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Forced symmetry breaking of homoclinic cycles

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Abstract. We consider two equivariant equations admitting structurally stable heteroclinic cycles. These equations stem from mode equations for the Rayleigh–Bénard convection and a model for turbulent layers in wall regions with riblets. Breaking the symmetry causes several different bifurcations to occur which can be explained by bifurcations of codimension two of homoclinic orbits for non-symmetric systems. In particular, stable periodic solutions of different symmetry type, other complicated heteroclinic cycles or geometric Lorenz attractors may emanate. Moreover, we delevop stability criteria for the bifurcating periodic solutions. In general, their stability type differs from the stability properties of the original heteroclinic cycle.

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1. Introduction

During the last ten years many attempts have been made to explain various kinds of intermittent behaviour in dynamical processes with the help of heteroclinic cycles. In differential equations which are invariant under the action of certain symmetry groups, heteroclinic cycles may appear for open ranges of parameter values. In other words, cycles can be structurally stable (codimension 0) within the class of invariant differential equations, whereas in generic, non-symmetric equations, one has to adjust at least one parameter (codimension 1) in order to observe homoclinic orbits or heteroclinic cycles to equilibria. Of course, the heteroclinic cycles found in equivariant systems are, in contrast to hyperbolic equilibria or periodic orbits, not robust under small symmetry breaking perturbations.

Orbits close to heteroclinic cycles will spent long time periods near the stationary states of the cycle and will spontaneously, in a bursting-like event, leave the stationary state and approach another one where they will again remain for long time periods. However, if the cycle is asymptotically stable, this intermittent behaviour will become slower; the time spent near the stationary states will approach infinity.

In the present work we try to capture some of the main features of symmetry breaking effects on heteroclinic cycles. We will, most of the time, restrict ourselves to two examples which are at the heart of many other cycles. The first example is a heteroclinic cycle with tetrahedral symmetry which was discovered by Busse and Clever [BC79] in a model for

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intermittent behaviour of convection rolls in rotational invariant Rayleigh–Bénard convection. The importance of studying symmetry breaking effects in this model when considering deviations from the Boussinesq approximation was already pointed out by [Swi84].

The other example arises in an $\mathcal{O}(2)$-mode interaction, studied by Armbruster *et al* [AGH88] and Proctor and Jones [PJ88]. Some attempts to study the influence of symmetry breaking were made by Campbell and Holmes [CH92]. In these two examples we discover several phenomena which can be attributed to homoclinic bifurcations in generic non-symmetric vector fields. However, these bifurcations appear with codimension zero in our examples while they are of codimension two and higher in the non-symmetric context. Nevertheless we need two parameters to unfold these bifurcations even in equivariant equations. We show that asymptotic stability of bifurcating periodic orbits is not equivalent to stability of the cycle! As far as possible we tried to give lists of stability properties of the bifurcating periodic orbits. In many cases shift dynamics occur for open ranges of parameter values. The bifurcating orbits are intermittent in the sense that they spend long time intervals near equilibria. Moreover, intermittency is sustained (it does not slow down as for asymptotically stable cycles) by the imperfection of the symmetry.

Let us briefly explain how this paper is organized. We first collect some basic aspects of heteroclinic cycles in equivariant differential equations. In section 3 we present the setting of our two major examples. In particular we will analyse the existence and stability of heteroclinic cycles. In section 4, we state our main results on symmetry breaking bifurcations, which will be proved in section 6. Before going to the proofs, we give a short summary of results on bifurcations from generic, non-symmetric, homoclinic orbits, which we will use in our proofs. In section 7 we show how symmetry breaking may also lead to chaotic behaviour of Lorenz-like attractors. Shift dynamics are encoded by itineraries in the cycle—in contrast to the encoding by return times in the previous sections. We will conclude with a discussion of some solved and unsolved problems in the theory around symmetric heteroclinic cycles.

2. Equivariance, heteroclinic cycles and symmetry breaking

2.1. Equivariant ODE's

We study differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = f(u(t)) \qquad u \in \mathbb{R}^n \tag{2.1}$$

with smooth $f \in C^l(\mathbb{R}^n)$, l sufficiently large. The vector field f is supposed to be equivariant with respect to a finite subgroup Γ of $\mathcal{O}(n)$, the group of orthogonal $n \times n$-matrices, which means that for any $\gamma \in \Gamma$ we have

$$f(\gamma u) = \gamma f(u) \qquad \text{for all } u \in \mathbb{R}^n\,.$$

Due to the equivariance, for any solution $u(t)$ of (2.1), $\gamma u(t)$ is also a solution. The isotropy subgroup of a point $p \in \mathbb{R}^n$ will be denoted by

$$G_p = \{\gamma \in \Gamma \mid \gamma p = p\}\,.$$

Note that points on a time orbit of (2.1) all have the same isotropy group whereas the isotropy groups of points on a group orbit $\{\gamma p \mid \gamma \in \Gamma\} = \Gamma p$ are conjugated

$$G_{\gamma p} = \gamma G_p \gamma^{-1}\,.$$

Given an isotropy subgroup G, the linear subspace

$$\mathrm{Fix}(G) = \{p \in \mathbb{R}^n \mid \gamma p = p \text{ for all } \gamma \in G\}$$

is called the fixed point subspace of G. As time preserves isotropy, fixed point subspaces are flow invariant.

