_\rho(t), \bar{\mu}_\rho)$ in the tube

$$\{(q, \mu) \in C^0([T_-, T_+], R(Q_\rho)) \times \mathrm{R};\ |\mu| + \sup_{t \in [T_-, T_+]} |q(t) - h(t)|_\alpha \leq \eta\},$$

and

$$|\bar{\mu}_\rho| + \sup_{t \in [T_-, T_+]} |\bar{h}_\rho(t) - h(t)|_\alpha \leq C(e^{-2\lambda^s T_+} + e^{2\lambda^u T_-} + \sup_{t \in \mathrm{R}} |(\mathrm{id} - Q_\rho) h(t)|_\alpha),$$

for numbers λ^s and λ^u as in Corollary 2.

We point out that super-convergence in the parameter occurs. Indeed, following the proof given in [29], we have

$$|\bar{\mu}_\rho| \leq C(e^{-(2\lambda^s + \lambda^u)T_+} + e^{(2\lambda^u + \lambda^s)T_-} + \sup_{t \in \mathrm{R}} |(\mathrm{id} - Q_\rho) h(t)|_\alpha).$$

2.5 Reversible Systems

In applications, elliptic equations are often time-reversible. Here, we account for this property, and adapt the algorithms described above to reversible systems. Consider equation (2.1)

$$\dot{u} = Au + f(u), \quad u \in X^\alpha. \tag{2.9}$$

Time-reversibility is encoded in the following hypothesis.

Hypothesis (R) *Suppose that $S \in \mathcal{L}(X)$ is a bounded operator such that*

(i) S anti-commutes with A and f, that is, $SA = -AS$ and $f(Su) = -Sf(u)$ on $D(A)$.

(ii) $S^2 = \mathrm{id}$.

(iii) S commutes with Q_ρ for all ρ.

We remark that $S \in \mathcal{L}(X^\alpha)$ on account of (R)(i). Finally, we assume that the homoclinic solution $h(t)$ is symmetric and a certain transversality condition is satisfied.

Hypothesis (H4)

(i) $Sh(0) = h(0)$, that is, $h(0) \in \mathrm{Fix}(S)$.

(ii) $\mathrm{Fix}(S) \oplus R(\phi_+^s(0, 0)) = X^\alpha$.

Here, $\phi_+^s(t,\tau)$ denotes the stable evolution of the variational equation (2.2) about $h(t)$; see Theorem 5 below. We then solve the boundary-value problem

$$\dot{u} = Au + Q_\rho f(u), \qquad (\mathrm{id} - S)u(0) = 0, \qquad R_{+,\rho}(u(T_+)) = 0, \tag{2.10}$$

on the interval $[0, T_+]$.

Hypothesis (T2) *Suppose that $\hat{X}_\rho$ are closed subspaces of X^α. Assume that $R_{+,\rho} \in C^2(X^\alpha \times \mathrm{R}, \hat{X}_\rho)$ such that DR_ρ and $D^2 R_\rho$ are bounded in a small ball centered at $(p_\rho(\mu_\rho), \mu_\rho)$ in $X^\alpha \times \mathrm{R}$ uniformly in ρ. Furthermore, $D_u R_{+,\rho}(p_\rho(\mu_\rho), \mu_\rho)|_{R(P_{+,\rho}(\mu_\rho))}$ is invertible, and the inverse is bounded uniformly in ρ.*

Theorem 4 *Assume that (A1)–(A3), (H1)–(H2), (H4), (C), (Q) and (R) are met. Suppose that $R_{+,\rho}$ satisfies (T2). The boundary-value problem (2.10) has then a unique solution $\bar{h}_\rho$ for all T_+ sufficiently large and ρ small enough. Furthermore, $\bar{h}_\rho(0) \in \mathrm{Fix}(S)$ is symmetric, and*

$$\sup_{t\in[0,T_+]} |\bar{h}_\rho(t) - h(t)|_\alpha \;\; \leq \;\; C(|R_{+,\rho}(h(T_+))|_\alpha + \sup_{t\in\mathrm{R}} |(\mathrm{id} - Q_\rho)h(t)|_\alpha).$$

The statements of Theorem 1 and 3 are also true for (2.10) if adapted appropriately.

Corollary 4 *Assume that (A1)–(A3), (H1)–(H2), and (R)(i) and (ii) are met. There is then a constant $\tau_* > 0$ such that (2.9) has a periodic orbit u_τ with minimal period τ for any $\tau > \tau_*$. Furthermore,*

$$\sup_{t\in[-\frac{1}{2}\tau,\frac{1}{2}\tau]} |u_\tau(t) - h(t)|_\alpha \leq Ce^{-\lambda^s\tau},$$

where the constant λ^s is as in Corollary 2.

Proof. We apply Theorem 4 with $R_+(u) = \frac{1}{2}(\mathrm{id} - S)u$ and $\hat{X} = R(\mathrm{id} - S)$. By Dunford-Taylor calculus and (R)(i), we have $SP_+ = P_- S$. Moreover, due to (R)(ii), $v \in R(\mathrm{id} - S)$ implies $Sv = -v$. Using these facts, it is then straightforward to show that $P_+(\mathrm{id} - S)|_{R(P_+)} = \mathrm{id}_{R(P_+)}$ and $(\mathrm{id} - S)P_+|_{R(\mathrm{id} - S)} = \mathrm{id}_{R(\mathrm{id} - S)}$. Hence, Hypothesis (T2) is satisfied. It follows as in [31] that the solution of this boundary-value problem is periodic. Note that $\lambda^s = \lambda^u$ due to reversibility. ∎

2.6 Computation of Heteroclinic Orbits

We emphasize that the results presented thus far also apply to heteroclinic orbits, that is, solutions connecting two different equilibria $p_\pm$ as $t \to \pm\infty$. Here, we briefly outline the necessary changes. Suppose that $p_\pm$ are hyperbolic equilibria of (2.1) which satisfy Hypothesis (H1). Furthermore, assume that $h(t)$ satisfies (H2) but with $\lim_{t\to\pm\infty} h(t) = p_\pm$. In particular, (H2) implies that the heteroclinic orbit $h(t)$ is isolated. Next, we assume that (A1)–(A3) are met. As a consequence of (H2), (A3) and [26, Corollary 1], the adjoint variational equation (2.3)

$$\dot{v} = -(A^* + D_u f(h(t), 0)^*)v$$

about the heteroclinic orbit $h(t)$ has only finitely many, linearly independent bounded solutions $\psi_j(t)$ for $j = 1, ..., m$ on R. Hypothesis (H3) is then replaced by the following assumption.

Hypothesis (H5) *Let $\mu \in \mathrm{R}^m$ and assume that the $m \times m$ matrix M with entries $M_{ij} := \int_{-\infty}^{\infty} \langle \psi_i(t), D_{\mu_j} f(h(t), 0) \rangle \, dt$ is invertible.*

Note that Hypothesis (H5) is automatically met if $m = 0$, that is, if the heteroclinic orbit is transversely constructed.

With Hypothesis (H3) replaced by (H5), Theorem 1 remains true. Let $p_{\pm,\rho}(\mu)$ denote the perturbed equilibria for (2.4). We denote the spectral projections of $A + D_u f(p_-, 0)$ and $A + Q_\rho D_u f(p_{-,\rho}(\mu), \mu)$ onto the stable eigenspaces corresponding to eigenvalues with negative real part by P_- and $P_{-,\rho}(\mu)$, respectively. Similarly, P_+ and $P_{+,\rho}(\mu)$ are the spectral projections of $A + D_u f(p_+, 0)$ and $A + Q_\rho D_u f(p_{+,\rho}(\mu), \mu)$, respectively, onto the eigenspaces corresponding to eigenvalues with positive real part.

Suppose that the boundary conditions are given by

$$R_\rho(u_+, u_-, \mu) = (R_{+,\rho}(u_+, \mu), R_{-,\rho}(u_-, \mu)) \in R(P_{+,\rho}(\mu_\rho)) \times R(P_{-,\rho}(\mu_\rho)).$$

We assume that $D_u R_{\pm,\rho}(p_{\pm,\rho}(\mu_\rho), \mu_\rho)|_{R(P_{\pm,\rho}(\mu_\rho))}$ is invertible, and the inverse is bounded uniformly in ρ. If Hypothesis (T1)(ii) is replaced by this assumption, the results in the previous sections remain true.

3 Exponential Dichotomies – an Excursion

Here, we summarize the results in [26] which are the key to the proofs of the theorems presented in the last section.

Assume that the operator A is as in Section 2, that is, $A : D(A) \subset X \to X$ is a closed operator such that its domain $D(A)$ is dense in X. Furthermore, A satisfies Hypotheses (A1) and (A2). Moreover, let $B \in C^0(\mathrm{R}, \mathcal{L}(X^\alpha, X))$ be a continuous family of operators.

Consider the differential equation

$$\dot{v} = (A + B(t))\, v. \tag{3.1}$$

We are particularly interested in solutions $v(t)$ with some prescribed exponential behavior for $t \in \mathrm{R}^+$ and $t \in \mathrm{R}^-$.

Definition *(Exponential Dichotomy)*
Equation (3.1) has an exponential dichotomy in X^α on the interval $J \subset \mathrm{R}$ if there exist positive constants C and κ, and operators $\phi^s(t,\tau)$ and $\phi^u(\tau,t)$ in $\mathcal{L}(X^\alpha)$ defined for $t \geq \tau$ with $t,\tau \in J$ such that the following is true.

(i) *For any $v \in X^\alpha$, $\phi^s(t,\tau)v$ is a solution of (3.1) for $t \geq \tau$ in J. Similarly, $\phi^u(t,\tau)v$ is a solution of (3.1) for $t \leq \tau$ in J.*

(ii) *For any $v \in X^\alpha$, $\phi^s(t,\tau)v$ and $\phi^u(\tau,t)v$ are continuous in $t \geq \tau$ in J.*

(iii) $\|\phi^s(t,\tau)\|_\alpha + \|\phi^u(\tau,t)\|_\alpha \leq Ce^{-\kappa(t-\tau)}$ *for all $t \geq \tau$ in J.*

(iv) $\phi^s(t,\tau)\phi^s(\tau,s) = \phi^s(t,s)$ *for all $t \geq \tau \geq s$ in J, and the analogous property for $\phi^u(\tau,t)$.*

Note that the operators $P(t) = \phi^s(t,t)$ are projections. We assume that $B(t)$ is small for large $|t|$. The constant ϵ is specified in Theorem 5 below.