2.2. *Structurally stable heteroclinic cycles*

Suppose that p_0 and p_1 are equilibria of (2.1) and that there exists a heteroclinic orbit $q(t)$ that connects p_0 to p_1. If $p_1 = \sigma p_0$ for some $\sigma \in \Gamma$, we call the set $\Gamma q(t) \cup \Gamma p_0$ a homoclinic cycle [MCG89, definition 2.1.], in fact, as Γ is finite, the sequence $(\sigma^k q(t))_{0 \leqslant k \leqslant N-1}$ will form a closed cycle joining the equilibria $\sigma^k p_0$, where N is such that $\sigma^N p_0 = p_0$. On the other hand, in the quotient space $\mathbb{R}^n/\Gamma$, the cycle is just a homoclinic orbit to the equilibrium $[p_0]$. In generic dynamical systems, homoclinic orbits are a codimension-one phenomenon as the intersection of stable and unstable manifold is of codimension one at least. In the class of equivariant dynamical systems, homoclinic cycles may be structurally stable. The intersection of stable and unstable manifolds might be transverse in a fixed point subspace. *We call the cycle structurally stable, if there exists* $\Sigma = \mathrm{Fix}(G)$ *such that* $q(t) \in \Sigma$ and $(W^u(p_0) \cap \Sigma)$ intersects $(W^s(p_1) \cap \Sigma)$ transversely in Σ. In proposition 4.1, we will find another way to express this property. In the literature many examples of homoclinic cycles have been found in local steady-state bifurcations with various symmetries. Asymptotic stability conditions have also been derived [KM91].

2.3. *Symmetry breaking*

We are particularly interested in situations where a dynamical system is close to a symmetric one. Let us therefore assume that the vector field f depends on a parameter $f = f(\varepsilon, u)$, $\varepsilon \in \mathbb{R}^k$ and that $f(0, \cdot)$ is equivariant with respect to Γ, but $f(\varepsilon, \cdot)$ only with respect to some subgroup $H \leqslant \Gamma$ for $\varepsilon \neq 0$. A lot of issues of local bifurcations which appear for $\varepsilon = 0$ will persist for $\varepsilon \neq 0$, such as hyperbolic equilibria or periodic orbits. Perturbations of heteroclinic cycles have recently been studied in the context of symmetry breaking. The perturbed, H-equivariant flow ($\varepsilon \neq 0$) might not possess invariant fixed-point subspaces which ensure structural stability of the cycle. Up to now we have tried to show persistence of some kind of recurrent dynamics [Mel89] or existence of periodic orbits [Cho92], [CF92], [Swi84], [Sch91]. Unicity of periodic orbits is rarely known. We will try to give a more detailed description of the dynamics in the neighbourhood of the homoclinic cycle. To this aim, we reduce the bifurcations of the symmetric homoclinic cycle to the investigation of homoclinic orbits in generic systems defined on the space of group orbits. The interesting issue that comes up is that the non-symmetric generic bifurcations we have to study are of codimension two, therefore the easiest symmetry breaking unfolding of the homoclinic cycle in an equivariant generic codimension zero situation already requires two or even more parameters.

3. Two examples

3.1. *Tetrahedral symmetry*

Here we consider the irreducible representation of the group $\mathbb{T} \oplus \mathbb{Z}_2$ on $\mathbb{R}^3$. It is generated by a reflection

$$\kappa = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and a rotation

$$\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

In particular, coordinate planes and axes are invariant under the flow of a $\mathbb{T} \oplus \mathbb{Z}_2$-equivariant vector field. An instability of the zero solution of such a dynamical system is described by the following third-order polynomial vector field:

$$\begin{aligned} \dot{x} &= \lambda x + x(ax^2 + by^2 + cz^2) \\ \dot{y} &= \lambda y + y(ay^2 + bz^2 + cx^2) \\ \dot{z} &= \lambda z + z(az^2 + bx^2 + cy^2). \end{aligned} \tag{3.1}$$

On the coordinate axes, the equilibrium $p_0 = (\sqrt{-\frac{\lambda}{a}}, 0, 0)$ bifurcates from the origin, together with its group orbit for $\frac{\lambda}{a} < 0$. Existence of homoclinic cycles is guaranteed by the following lemma:

Lemma 3.1. *Consider equation (3.1) with $a < 0$ and $\lambda > 0$. Then if and only if $b < a < c$ or $c < a < b$, there exists a heteroclinic orbit, connecting p_0 to $\sigma p_0 = (0, 0, \sqrt{-\frac{\lambda}{a}})$ or $\sigma^2 p_0 = (0, \sqrt{-\frac{\lambda}{a}}, 0)$, respectively.*

The cycle is asymptotically stable if and only if $2a > b + c$.