Hypothesis (D1) *There are numbers $\vartheta > 0$ and $t_* > 0$ such that $B \in C^{0,\vartheta}(\mathrm{R}, \mathcal{L}(X^\alpha, X))$ and $\|B(t)\|_{\mathcal{L}(X^\alpha,X)} \leq \epsilon$ for all $|t| \geq t_*$.*

Finally, we assume forward and backward uniqueness of solutions of equation (3.1) on the interval R.

Hypothesis (D2) *The only bounded solution $v(t)$ of (3.1) on the intervals R^+ or R^- with $v(0) = 0$ is the trivial solution $v(t) = 0$. Similarly, the only bounded solution $w(t)$ of the adjoint equation $\dot{w} = -(A + B(t))^* w$ on R^+ or R^- with $w(0) = 0$ is $w(t) = 0$.*

We then have the following existence result for exponential dichotomies of (3.1).

Theorem 5 ([26]) *Suppose that Hypotheses (A1)–(A2) and (D2) are satisfied. There is then a constant $\epsilon_0 > 0$ such that (3.1) has an exponential dichotomy on R^+ provided Hypothesis (D1) is met with $\epsilon = \epsilon_0$.*

(i) The projections $P(t) = \phi^s(t,t)$ are Hölder continuous in $t \in \mathrm{R}^+$ with values in $\mathcal{L}(X^\alpha)$.

(ii) The operator $\phi^s(t,\tau)$ has a bounded extension to X satisfying $\phi^s(t,\tau)\phi^s(\tau,s) = \phi^s(t,s)$ for all $t \geq \tau \geq s \geq 0$.

(iii) $\phi^s(t,\tau) \in \mathcal{L}(X, X^\alpha)$ for $t > \tau$ and $\|\phi^s(t,\tau)\|_{\mathcal{L}(X,X^\alpha)} \leq C(t-\tau)^{-\alpha}\, e^{-\kappa(t-\tau)}$.

Analogous properties hold for $\phi^u(\tau,t)$ with $t \geq \tau \geq 0$.

The same results is true with R^+ replaced by R^-. We denote the evolution operators by $\phi^s_+(t,\tau)$ and $\phi^u_+(\tau,t)$ for $t \geq \tau \geq 0$, and by $\phi^s_-(\tau,t)$ and $\phi^u_-(t,\tau)$ for $t \leq \tau \leq 0$. Finally, we compare the evolution operators for two different equations.

Lemma 3.1 ([26]) *Suppose $B_1(t)$ and $B_2(t)$ satisfy the assumptions of Theorem 5 on $J = \mathrm{R}^+$. There exist positive numbers C and η such that the following is true. If*

$$\sup_{t\geq 0} \|B_1(t) - B_2(t)\|_{\mathcal{L}(X^\alpha,X)} < \eta,$$

the projections $P_j(t)$, $j \in \{1,2\}$, which correspond to the equations $\dot{v} = (A + B_j(t))v$ satisfy the estimate

$$\sup_{t\geq 0} \|P_1(t) - P_2(t)\|_{\mathcal{L}(X^\alpha)} \leq C\eta.$$

4 The Galerkin Approximation

In this section, Theorem 1 is proved. Throughout, C denotes various different constants all independent of T_- and T_+. We will use the following version of Banach's fixed point theorem.

Lemma 4.1 *Suppose that Y and $\hat{Y}$ are Banach spaces and $G : Y \to \hat{Y}$ is a C^1-function. Assume that there exists a linear, bounded and invertible operator $L : Y \to \hat{Y}$, an element $y_0 \in Y$, and numbers $\eta > 0$ and $0 < \kappa < 1$ such that*

(i) $\|\mathrm{id} - L^{-1}DG(y))\| \leq \kappa$ for all $y \in B_\eta(y_0)$,

(ii) $|L^{-1}G(y_0)| \leq (1-\kappa)\eta$.

There exists then a unique point $y_* \in B_\eta(y_0)$ with $G(y_*) = 0$ and

$$|y_0 - y_*| \leq (1-\kappa)^{-1}\,|L^{-1}G(y_0)|, \qquad \|DG(y)^{-1}\| \leq (1+\kappa)\,\|L^{-1}\|,$$

uniformly in $y \in B_\eta(y_0)$.

Here, $B_\eta(y)$ is the ball with center y and radius η in Y. We start with a useful consequence of Hypothesis (Q).

Lemma 4.2 *Assume that (Q) is satisfied and let $K \in \mathcal{L}(X^\alpha, X)$ be a compact operator. Then $\|(\mathrm{id} - Q_\rho)K\|_{\mathcal{L}(X^\alpha,X)} \to 0$ as $\rho \to 0$.*

Proof. We argue by contradiction. Suppose that there are elements $v_n \in X^\alpha$ and $\rho_n > 0$ with $|v_n|_\alpha = 1$ and $\rho_n \to 0$ as $n \to \infty$ such that $|(\mathrm{id} - Q_{\rho_n})Kv_n|_0 \geq \delta > 0$. After choosing a subsequence, we have $Kv_n \to w$ in X since K is compact. Hence,

$$\begin{aligned} |(\mathrm{id} - Q_{\rho_n})Kv_n|_0 &\leq |(\mathrm{id} - Q_{\rho_n})w|_0 + |(\mathrm{id} - Q_{\rho_n})(Kv_n - w)|_0 \\ &\leq |(\mathrm{id} - Q_{\rho_n})w|_0 + C|Kv_n - w|_0 \to 0, \end{aligned}$$

as $n \to \infty$, due to (Q)(ii). This is a contradiction. ∎

Note that $D_u f(u,\mu)$ is compact for any (u,μ) provided Hypothesis (C) is satisfied.

4.1 Persistence of the Equilibrium

In order to show that the equilibrium persists, consider

$$\begin{aligned} &(A + D_u f(p_0, 0))^{-1}(A(p_0 + u) + Q_\rho f(p_0 + u, \mu)) \\ &= u + (A + D_u f(p_0,0))^{-1}\Big(Q_\rho(f(p_0 + u, \mu) - f(p_0, 0) - D_u f(p_0, 0)u) \\ &\quad + (\mathrm{id} - Q_\rho)(f(p_0, 0) + D_u f(p_0, 0)u)\Big) \\ &=: G_\rho(u, \mu). \end{aligned}$$

It suffices to seek zeroes of $G_\rho(u,\mu)$ near $(p_0, 0)$. The map G is smooth in (u, μ) as a map from $X^\alpha \times \mathrm{R}$ to X^α and satisfies $G_0(0,0) = 0$ as well as $D_u G_0(0,0) = \mathrm{id}$. Furthermore, using $Ap_0 + f(p_0, 0) = 0$,

$$\begin{aligned} |G_\rho(0,\mu)|_\alpha &\leq C|(A + D_u f(p_0,0))^{-1}(Q_\rho(f(p_0,\mu) - f(p_0,0)) + (\mathrm{id} - Q_\rho)f(p_0,0))|_\alpha \\ &\leq C(|\mu| + |(A + D_u f(p_0,0))^{-1}A(\mathrm{id} - Q_\rho)p_0|_\alpha) \\ &\leq C(|\mu| + |(\mathrm{id} - Q_\rho)p_0|_\alpha), \end{aligned}$$

and, due to (C) and Lemma 4.2,

$$\|D_u G(u,\mu,\rho) - \mathrm{id}\,\|_\alpha \leq C\|D_u f(p_0 + u, \mu) - D_u f(p_0, 0) + (\mathrm{id} - Q_\rho)D_u f(p_0,0)\|_\alpha < \frac{1}{2}$$

for all (u,μ,ρ) in a ball in $X^\alpha \times \mathrm{R}^2$ centered at the origin with sufficiently small radius η. We apply Lemma 4.1 for any (μ, ρ) in $B_\eta(0) \subset \mathrm{R}^2$ with $L = \mathrm{id}$. Hence, there exists a unique zero $p_\rho(\mu) \in B_\eta(p_0) \subset X^\alpha$ of $G_\rho(\cdot,\mu)$, with $p_0(0) = p_0$ and

$$|p_\rho(\mu) - p_0|_\alpha \leq C(|(\mathrm{id} - Q_\rho)p_0|_\alpha + |\mu|).$$

Furthermore, $p_\rho(\mu)$ is smooth in μ. By construction, $p_\rho(\mu)$ are equilibria of (2.4). By Hypothesis (C), Theorem 5, and Lemma 4.2, the equation

$$\dot{v} = (A + Q_\rho D_u f(p_\rho(\mu), \mu))v$$

has an exponential dichotomy on R with projections $P_{+,\rho}(\mu)$ and $P_{-,\rho}(\mu)$. This proves the first, and part of the third claim in Theorem 1.

4.2 Persistence of the Homoclinic Orbit

Next, we introduce a new variable v by

$$u(t) = h(t) + v(t), \tag{4.1}$$

and write equation (2.4), that is $\dot{u} = Au + Q_\rho f(u, \mu)$, in the form

$$\begin{aligned} \dot{v} &= (A + D_u f(h(t), 0))v + F_\rho(t, v, \mu) \\ &= (A + D_u f(h(t), 0))v + D_\mu f(h(t), 0)\mu + \hat{F}_\rho(t, v, \mu), \end{aligned} \tag{4.2}$$

with

$$\begin{aligned} \hat{F}_\rho(t, v, \mu) \;:=\; & -(\mathrm{id} - Q_\rho)\Big(D_u f(h(t), 0)v + D_\mu f(h(t), 0)\mu + f(h(t), 0)\Big) \\ & + Q_\rho\Big(f(h(t) + v, \mu) - f(h(t), 0) - D_u f(h(t), 0)v - D_\mu f(h(t), 0)\mu\Big). \end{aligned}$$

Due to Hypothesis (C) and Lemma 4.2, we have the estimate

$$\|D_{(u,\mu)}\hat{F}_\rho(t, v, \mu)\|_{\mathcal{L}(X^\alpha, X)} \le C(|v|_\alpha + |\mu|) + g(\rho), \tag{4.3}$$

for some function $g(\rho)$ with $g(\rho) \to 0$ as $\rho \to 0$.

On account of Theorem 5 and Hypotheses (A1)–(A3), we know that equation (2.2)

$$\dot{v} = (A + D_u f(h(t), 0))v$$

has exponential dichotomies on R^+ and R^-. As in Theorem 5, we denote the solution operators of this equation by $\phi_+^s(t, \tau)$ and $\phi_+^u(\tau, t)$ for $t \ge \tau \ge 0$, and by $\phi_-^s(\tau, t)$ and $\phi_-^u(t, \tau)$ for $t \le \tau \le 0$.

Solutions of the nonlinear equation (4.2) are bounded on R if, and only if, there exist $(b_+, b_-) \in R(\phi_+^s(0, 0)) \times R(\phi_-^u(0, 0))$ such that

$$\begin{aligned} v_+(t) &= \phi_+^s(t, 0)b_+ + \int_0^t \phi_+^s(t, \tau)F_\rho(\tau, v_+(\tau), \mu)\, d\tau + \int_\infty^t \phi_+^u(t, \tau)F_\rho(\tau, v_+(\tau), \mu)\, d\tau, \\ v_-(t) &= \phi_-^u(t, 0)b_- + \int_0^t \phi_-^u(t, \tau)F_\rho(\tau, v_-(\tau), \mu)\, d\tau + \int_{-\infty}^t \phi_-^s(t, \tau)F_\rho(\tau, v_-(\tau), \mu)\, d\tau, \\ v_+(0) &= v_-(0) \\ 0 &= \langle \varphi, v_+(0) \rangle. \end{aligned}$$

Here, $\varphi \in (X^\alpha)^*$ is chosen such that $\langle \varphi, \dot h(0)\rangle = 1$. The last equation takes care of the translational invariance of (2.4). In the first and second equation, we have $t \in \mathrm{R}^+$ and $t \in \mathrm{R}^-$, respectively. We remark that it suffices to seek weak solutions of (4.2) since any weak solution is actually a strong solution, see [26, Lemma 3.1]. Let

$$G_\rho(b_+, b_-, v_+, v_-, \mu) :=$$
$$\begin{pmatrix} v_+(t) - \phi_+^s(t,0)b_+ - \int_0^t \phi_+^s(t,\tau)F_\rho(\tau, v_+(\tau),\mu)\,d\tau - \int_\infty^t \phi_+^u(t,\tau)F_\rho(\tau, v_+(\tau),\mu)\,d\tau \\ v_-(t) - \phi_-^u(t,0)b_- - \int_0^t \phi_-^u(t,\tau)F_\rho(\tau, v_-(\tau),\mu)\,d\tau - \int_{-\infty}^t \phi_-^s(t,\tau)F_\rho(\tau, v_-(\tau),\mu)\,d\tau \\ b_+ - b_- - \int_0^\infty \phi_+^u(0,\tau)F_\rho(\tau, v_+(\tau),\mu)\,d\tau - \int_{-\infty}^0 \phi_-^s(0,\tau)F_\rho(\tau, v_-(\tau),\mu)\,d\tau \\ \langle \varphi, \phi_+^s(0,0)b_+ - \int_0^\infty \phi_+^u(0,\tau)F_\rho(\tau, v_+(\tau),\mu)\,d\tau\rangle \end{pmatrix},$$

and consider $G_\rho : Y \to \hat Y$ for fixed ρ as a map defined on the spaces

$$\begin{aligned} Y &:= R(\phi_+^s(0,0)) \times R(\phi_-^u(0,0)) \times C^0(\mathrm{R}^+, X^\alpha) \times C^0(\mathrm{R}^-, X^\alpha) \times \mathrm{R}, \\ \hat Y &:= C^0(\mathrm{R}^+, X^\alpha) \times C^0(\mathrm{R}^-, X^\alpha) \times X^\alpha \times \mathrm{R}. \end{aligned}$$

Note that G_ρ is well defined and smooth in $(b_+, b_-, v_+, v_-, \mu)$. We exploit the splitting (4.2) of F_ρ into the linear term $D_\mu f(h(t),0)\mu$ and the quadratic term $\hat F_\rho$, see (4.3). Therefore, consider

$$G_\rho(b_+, b_-, v_+, v_-, \mu) = L(b_+, b_-, v_+, v_-, \mu)$$
$$- \begin{pmatrix} \int_0^t \phi_+^s(t,\tau)\hat F_\rho(\tau, v_+(\tau),\mu)\,d\tau + \int_\infty^t \phi_+^u(t,\tau)\hat F_\rho(\tau, v_+(\tau),\mu)\,d\tau \\ \int_0^t \phi_-^u(t,\tau)\hat F_\rho(\tau, v_-(\tau),\mu)\,d\tau + \int_{-\infty}^t \phi_-^s(t,\tau)\hat F_\rho(\tau, v_-(\tau),\mu)\,d\tau \\ \int_0^\infty \phi_+^u(0,\tau)\hat F_\rho(\tau, v_+(\tau),\mu)\,d\tau + \int_{-\infty}^0 \phi_-^s(0,\tau)\hat F_\rho(\tau, v_-(\tau),\mu)\,d\tau \\ \langle \varphi, \int_0^\infty \phi_+^u(0,\tau)\hat F_\rho(\tau, v_+(\tau),\mu)\,d\tau\rangle \end{pmatrix},$$

where the linear part $L : Y \to \hat Y$ is bounded and given by

$$L(b_+, b_-, v_+, v_-, \mu) =$$
$$\begin{pmatrix} v_+ - \phi_+^s(\cdot,0)b_+ - \mu\Big(\int_0 \phi_+^s(\cdot,\tau)D_\mu f(h(\tau),0)\,d\tau + \int_\infty \phi_+^u(\cdot,\tau)D_\mu f(h(\tau),0)\,d\tau\Big) \\ v_- - \phi_-^u(\cdot,0)b_- - \mu\Big(\int_0 \phi_-^u(\cdot,\tau)D_\mu f(h(\tau),0)\,d\tau + \int_{-\infty} \phi_-^s(\cdot,\tau)D_\mu f(h(\tau),0)\,d\tau\Big) \\ b_+ - b_- - \mu\Big(\int_0^\infty \phi_+^u(0,\tau)D_\mu f(h(\tau),0)\,d\tau + \int_{-\infty}^0 \phi_-^s(0,\tau)D_\mu f(h(\tau),0)\,d\tau\Big) \\ \langle \varphi, \phi_+^s(0,0)b_+ - \mu\int_0^\infty \phi_+^u(0,\tau)D_\mu f(h(\tau),0)\,d\tau\rangle \end{pmatrix}.$$

Note that the last two components of L do not depend on (v_+, v_-). The linear operator

$$\Phi_0 : R(\phi_+^s(0,0)) \times R(\phi_-^u(0,0)) \to X^\alpha, \quad \Phi_0(b_+, b_-) = b_+ - b_-,$$

is a Fredholm operator with index zero. Its null space and range are given by

$$\begin{aligned} N(\Phi_0) &= \mathrm{span}\{(\dot h(0), \dot h(0))\} \subset R(\phi_+^s(0,0)) \times R(\phi_-^u(0,0)), \\ R(\Phi_0) &= \{v \in X^\alpha;\ \langle \psi(0), v\rangle = 0\}. \end{aligned}$$

On the other hand,

$$\langle \varphi, \phi^s_+(0,0)\dot h(0)\rangle = \langle \varphi, \dot h(0)\rangle = 1,$$

and

$$\begin{aligned}&\int_{-\infty}^{\infty} \langle \psi(\tau), D_\mu f(h(\tau),0)\rangle \, d\tau \\ &= \Big\langle \psi(0), \int_0^\infty \phi^u_+(0,\tau) D_\mu f(h(\tau),0)\, d\tau + \int_{-\infty}^0 \phi^s_-(0,\tau) D_\mu f(h(\tau),0)\, d\tau \Big\rangle.\end{aligned}$$

Hence, as a consequence of Hypothesis (H3), the operator L is continuously invertible.

Due to the estimate (4.3), we can apply Lemma 4.1 for any small fixed ρ to the map G_ρ with $y_0 = (0,0,0,0)$. Hence, we obtain the existence and uniqueness statements in (ii) and (iii) of Theorem 1. It is straightforward to show that $h_\rho(t)$ is homoclinic to the hyperbolic equilibrium $p_\rho(\mu_\rho)$.

The estimate given in (ii) follows from Lemma 4.1 provided we can prove that

$$|G_\rho(0,0,0,0)|_{\hat Y} \le C \sup_{t\in \mathrm{R}} |(\mathrm{id} - Q_\rho) h(t)|_\alpha, \tag{4.4}$$

where

$$G_\rho(0,0,0,0,0) = \begin{pmatrix} \int_0^t \phi^s_+(t,\tau)(\mathrm{id} - Q_\rho) f(h(\tau),0)\, d\tau + \int_\infty^t \phi^u_+(t,\tau)(\mathrm{id} - Q_\rho) f(h(\tau),0)\, d\tau \\ \int_0^t \phi^u_-(t,\tau)(\mathrm{id} - Q_\rho) f(h(\tau),0)\, d\tau + \int_{-\infty}^t \phi^s_-(t,\tau)(\mathrm{id} - Q_\rho) f(h\tau),0)\, d\tau \\ \int_0^\infty \phi^u_+(0,\tau)(\mathrm{id} - Q_\rho) f(h(\tau),0)\, d\tau + \int_{-\infty}^0 \phi^s_-(0,\tau)(\mathrm{id} - Q_\rho) f(h(\tau),0)\, d\tau \\ \langle \varphi, \int_0^\infty \phi^u_+(0,\tau)(\mathrm{id} - Q_\rho) f(h(\tau),0)\, d\tau \rangle \end{pmatrix}.$$