Proof. Suppose $b < a < c$ (otherwise interchange y and z). The eigenvalues of the linearization around p_0 are

$$\mu_x = -2\lambda \qquad \mu_y = \lambda\left(1 - \frac{c}{a}\right) \qquad \mu_z = \lambda\left(1 - \frac{b}{a}\right). \tag{3.2}$$

The equilibrium p_0 is a saddle in the directions transverse to the x-axis under the condition $b < a < c$. Its unstable manifold is included in the xz-plane, where σp_0 is stable. It is therefore sufficient to show that orbits remain bounded and that no mixed modes exist in the xz-plane. Rescaling time and (x, y, z), we can arrange to have $a = -1$.

As $b < 0$, it is easily seen that $0 \leqslant x \leqslant \sqrt{\lambda}$ is forward invariant in the xz-plane. This implies also that $\dot{z} \leqslant z(\lambda - z^2 + \max(b, 0) \cdot \lambda)$ and that therefore z also stays bounded.

Looking for mixed modes, we have to solve

$$\lambda - x^2 + cz^2 = 0 \qquad \lambda - z^2 + bx^2 = 0$$

but the unique solution (x^2, z^2) of this linear system is not positive when $b < -1 < c$.

In case the assumptions of the lemma are not satisfied, p_0 will be stable (or unstable) in the y- and z-directions and therefore no cycle can occur.

Asymptotic stability conditions follow from [KM91, theorem 4.1]. They show that one can neglect the stable radial eigenvalue μ_x and $\mu_z < -\mu_y$ is equivalent to $2a > b + c$. □

This system of equations was first considered by Busse and Clever [BC79] as a model for a planar rotationally invariant Rayleigh–Bénard problem, where x, y and z model the amplitudes of the three dominating convection rolls which can be obtained by a rotation of $2\pi/3$ from each other. Later, Guckenheimer and Holmes [GH88] showed the existence of structurally stable and asymptotically stable homoclinic cycles in these equations. First attempts to study symmetry breaking phenomena were made by Swift [Swi84] who considered a non-Boussinesq approximation to the Rayleigh–Bénard problem which involved symmetry breaking from $\mathbb{T} \oplus \mathbb{Z}_2$ to $\mathbb{T}$.

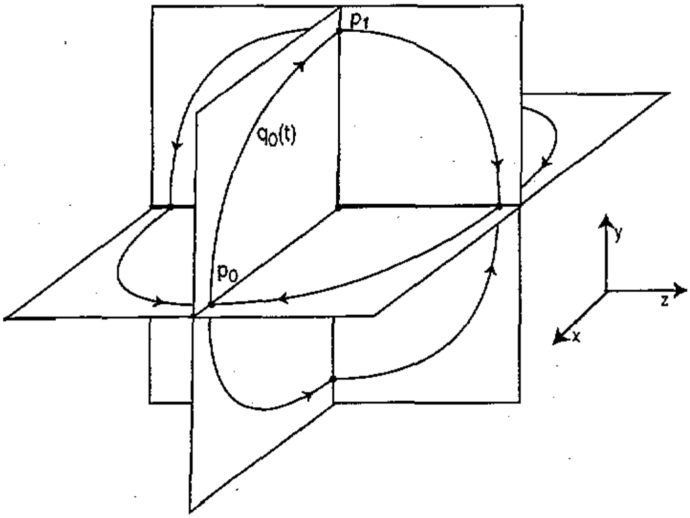

Figure 1. A homoclinic cycle with $\mathbb{T} \oplus \mathbb{Z}_2$-symmetry in $\mathbb{R}^3$.

We also study the slightly generalized problem where two $\mathbb{T}$-modes are coupled: in a fourth space dimension ξ the element κ is supposed to act as $-id$, σ should act trivially. The unfolding of the zero solution is then a codimension two problem, governed by the system of four differential equations

$$\begin{aligned}
\dot{x} &= \lambda x + x(ax^2 + by^2 + cz^2) + d\xi yz \\
\dot{y} &= \lambda y + y(ay^2 + bz^2 + cx^2) + d\xi xz \\
\dot{z} &= \lambda z + z(az^2 + bx^2 + cy^2) + d\xi xy \\
\dot{\xi} &= \nu\xi + e\, xyz - \xi^3 .
\end{aligned} \tag{3.3}$$

Note that the dynamics in the coordinate planes, where the homoclinic cycle lies, remain unchanged. Asymptotic stability of the cycle is guaranteed by the additional assumption $\nu < 0$ (in the terminology of [KM91], ξ is a 'transverse' direction).