In order to prove (4.4), it suffices to show that

$$\Big| \int_0^t \phi^s_+(t,\tau)(\mathrm{id} - Q_\rho) f(h(\tau),0)\, d\tau \Big|_\alpha \le C \sup_{\tau\in \mathrm{R}} |(\mathrm{id} - Q_\rho) h(\tau)|_\alpha \tag{4.5}$$

for some constant C independently of $t \ge 0$, and similar estimates for the other integrals. We use the fact that $h(t)$ satisfies $\dot h = Ah + f(h,0)$ for $t \in \mathrm{R}$, that is,

$$h \in C^1(\mathrm{R}, X) \cap C^0(\mathrm{R}, D(A)). \tag{4.6}$$

Therefore, using (Q)(i), we have

$$\begin{aligned}&\int_0^t \phi^s_+(t,\tau)(\mathrm{id} - Q_\rho) f(h(\tau),0)\, d\tau \\ &= \int_0^t \phi^s_+(t,\tau)(\mathrm{id} - Q_\rho)(\frac{d}{d\tau} h(\tau) - Ah(\tau))\, d\tau \\ &= \int_0^t \Big(- \Big(\frac{d}{d\tau}\phi^s_+(t,\tau)\Big)(\mathrm{id} - Q_\rho) h(\tau) - \phi^s_+(t,\tau) A (\mathrm{id} - Q_\rho) h(\tau)\Big)\, d\tau\end{aligned}$$

$$
\begin{aligned}
&+\phi_+^s(t,t)(\mathrm{id}-Q_\rho)h(t)\\
=\;& \int_0^t \Big(\phi_+^s(t,\tau)(A+D_u f(h(\tau),0))(\mathrm{id}-Q_\rho)h(\tau) - \phi_+^s(t,\tau)A(\mathrm{id}-Q_\rho)h(\tau)\Big)\,d\tau\\
&+\phi_+^s(t,t)(\mathrm{id}-Q_\rho)h(t)\\
=\;& \int_0^t \phi_+^s(t,\tau)D_u f(h(\tau),0)(\mathrm{id}-Q_\rho)h(\tau)\,d\tau + \phi_+^s(t,t)(\mathrm{id}-Q_\rho)h(t).
\end{aligned}
$$

Note that integration by parts and taking the derivative $\frac{d}{d\tau}\phi_+^s(t,\tau)$ is allowed on account of (4.6). It is now straightforward to obtain the aforementioned estimate (4.5). The other estimates are obtained in an analogous fashion, and we omit the details. This proves the claim (4.4).

Finally, we show the homoclinic orbits $h_\rho(t)$ of (2.4) are non-degenerate.

Lemma 4.3 *The only bounded solution, up to constant multiples, of the variational equation*

$$\dot v = (A + Q_\rho D_u f(h_\rho(t), \mu_\rho))v$$

about $h_\rho(t)$ is given by $\dot h_\rho(t)$. In other words, the solutions $h_\rho(t)$ are non-degenerate.

Proof. On account of Theorem 5, the variational equation

$$\dot v = Av + Q_\rho D_u f(h_\rho(t), \mu_\rho)v$$

has an exponential dichotomy on R^+ and R^- with solution operators $\phi_{+,\rho}^s(t,\tau)$ and $\phi_{+,\rho}^u(\tau,t)$ for $t \ge \tau \ge 0$, and $\phi_{-,\rho}^s(\tau,t)$ and $\phi_{-,\rho}^u(t,\tau)$ for $t \le \tau \le 0$. On account of Hypothesis (C), and Lemmata 3.1 and 4.2, $\phi_{+,\rho}^s(0,0)$ and $\phi_{-,\rho}^u(0,0)$ are close to $\phi_+^s(0,0)$ and $\phi_-^u(0,0)$, respectively, in the $\mathcal{L}(X^\alpha)$-norm. Therefore, $\dot h_\rho(t)$ is the only bounded solution, up to constant multiples, of equation (2.4). ■

It is a consequence of the proof of Lemma 4.3 that $\dot h_\rho(\cdot) \in C^0(\mathrm{R}, X^\alpha)$. Indeed, $\dot h_\rho(0) \in R(\phi_{+,\rho}^s(0,0))$, and therefore $\dot h_\rho(0) \in X^\alpha$. Furthermore, $\dot h_\rho(t) = \phi_{+,\rho}^s(t,0)\dot h_\rho(0)$ for $t > 0$ is continuous in t as a function into X^α by Theorem 5. Since the choice of $t = 0$ is arbitrary, we see that in fact $\dot h_\rho(\cdot) \in C^0(\mathrm{R}, X^\alpha)$.

5 The Truncated Boundary-Value Problem

In this section, we prove Theorem 2. Again, C denotes various different constants independent of T_- and T_+.

5.1 The Nonlinear Equation

We exploit the transformation $u(t) = h_\rho(t) + v(t)$ and $\mu = \mu_\rho + \nu$. The function $v(t)$ then satisfies the equation

$$\begin{aligned} \dot{v} &= (A + D_u f(h(t),0))v + F_\rho(t,v,\nu) \\ &= (A + D_u f(h(t),0))v + D_\mu f(h_\rho(t),\mu_\rho)\nu + \hat{F}_\rho(t,v,\nu), \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} \hat{F}_\rho(t,v,\nu) &= Q_\rho\Big(f(h_\rho(t)+v,\mu_\rho+\nu) - f(h_\rho(t),\mu_\rho) - D_u f(h(t),0)v\Big) \\ & \quad -(\mathrm{id} - Q_\rho)D_u f(h(t),0)v - D_\mu f(h_\rho(t),\mu_\rho)\nu. \end{aligned}$$

The derivative $D_{(v,\nu)}\hat{F}_\rho(t,v,\nu)$ is given by

$$\begin{aligned} & D_{(v,\nu)}\hat{F}_\rho(t,v,\nu) = \\ & \quad \Big[Q_\rho(D_u f(h_\rho(t)+v,\mu_\rho+\nu) - D_u f(h(t),0)) - (\mathrm{id} - Q_\rho)D_u f(h(t),0), \\ & \quad Q_\rho(D_\mu f(h_\rho(t)+v,\mu_\rho+\nu) - D_\mu f(h_\rho(t),\mu_\rho)) - (\mathrm{id} - Q_\rho)D_\mu f(h_\rho(t),\mu_\rho)\Big]. \end{aligned}$$

Due to Theorem 1, Hypothesis (C), and Lemma 4.2, we obtain the estimate

$$\|D_{(v,\mu)}\hat{F}_\rho(t,v,\mu)\|_{\mathcal{L}(X^\alpha \times \mathrm{R}, X)} \leq C(|v|_\alpha + |\nu|) + g(\rho), \tag{5.2}$$

uniformly in ρ for (v,ν) in a ball centered at zero of sufficiently small radius η in $X^\alpha \times \mathrm{R}$. Here, the function $g(\rho) \to 0$ as $\rho \to 0$. Let

$$\begin{aligned} a = (a_+, a_-) &\in X_a := R(P_+) \times R(P_-), \\ b = (b_+, b_-) &\in X_b := R(\phi_+^s(0,0)) \times R(\phi_-^u(0,0)). \end{aligned} \tag{5.3}$$

We define the maps

$$\begin{aligned} I_{+,T,\rho} &: X_a \times X_b \times C^0([0,T_+],X^\alpha) \times \mathrm{R} \to C^0([0,T_+],X^\alpha), \\ I_{-,T,\rho} &: X_a \times X_b \times C^0([T_-,0],X^\alpha) \times \mathrm{R} \to C^0([T_-,0],X^\alpha), \end{aligned}$$

by

$$\begin{aligned} & I_{+,T,\rho}(a,b,v_+,\nu)(t) = \phi_+^u(t,T_+)a_+ + \phi_+^s(t,0)b_+ \\ & \quad + \int_{T_+}^t \phi_+^u(t,\tau)F_\rho(\tau,v_+(\tau),\nu)\,d\tau + \int_0^t \phi_+^s(t,\tau)F_\rho(\tau,v_+(\tau),\nu)\,d\tau, \end{aligned} \tag{5.4}$$

and the analogous expression for $I_{-,T,\rho}(a,b,v_-,\nu)$. Note that both maps are smooth. Any bounded solution of (5.1) satisfies the integral equation

$$\begin{aligned} 0 &= v_+(t) - I_{+,T,\rho}(a,b,v_+,\nu)(t), \\ 0 &= v_-(t) - I_{-,T,\rho}(a,b,v_-,\nu)(t), \end{aligned} \tag{5.5}$$

together with the equation $v_+(0) = v_-(0)$ for some (a, b). Here, $t \in [0, T_+]$ in the first, and $t \in [T_-, 0]$ in the second equation in (5.5). In addition, we have to solve the phase and boundary conditions

$$\begin{aligned} R_\rho(h_\rho(T_+) + v_+(T_+), h_\rho(T_-) + v_-(T_-), \mu_\rho + \nu) &= 0, \\ J_{T,\rho}(h_\rho + V(v_+, v_-), \mu_\rho + \nu) &= 0, \end{aligned} \tag{5.6}$$

where the linear, bounded operator

$$V : C^0([0, T_+], X^\alpha) \times C^0([T_-, 0], X^\alpha) \to C^0([T_-, T_+], X^\alpha)$$

is defined by

$$V(v_+, v_-)(t) = \begin{cases} v_+(t) + v_-(0) - v_+(0) & t > 0, \\ v_-(t) & t \le 0. \end{cases} \tag{5.7}$$

For $v \in X^\alpha$, we expand the boundary conditions

$$\begin{aligned} R_\rho(h_\rho(T_+) + v_+, h_\rho(T_-) + v_-, \mu_\rho + \nu) &= R_\rho(h_\rho(T_+), h_\rho(T_-), \mu_\rho) \\ &+ D_{u_+, u_-, \mu} R_\rho(h_\rho(T_+), h_\rho(T_-), \mu_\rho)(v_+, v_-, \nu) \\ &+ \hat{R}_\rho(h_\rho(T_+), h_\rho(T_-), v_+, v_-, \nu), \end{aligned} \tag{5.8}$$

with

$$\|D_{(v_+, v_-, \nu)} \hat{R}_\rho(h_\rho(T_+), h_\rho(T_-), v_+, v_-, \nu)\| \le C(|v_+|_\alpha + |v_-|_\alpha + |\nu|), \tag{5.9}$$

for (v_+, v_-, ν) in a ball with small radius centered at zero in $X^\alpha \times X^\alpha \times \mathrm{R}$, independently of ρ. Similarly, for $v \in C^0(T, X^\alpha)$, we have

$$\begin{aligned} J_{T,\rho}(h_\rho + v, \mu_\rho + \nu) &= J_{T,\rho}(h_\rho, \mu_\rho) \\ &+ D_v J_{T,\rho}(h_\rho, \mu_\rho) v + D_\mu J_{T,\rho}(h_\rho, \mu_\rho)\nu + \hat{J}_{T,\rho}(h_\rho, v, \nu), \end{aligned} \tag{5.10}$$

with

$$\|D_{(v,\nu)} \hat{J}_{T,\rho}(h_\rho, v, \nu)\| \le C(|v|_\alpha + |\nu|), \tag{5.11}$$

for (v, ν) in a ball with small radius centered at zero in $C^0(T, X^\alpha) \times \mathrm{R}$, independently of ρ. We consider the nonlinear equation

$$G_{T,\rho} : Y \to \hat{Y}, \qquad G_{T,\rho}(a, b, v_+, v_-, \nu) = 0 \tag{5.12}$$

with

$$\begin{aligned} Y &= X_a \times X_b \times C^0([0, T_+], X^\alpha) \times C^0([T_-, 0], X^\alpha) \times \mathrm{R} \\ \hat{Y} &= C^0([0, T_+], X^\alpha) \times C^0([T_-, 0], X^\alpha) \times X^\alpha \times X^\alpha \times \mathrm{R}, \end{aligned}$$

defined by the right-hand side of (5.5), the continuity equation

$$0 = I_{+,T,\rho}(a, b, v_+, \nu)(0) - I_{-,T,\rho}(a, b, v_-, \nu)(0)$$

the first two equations in (5.6), and the equation

$$0 \quad = \quad J_{T,\rho}(h_\rho + V(I_{+,T,\rho}(a, b, v_+, \nu), I_{-,T,\rho}(a, b, v_-, \nu)), \mu_\rho + \nu).$$

It is a consequence of the above discussion that G is well defined and smooth. Furthermore, due to the estimates (5.2), (5.9) and (5.11), we can solve equation (5.12) in a ball centered at the origin with small radius η uniformly for any sufficiently small ρ provided the linearized operator at $(a, b, v_+, v_-, \nu) = 0$ is invertible uniformly in T and ρ. The arguments are analogous to those presented in the last section, whence we omit them. Note that the error estimate in Theorem 2 follows from

$$G_{T,\rho}(0) = \Big(0, 0, 0, R_\rho(h_\rho(T_+), h_\rho(T_-), \mu_\rho), J_{T,\rho}(h_\rho, \mu_\rho)\Big)$$

and Lemma 4.1. Indeed, replacing $h_\rho(\cdot)$ by $h_\rho(\cdot + \gamma_{T,\rho})$ for some small $\gamma_{T,\rho}$, we can achieve that $J_{T,\rho}(h_\rho, \mu_\rho) = 0$.

5.2 The Linearized Boundary-Value Problem

It remains to show that the operator $L_{T,\rho} = DG_{T,\rho}(0)$ is invertible as a map from Y to $\hat{Y}$. Let

$$\begin{aligned}\hat{I}_{+,T,\rho}(a, b, \nu)(t) &= \phi^u_+(t, T_+)a_+ + \phi^s_+(t, 0)b_+ \\ &+\nu\Big(\int_{T_+}^t \phi^u_+(t, \tau)D_\mu f(h_\rho(\tau), \mu_\rho)\, d\tau + \int_0^t \phi^s_+(t, \tau)D_\mu f(h_\rho(\tau), \mu_\rho)\, d\tau\Big),\end{aligned} \tag{5.13}$$

$$\begin{aligned}\hat{I}_{-,T,\rho}(a, b, \nu)(t) &= \phi^s_-(t, T_-)a_- + \phi^u_-(t, 0)b_- \\ &+\nu\Big(\int_{T_-}^t \phi^s_-(t, \tau)D_\mu f(h_\rho(\tau), \mu_\rho)\, d\tau + \int_0^t \phi^u_-(t, \tau)D_\mu f(h_\rho(\tau), \mu_\rho)\, d\tau\Big),\end{aligned} \tag{5.14}$$

for $t \in [0, T_+]$ and $t \in [T_-, 0]$, respectively. The linear operators $\hat{I}$ are bounded from $X_a \times X_b \times \mathrm{R}$ into $C^0([0, T_+], X^\alpha)$ and $C^0([T_-, 0], X^\alpha)$, respectively. We then have

$$L_{T,\rho}(a, b, v_+, v_-, \nu) = \tag{5.15}$$
$$\begin{pmatrix} v_+ - \hat{I}_{+,T,\rho}(a, b, \nu) \\ v_- - \hat{I}_{-,T,\rho}(a, b, \nu) \\ \hat{I}_{+,T,\rho}(a, b, \nu)(0) - \hat{I}_{-,T,\rho}(a, b, \nu)(0) \\ DR_\rho(h_\rho(T_+), h_\rho(T_-), \mu_\rho)\Big(\hat{I}_{+,T,\rho}(a, b, \nu)(T_+), \hat{I}_{-,T,\rho}(a, b, \nu)(T_-), \nu\Big) \\ D_v J_{T,\rho}(h_\rho, \mu_\rho)V(\hat{I}_{+,T,\rho}(a, b, \nu), \hat{I}_{-,T,\rho}(a, b, \nu)) + D_\mu J_{T,\rho}(h_\rho + v, \mu_\rho)\nu \end{pmatrix}.$$

We have to show that the equation

$$L_{T,\rho}(a,b,v_+,v_-,\nu) = (g_+,g_-,c,r,j) \tag{5.16}$$

has a unique solution $(a,b,v_+,v_-,\nu) \in Y$ for any $(g_+,g_-,c,r,j) \in \hat{Y}$, and

$$|(a,b,v_+,v_-,\nu)|_Y \le C|(g_+,g_-,c,r,j)|_{\hat{Y}}$$

for some positive constant C independent of ρ and T. Inspecting the definition (5.15) of $L_{T,\rho}$, it is clear that we can solve the first two components in (5.16) for (v_+,v_-) such that $(v_+,v_-) = W_1(a,b,\nu,g_+,g_-)$ with $\|W_1\| \le C$.

Next, consider the boundary condition

$$r \;=\; DR_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho)(w_+,w_-,\nu), \tag{5.17}$$

with

$$\begin{aligned} w_+ &= \phi_+^u(T_+,T_+)a_+ + \phi_+^s(T_+,0)b_+ + \nu\int_{T_+}^0 \phi_+^s(T_+,\tau)D_\mu f(h_\rho(\tau),\mu_\rho)\,d\tau, \\ w_- &= \phi_-^s(T_-,T_-)a_- + \phi_-^u(T_-,0)b_- + \nu\int_{T_-}^0 \phi_-^u(T_-,\tau)D_\mu f(h_\rho(\tau),\mu_\rho)\,d\tau. \end{aligned}$$

The key is that

$$\begin{aligned} D_{u_+,u_-}R_\rho(h_\rho(T_+),h_\rho(T_-),\mu_\rho)(\phi_+^u(T_+,T_+)|_{R(P_+)},\phi_-^s(T_-,T_-)|_{R(P_-)}) \\ : R(P_+)\times R(P_-) \to X^\alpha \end{aligned}$$

is invertible uniformly in ρ due to Hypothesis (T1)(ii). Indeed, the projections $\phi_+^u(T_+,T_+)$ and P_+ as well as $\phi_-^s(T_-,T_-)$ and P_- are close to each other for all $|T_-|$, T_+ sufficiently large and ρ small enough due to Hypothesis (C), and Lemmata 3.1 and 4.2.

Therefore, we can solve (5.17) for $a=(a_+,a_-)$ and obtain $a = W_2(b,\nu,r)$ with $\|W_2\| \le C$ independently of T_-, T_+ and ρ. Actually, we obtain the better estimate

$$|W_2(b,\nu,r)|_\alpha \le C(e^{-\kappa T_+}|b_+|_\alpha + e^{\kappa T_-}|b_-|_\alpha + |\nu| + |r|_\alpha),$$

since

$$|\phi_+^s(T_+,0)b_+|_\alpha \le Ce^{-\kappa T_+}|b_+|_\alpha,$$

and the analogous estimate for $\phi_-^u(T_-,0)$ by Theorem 5.

In the next step, we apply these estimates to the operator $V(\hat{I}_{+,T,\rho}(a,b,\nu),\hat{I}_{-,T,\rho}(a,b,\nu))$ appearing in the phase condition; see also (5.7) for its definition. Using (5.13), (5.14) and the estimates for a, we obtain the expansion

$$V(\hat{I}_{+,T,\rho},\hat{I}_{-,T,\rho})(a,b,\nu)(t) = \begin{cases} \phi_+^s(t,0)b_+ + b_- - b_+ + W_3(b,\nu,r)(t) & t>0, \\ \phi_-^u(t,0)b_- + W_3(b,\nu,r)(t) & t\le 0, \end{cases}$$

with

$$|W_3(b,\nu,r)(t)|_\alpha \le C(e^{-\kappa T_+}|b_+|_\alpha + e^{\kappa T_-}|b_-|_\alpha + |\nu| + |r|_\alpha).$$

According to the results in Section 4.2, we may write

$$(b_+, b_-) = (\hat{b}_+, \hat{b}_-) + \gamma(\dot{h}(0), \dot{h}(0)), \qquad (\hat{b}_+, \hat{b}_-) \in \hat{X}_b^\alpha$$

with $\hat{X}_b^\alpha \oplus \operatorname{span}\{(\dot{h}(0), \dot{h}(0))\} = X_b^\alpha$. We obtain

$$V(\hat{I}_{+,T,\rho}, \hat{I}_{-,T,\rho})(a,b,\nu)(t) = \gamma \dot{h}(t) + W_4(b,\nu,r)(t),$$

with

$$|W_4(b,\nu,r)(t)|_\alpha \le C(e^{-\kappa|T|}|\gamma| + |\hat{b}_+|_\alpha + |\hat{b}_-|_\alpha + |\nu| + |r|_\alpha),$$

and $|T| := \min(|T_-|, T_+)$. The phase condition and continuity equation are then given by

$$\begin{aligned} j &= \gamma D_v J_{T,\rho}(h_\rho, \mu_\rho)\dot{h} + W_5(b,\nu,r), \\ c &= \phi_+^u(0,T_+)W_{2,+}(b_+,\nu,r_+) - \phi_-^s(0,T_-)W_{2,-}(b_-,\nu,r_-) + \hat{b}_+ - \hat{b}_- \\ &\quad -\nu\Big(\int_0^{T_+} \phi_+^u(0,\tau)D_\mu f(h_\rho(\tau),\mu_\rho)\,d\tau + \int_{T_-}^0 \phi_-^s(0,\tau)D_\mu f(h_\rho(\tau),\mu_\rho)\,d\tau\Big), \end{aligned} \tag{5.18}$$

with

$$|W_5(b,\nu,r)| \le C(e^{-\kappa T_+}|b_+|_\alpha + e^{\kappa T_-}|b_-|_\alpha + |\nu| + |r|_\alpha).$$

Note that we have the estimates

$$\begin{aligned} &|\phi_+^u(0,T_+)W_{2,+}(b_+,\nu,r_+) - \phi_-^s(0,T_-)W_{2,-}(b_-,\nu,r_-)|_\alpha \\ \le\ & C(e^{-\kappa T_+} + e^{\kappa T_-})(|b_+|_\alpha + |b_-|_\alpha + |\nu| + |r|_\alpha) \end{aligned}$$

due to Theorem 5. Moreover,

$$\int_{T_-}^{T_+} \langle \psi(t), D_\mu f(h_\rho(t),\mu_\rho)\rangle\,dt$$

is bounded away from zero due to Hypothesis (H3), Theorem 1 and the fact that $\psi(t)$ converges to zero exponentially. Therefore, by the same arguments as in Section 4.2 using Theorem 1 and Hypothesis (T1)(i), we can solve (5.18) for $(\hat{b},\gamma,\nu)$.

5.3 Proof of Theorem 4

The proof is a consequence of the proof of Theorem 2. In fact, we only need to consider functions v_+ and variables a_+ and b_+. It is straightforward to see that b_+ can be used to solve the boundary condition $(\mathrm{id} - S)u(0) = 0$ due to the transversality condition (R)(ii). We omit the details.

6 The Finite-Dimensional Boundary-Value Problem

In this section, we prove Theorem 3. We embed the boundary-value problem on $R(Q_\rho)$ into a larger one defined on X^α, and then apply Theorem 2. Any element u in X^α can be written according to

$$u = q + w, \qquad (q,w) \in R(Q_\rho) \times N(Q_\rho).$$

Using this decomposition, we have

$$A + Q_\rho D_u f(p_\rho(\mu),\mu) = \begin{pmatrix} (A + Q_\rho D_u f(p_\rho(\mu),\mu))|_{R(Q_\rho)} & Q_\rho D_u f(p_\rho(\mu),\mu)|_{N(Q_\rho)} \\ 0 & A|_{N(Q_\rho)} \end{pmatrix}.$$

The spectral projections of A, $A + Q_\rho D_u f(p_\rho(\mu),\mu)$ and $(A + Q_\rho D_u f(p_\rho(\mu),\mu))|_{R(Q_\rho)}$ are denoted by $\hat{P}_\pm$, $P_{\pm,\rho}(\mu)$ and $Q_{\pm,\rho}(\mu)$, respectively; see Hypotheses (A1) and (H1). On account of Hypothesis (Q), we then have

$$P_{\pm,\rho}(\mu) = \begin{pmatrix} Q_{\pm,\rho}(\mu) & D_{\pm,\rho}(\mu) \\ 0 & (\mathrm{id} - Q_\rho)\hat{P}_\pm \end{pmatrix},$$

for some bounded operators $D_{\pm,\rho}(\mu)$.

The equation $\dot{u} = Au + Q_\rho f(u,\mu)$ is equivalent to

$$\dot{q} = Aq + Q_\rho f(q+w,\mu), \qquad \dot{w} = Aw. \tag{6.1}$$

We include the phase and boundary conditions

$$\begin{aligned} \tilde{J}((q,w),\mu) &= \int_{T_-}^{T_+} \langle \dot{h}_\rho(t), (q(t) + w(t) - h_\rho(t)\rangle_X \, dt, \\ \tilde{R}_+((q,w)(T_+),\mu) &= P_{+,\rho}(\mu_\rho) P_{+,\rho}(\mu)(q(T_+) + w(T_+) - p_\rho(\mu)) \\ \tilde{R}_-((q,w)(T_-),\mu) &= P_{-,\rho}(\mu_\rho) P_{-,\rho}(\mu)(q(T_-) + w(T_-) - p_\rho(\mu)). \end{aligned} \tag{6.2}$$

We first prove that (6.1–6.2) has a unique solution. Using Remark 2.1, it is straightforward to show that (6.2) satisfies Hypothesis (T1). For instance, for $\mu = \mu_\rho$,

$$D_u \tilde{R}_+(p_\rho(\mu_\rho),\mu_\rho)\Big|_{R(P_{\pm,\rho}(\mu_\rho))} = P_{\pm,\rho}(\mu_\rho)\Big|_{R(P_{\pm,\rho}(\mu_\rho))}$$

which is clearly invertible as an operator into $R(P_{+,\rho}(\mu_\rho))$. Therefore, this operator remains invertible for μ close to μ_ρ with uniform inverse. The same argument applies to the derivative of the second boundary condition. Hence, Theorem 2 applies, and (6.1–6.2) has a unique solution.

To finish the argument, we observe that any solution (q,w) of (6.1–6.2) has necessarily $w = 0$. Indeed, w has to satisfy

$$\dot{w} = Aw, \qquad \hat{P}_+ w(T_+), \qquad \hat{P}_- w(T_-) = 0.$$

Since $A|_{N(Q_\rho)}$ is hyperbolic, $w = 0$ is the only solution. With $w = 0$, it is easy to see that (6.1–6.2) and (2.6–2.8) coincide. Hence, $(q, w) = (q, 0)$ satisfies (6.1–6.2) if, and only if, q is a solution of (2.6–2.8).

Finally, we have

$$|R_{+,\rho}(h(T_+), 0)|_\alpha \leq C|h(T_+), 0)|_\alpha^2 \leq Ce^{-2\lambda^s T_+},$$

and the analogous estimate for $R_{+,\rho}(h(T_+), 0)$. This completes the proof of Theorem 3.

7 Semilinear Elliptic Equations

Here, we show that elliptic equations on infinite cylinders are included in the abstract set-up of the earlier sections. We refer to [26] for more details. Furthermore, we comment on the satisfaction of the hypotheses of Theorems 1 and 2, and discuss particular discretizations.

Let Y be a Hilbert space and $L : D(L) \subset Y \to Y$ a densely defined, positive definite, self-adjoint operator with compact resolvent. In most application, we have $Y = L^2(\Omega)$ for some bounded domain Ω and $L = -\Delta$ on Ω together with Dirichlet boundary conditions, say, so that $D(L) = H^2(\Omega) \cap H_0^1(\Omega)$. We denote the fractional power spaces associated with L by Y^α. In particular, $Y^1 = D(L)$. Suppose that

$$g : Y^{\frac{1+\alpha-\epsilon}{2}} \times Y^{\frac{\alpha-\epsilon}{2}} \to Y$$

is a nonlinearity of class C^2 for some $\alpha \in [0, 1)$ and $\epsilon > 0$. Consider the abstract elliptic equation

$$u_{xx} - Lu = g(u, u_x), \qquad x \in \mathrm{R} \tag{7.1}$$

for $u \in Y^\alpha$. We reformulate (7.1) as the first-order equation

$$\frac{d}{dx}v = Av + G(v) \tag{7.2}$$

with $v = (u, u_x)$ and $G(v) = (0, g(v))$. Here,

$$A = \begin{pmatrix} 0 & \mathrm{id} \\ L & 0 \end{pmatrix} : Y^1 \times Y^{\frac{1}{2}} \to Y^{\frac{1}{2}} \times Y.$$

In particular, Hypothesis (A1) is met; in fact, the projections $\hat{P}_\pm$ are given by

$$\hat{P}_\pm = \frac{1}{2} \begin{pmatrix} \mathrm{id} & \pm L^{-\frac{1}{2}} \\ \pm L^{\frac{1}{2}} & \mathrm{id} \end{pmatrix} : Y^{\frac{1}{2}} \times Y \to Y^{\frac{1}{2}} \times Y.$$

The fractional power spaces are $X^\alpha = Y^{\frac{1+\alpha}{2}} \times Y^{\frac{\alpha}{2}}$. The mapping $G : X^\alpha \subset X^{\alpha-\epsilon} \to X$ is C^2 since g is. It is also clear that A has compact resolvent whenever L has.

Therefore, Hypotheses (A1), (A2) and (C) are met. We refer to [8, Satz 5] for conditions guaranteeing that Hypothesis (A3) is met. Given a particular solitary-wave solution of such an elliptic system, hyperbolicity of equilibria (H1) and transverse unfolding (H3) are generic properties, at least if we allow for nonlinearities of the form $g(y, u, u_x, \nabla_y u, \mu)$.

In order to apply our results to concrete problems, we have to choose a discretization in the cross-section, corresponding to the projectors Q_ρ, and boundary conditions at $x = T_-$ and $x = T_+$. For elliptic equations (7.1), it is convenient to choose Q_ρ with $\rho \in \{1/k;\, k \in \mathrm{N}\}$ as the orthogonal Galerkin projections onto the first m eigenfunctions of L. Condition (Q) is then an immediate consequence of the completeness of the orthonormal system of eigenfunctions.

The choice of boundary conditions R turns out to be less evident in general, as the projectors $P_{+,\rho}$ and $P_{-,\rho}$ might be hard to compute. We emphasize here that, in general, simple Dirichlet boundary conditions $u(T_\pm) = p$ or Neumann boundary conditions $v(T_\pm) = 0$ will not work. Even for systems of equations on the line with no cross-section, that is $Y = \mathrm{R}^{2k}$, the dimensions of stable and unstable subspaces at the equilibrium may not coincide: $\dim R(P_+) \neq k$. Then Dirichlet as well as Neumann boundary conditions yield ill-posed problems. The only generic choice then seems to be given through periodic boundary conditions — or the actual computation of P_+. However, there are important cases where Dirichlet and Neumann conditions work. Examples are reversible systems or equations of variational type, which we now discuss in more detail.

If $g = g(u)$, then the system is reversible. Reversibility acts through $S(u, v) = (u, -v)$. The condition $(\mathrm{id} - S)u(0)$ in (2.10) reduces to $v = 0$, in other words, Neumann boundary conditions at $x = 0$. The hyperbolicity assumption (H1) is then equivalent to linear stability of the equilibrium $u(x, y) = p(y)$ for the parabolic equation $u_t = u_{xx} - Lu - g(u)$ on the cylinder. Due to the second-order structure, eigenfunctions of the linearization of (7.1) at the equilibrium are of the form $(u_k, \pm\sqrt{\lambda_k} u_k)$ where λ_k and u_k are eigenvalues and eigenfunctions, respectively, of $L + Dg(p)$. By hyperbolicity, $\lambda_k > 0$. We claim that we can choose Dirichlet or Neumann conditions at $x = T_+$ as well. Indeed, the stable subspace $R(P_-)$ is spanned by $(u_k, \sqrt{\lambda_k} u_k)$ and the spaces $\{(u, v);\, u = 0\}$ or $\{(u, v);\, v = 0\}$ are closed complements of this subspace. We summarize this result in the following proposition.

Proposition 1 *Assume that (H1), (H2) and (H4) are met. Furthermore, suppose that $g = g(u)$. Dirichlet and Neumann boundary condition then satisfy (T2).*

These arguments can be slightly generalized to elliptic equations with variational structure

$$u_{xx} = Lu + cu_x + \nabla F(u),$$

where heteroclinic orbits connecting stable equilibria are of interest. Again stability is with respect to the linearization of the associated parabolic problem in the infinite cylinder. Though this system is not reversible, a calculation similar to the one given above shows that Dirichlet or Neumann boundary conditions at $x = T_-$ and $x = T_+$ satisfy Hypothesis (T1)(ii) on the boundary conditions.

We remark that Corollaries 3 and 4 establish the existence of solutions of (7.1) which are periodic in x with arbitrarily large period and have the same profile in the cross-section as the solitary wave.

8 Numerical Simulations

In this section, we compare the theoretical predictions with numerical computations. Consider the elliptic equation

$$u_{xx} + u_{yy} + cu_x = u(1 + 2p - u) + p_{yy} - p(1 + p), \qquad (x, y) \in \mathrm{R} \times (-1, 1), \quad (8.1)$$

for $u \in \mathrm{R}$ with Neumann boundary conditions

$$u_y(x, \pm 1) = 0, \qquad x \in \mathrm{R}. \quad (8.2)$$

For the function $p(y)$, we take the polynomial $p(y) = (1 + y)^2(1 - y)^2$ which clearly satisfies $p_y(\pm 1) = 0$. Note that $p(y)$ satisfies (8.1–8.2) for any c. Furthermore, we have the explicit solitary wave

$$h(x, y) = p(y) + \frac{3}{2} \operatorname{sech}^2 \left(\frac{1}{2}x\right), \quad (8.3)$$

of (8.1) for $c = 0$. We write (8.1) as the first-order system

$$\frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -u_{yy} - cv + u(1 + 2p - u) + p_{yy} - p(1 + p) \end{pmatrix}, \quad (8.4)$$

in x, where $(u, v) \in H^1(-1, 1) \cap L^2(-1, 1)$.

It turns out that Hypotheses (A1)–(A3), (H1)–(H3) and (C) are satisfied with respect to the parameter c; see Section 7 and [26, Section 6.3]. Alternatively, we may fix $c = 0$. Equation (8.4) is then reversible with $S(u, v) = (u, -v)$, and Hypotheses (R)(i), (R)(ii) and (H4) are met.

8.1 Projection Boundary Conditions

The even eigenfunctions and corresponding eigenvalues of the linearization of (8.4) at $(p, 0)$ are given by

$$q_{\pm k}(y) = \begin{pmatrix} (1 + \pi^2 k^2)^{-\frac{1}{2}} \\ \pm 1 \end{pmatrix} \cos k\pi y \quad \text{and} \quad \lambda_{\pm k} = \pm\sqrt{1 + \pi^2 k^2}, \quad (8.5)$$

respectively, for $k \in \mathbb{N}_0$. We consider the Galerkin projection

$$Q_n = \sum_{k=-n}^{n} \langle q_k, \cdot \rangle_{H^1 \times L^2}, \tag{8.6}$$

which clearly satisfies Hypothesis (Q).

The Fourier series of the polynomial $p(y)$ is given by

$$p(y) = \frac{16}{15} + \frac{48}{\pi^4} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4} \cos k\pi y,$$

and

$$|(\mathrm{id} - Q_n)p|_{C^0} \approx n^{-4}(an + b) \tag{8.7}$$

for some positive numbers a and b.

We then solve the system

$$\begin{aligned}
\frac{d}{dx}\begin{pmatrix} u \\ v \end{pmatrix} &= Q_n \begin{pmatrix} v \\ -u_{yy} - cv + u(1 + 2p - u) + p_{yy} - p(1+p) \end{pmatrix}, \\
0 &= \int_{-T}^{T} \Big\langle Q_n(h_x, h_{xx})(x), (u,v)(x) - Q_n(h_x, h_{xx})(x) \Big\rangle_{L^2 \times L^2} dx, \\
0 &= Q_{+,n}(c)\Big((u,v)(T) - (p_n(c), 0)\Big), \\
0 &= Q_{-,n}(c)\Big((u,v)(-T) - (p_n(c), 0)\Big),
\end{aligned} \tag{8.8}$$

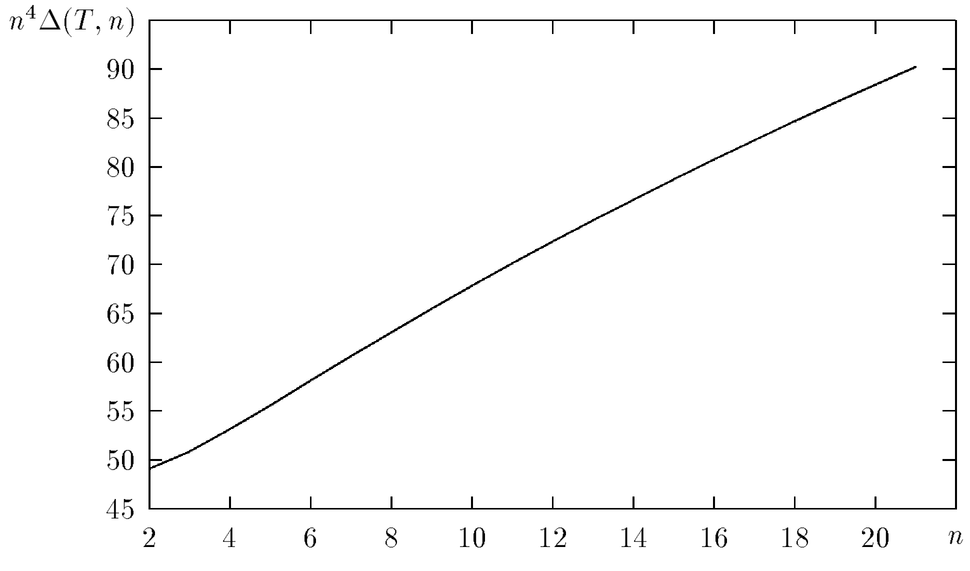

Figure 1: This plot contains the scaled error $n^4\Delta(T,n)$ versus the number n of Galerkin modes for the solution of (8.8) for fixed length $T = 15.0$ of the truncation interval. In this scaling, the error is a linear function of n, see (8.9).

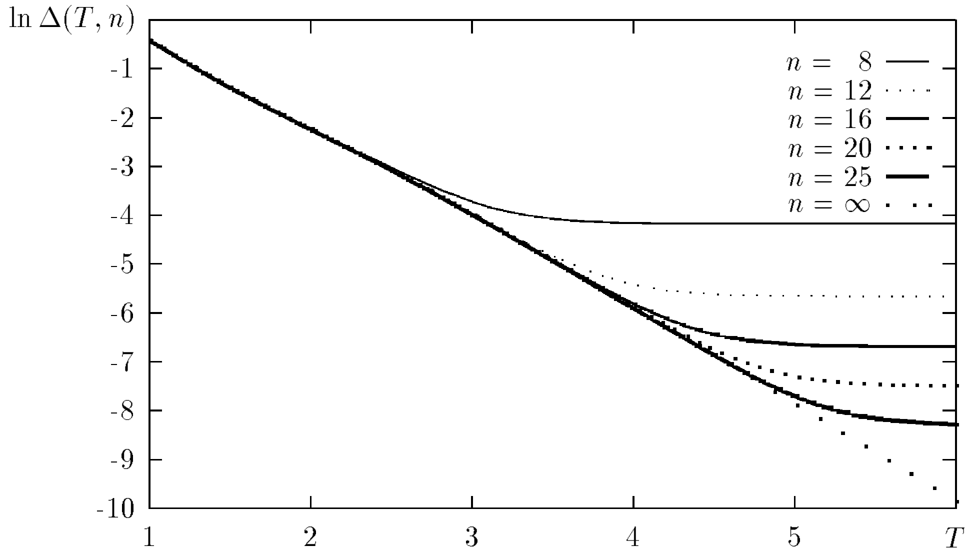

Figure 2: Here, the scaled error $\ln\Delta(T,n)$ versus the length T of the interval $(-T,T)$ for the solution of (8.8) is shown. For small T, the scaled error is then linear in T with slope -1.8 which is in agreement with the prediction of -2 in Theorem 3. For larger values of T, the error due to the Galerkin truncation becomes dominant. As expected, the remaining error is smaller for a larger number of modes. Also, since the error curves for different values of n are not shifted against each other, the picture confirms that the constant C appearing in (8.9) is independent of n. The case $n=\infty$ corresponds to setting $p(y)=0$ which demonstrates the error induced purely by the truncation of the interval.

on $(-T,T)$ with $(u,v)\in R(Q_n)$; see Section 2.4. Hypothesis (T1) is met, whence Theorem 3 is applicable. Therefore, the difference $\Delta(T,n)$ of the true solution $h(x,y)$ given in (8.3) and the solution $\bar{h}_n(x,y)$ of (8.8) can be estimated by

$$\begin{aligned}\Delta(T,n) &= \sup\{|\bar{h}_n(x,y)-h(x,y)|;\,(x,y)\in(-T,T)\times(-1,1)\} \qquad (8.9)\\ &\leq C\Big(e^{-2T}+\sup\{|(\mathrm{id}-Q_n)(0,p)(x,y)|;\,(x,y)\in(-T,T)\times(-1,1)\}\Big)\\ &\approx C(e^{-2T}+n^{-4}(an+b)),\end{aligned}$$

using the expression for the eigenvalues given in (8.5).

The boundary-value problem (8.8) is now solved using AUTO97, see [9], for various choices of T and n. The results of the numerical simulations are plotted in Figures 1 and 2. They confirm the theoretical error estimate (8.9).

Note that the residual $(\mathrm{id}-Q_n)(p_{yy}-p(1+p))$ on the right-hand side of (8.8) is actually of the order $\frac{1}{n}$. The error, however, is induced by the approximation of the true solution using Galerkin modes which is of the order $\frac{1}{n^3}$.

8.2 Neumann Boundary Conditions

Next, we consider the approximation

$$\begin{aligned} u_{xx} + u_{yy} &= u(1+2p-u) + p_{yy} - p(1+p), \qquad && (x,y) \in (0,T)\times(-1,1), \\ u_y(x,\pm 1) &= 0, && x \in (0,T), \\ u_x(0,y) = u_x(T,y) &= 0, && y \in (-1,1). \end{aligned} \tag{8.10}$$

Hypothesis (T2) is met and Theorem 4 applies with $Q_\rho \equiv \mathrm{id}$. Hence, the difference $\Delta(T)$ of the solution $h(x,y)$ given in (8.3) and the solution $\bar{h}(x,y)$ of (8.10) can be estimated by

$$\Delta(T) = \sup\{|\bar{h}(x,y) - h(x,y)|;\, (x,y) \in (-T,T)\times(-1,1)\} \le Ce^{-T} \tag{8.11}$$

using the expression for the eigenvalues given in (8.5).

We used second-order centered finite differences on a staggered grid with N_x horizontal and N_y vertical mesh points in order to solve (8.10). For the resulting equation on the grid, we employed a conjugated-gradient solver (without preconditioning) together with Newton's method. The difference of the associated solution $\bar{h}_N$ and the true solution h is denoted by $\Delta(T,N)$ where $N = (N_x, N_y)$. The results of the numerical simulations are shown in Figure 3. Again, the theoretical predictions of Theorem 4 are in good agreement with the computations.

For comparison, we also computed solutions of

$$\begin{aligned} u_{xx} + u_{yy} &= u(1+2p-u) + p_{yy} - p(1+p), \qquad && (x,y) \in (0,T)\times(-1,1), \\ u_y(x,\pm 1) &= 0, && x \in (0,T), \\ u_x(0,y) &= 0, && y \in (-1,1), \\ u_x(T,y) + u(T,y) - p(y) &= 0, && y \in (-1,1). \end{aligned} \tag{8.12}$$

These are the projection boundary conditions. The error is therefore expected to behave like e^{-2T} by Theorem 4; see Figure 4.

9 An Application to the von Kármán–Donnell Equations

As mentioned in the introduction, we consider the post buckling of an infinitely long cylindrical shell under axial compression as modeled by the von Kármán–Donnell equations. In [22, 23] and [24] solitary-waves were computed and it was shown that these solutions provide a good approximation to the localized buckling pattern observed in experiments. Here, we indicate how the proofs of Sections 4 and 5 may be adapted to this case and show

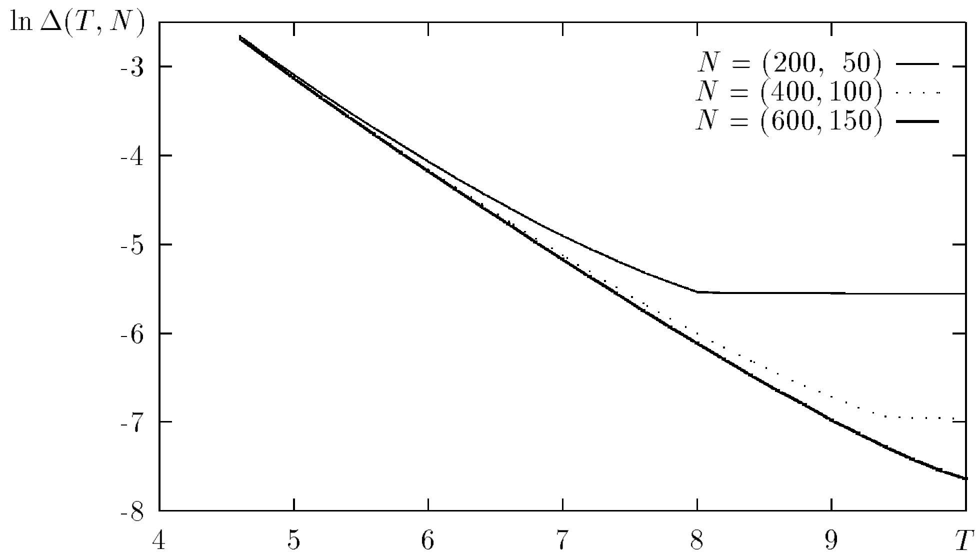

Figure 3: The scaled error $\ln \Delta(T,N)$ versus the length T of the interval $(-T,T)$ for the solution of (8.10) is shown. Here, $N = (N_x, N_y)$ is the number of horizontal and vertical grid points. For small T, the scaled error is linear in T with slopes of -0.86, -0.99 and -1.01. The slope predicted in Theorem 4 is -1. If T is large enough, the error due to the approximation with finite differences becomes dominant; again, the remaining error is smaller for a larger number of grid points.

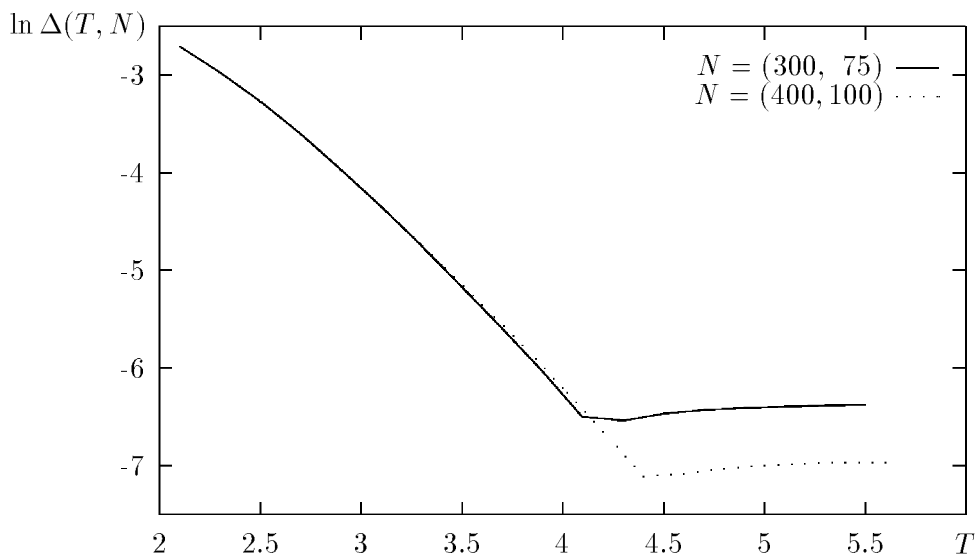

Figure 4: Here, the error for the solution of (8.12) is plotted. For small T, the scaled error is linear in T with slope -1.86; Theorem 4 predicts a slope of -2.

numerically that, for a fixed spatial truncation, the error in the truncation on the length of the cylinder scales in accordance with Theorem 4.

9.1 The von Kármán–Donnell Equations

The classical formulation for a thin cylindrical shell of radius r and thickness t is given by the von Kármán–Donnell equations:

$$\begin{aligned} \kappa^2\nabla^4 w + \lambda w_{xx} - \rho\phi_{xx} &= w_{xx}\phi_{yy} + w_{yy}\phi_{xx} - 2w_{xy}\phi_{xy} \\ \nabla^4\phi + \rho w_{xx} &= (w_{xy})^2 - w_{xx}w_{yy} \end{aligned} \tag{9.1}$$

where ∇^4 is the two-dimensional bi-harmonic operator, $x \in \mathrm{R}$ is the axial and $y \in [0, 2\pi r)$ is the circumferential coordinate, w is the outward radial displacement measured from an unbuckled state, and ϕ is a stress function [20]. Parameters appearing in (9.1) are the curvature, $\rho = 1/r$, the geometric constant, $\kappa^2 = t^2/12(1-\nu^2)$, where ν is Poisson's ratio, and loading parameter λ. Localized buckle patterns are observed and these are well approximated by a solitary wave in x, see [22, 24] and Figure 5.

We discretize the von Kármán–Donnell equations (9.1) in such a way as to exploit the natural symmetries in the problem. Experimentally a well defined number, s, of periodic waves is observed circumferentially [20, 33] in the buckled deformation, corresponding to an invariance under rotation of $2\pi/s$. Hence we write

$$w(x,y) = \sum_{m=0}^{\infty} a_m(x)\cos(ms\rho y); \;\; \phi(x,y) = \sum_{m=0}^{\infty} b_m(x)\cos(ms\rho y), \;\; s \in \mathrm{N}.$$

Substituting into the von Kármán–Donnell equations and taking the L^2 inner product with $\cos(ms\rho y)$, we obtain a system of nonlinear ODEs for the Fourier modes a_m and b_m for $m = 0, \ldots, \infty$. The Galerkin approximation is formed by taking $m = 0, \ldots, M-1$ for some finite M giving a system of $8M$ first-order ordinary differential equations. We may formally write the resulting set of ODEs as

$$a_m^{\mathrm{iv}} = F_{1,m}(a_m, a_m^{\mathrm{i}}, a_m^{\mathrm{ii}}, a_m^{\mathrm{iii}}, b_m, b_m^{\mathrm{i}}, b_m^{\mathrm{ii}}, b_m^{\mathrm{iii}}); \;\; b_m^{\mathrm{iv}} = F_{2,m}(a_m, a_m^{\mathrm{i}}, a_m^{\mathrm{ii}}, a_m^{\mathrm{iii}}, b_m, b_m^{\mathrm{i}}, b_m^{\mathrm{ii}}, b_m^{\mathrm{iii}}),$$

where superscripts denote differentiation with respect to x.

Note that $s = 1$ corresponds to the standard Galerkin approximation. Convergence as M is increased was examined numerically in [23] and it was found that $M = 6$ gives a reasonable compromise between accuracy and computation efficiency.

Experimentally, the observed buckle patterns tend to be *cross-symmetric* about a section $x = T/2$ that is

$$w(x,y) = w(T-x, y+\pi r/s) \qquad \& \qquad \phi(x,y) = \phi(T-x, y+\pi r/s). \tag{9.2}$$