Our main results describe the dynamics of systems of differential equations which are close to these three- (or four-) dimensional equations but do only possess less symmetry H, namely $H = \mathbb{T}$, $H = \mathbb{Z}_3$ generated by σ or $H = \mathbb{Z}_6$, generated by $\kappa\sigma$. Besides motivation by Swift's work on the non-Boussinesq case, one can see that these subgroups will reveal the most interesting phenomena—by breaking the cycle—but, nevertheless, permit a detailed study. As the different equilibria of the cycle lie on one group orbit of $H = \mathbb{T}$, the unfolding of the cycle can be described by a minimal number of parameters.

3.2. D_4-symmetry

Our second example is concerned with homoclinic cycles which bifurcate from the origin in D_4-equivariant systems of differential equations. Two reducible representations of D_4 are considered

(i) D_4 acting on $\mathbb{R}^3$ as $D_4^d \leqslant \mathcal{O}(3)$ (the twisted subgroup of $D_4 \oplus \mathbb{Z}_2$, cf [GSS88] for notation) with generators

$$\kappa = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \sigma = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(ii) D_4 acting on $\mathbb{R}^4$ as a subgroup of $\mathcal{O}(2)$ which acts via its $(l = 1, l = 2)$ representation [CH92], [AGH88] with generators

$$\kappa = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \text{and} \qquad \sigma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Note that in (ii), a generic steady-state bifurcation from the origin is already of codimension three. It is motivated by bursting phenomena in boundary layers of fluids, where a spanwise translational invariance of the problem is broken by introducing small riblets in the wall region. Equidistance in the partition of the riblets corresponds to a $\mathbb{Z}_k \leqslant \mathcal{O}(2)$ symmetry, a reflectional symmetry with respect to the centre line produces D_k-symmetry in the equations. The choice $k = 4$ corresponds to the experimental setting [CH92], [ALH90]. We are particularly interested in the dynamical phenomena which might occur when reflectional symmetry is broken, for example, because the riblets are not well centred.

The importance of the 3d-model is that it is the core of both, $\mathcal{O}(2)$ and D_4-mode couplings. Besides the $\mathbb{T} \oplus \mathbb{Z}_2$ symmetric cycle, it is the only homoclinic cycle which can be forced to exist by symmetry in $\mathbb{R}^3$ [Sch91]. Many dynamical questions are already exhibited when considering this 3d-model. The codimension-two steady-state bifurcation is determined by the third-order truncated system

$$\begin{aligned} \dot{x} &= \lambda x + dxz + x(ax^2 + by^2 + cz^2) \\ \dot{y} &= \lambda y - dyz + y(ay^2 + bx^2 + cz^2) \\ \dot{z} &= \nu z + \gamma(x^2 - y^2) + z(\alpha z^2 + \beta(x^2 + y^2)) . \end{aligned} \tag{3.4}$$

Note that the z-axis and the xz-coordinate plane are flow invariant. In the xz-plane, we have a reflection symmetry $x \to -x$. The yz-plane is conjugate to the xz-plane via σ.

By a suitable rescaling we can suppose $d = 1$, $\alpha = -1$ and $|\gamma| = 1$. For $\gamma > 0$, no cycle can exist [AGH88] and we may set $\gamma = -1$. On the z-axis, the equilibrium $p^+ = (0, 0, \sqrt{\nu})$ bifurcates from the origin, together with its symmetric $p^- = (0, 0, -\sqrt{\nu}) = \sigma p^+$. For some values of λ and ν there are four other equilibria, called mixed modes (MM), which lie in the coordinate planes $x = 0$ and $y = 0$, all on the same group orbit. We are interested in

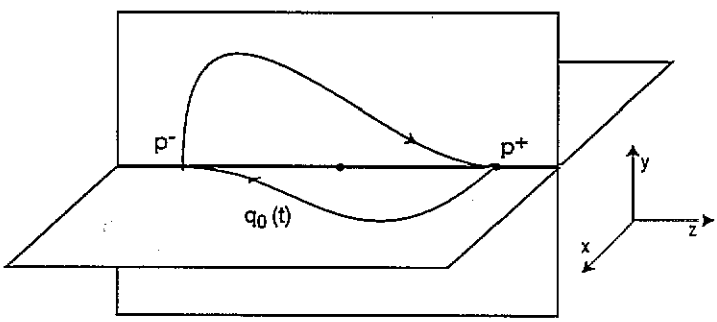

Figure 2. A homoclinic cycle with D_4^d-symmetry in $\mathbb{R}^3$.

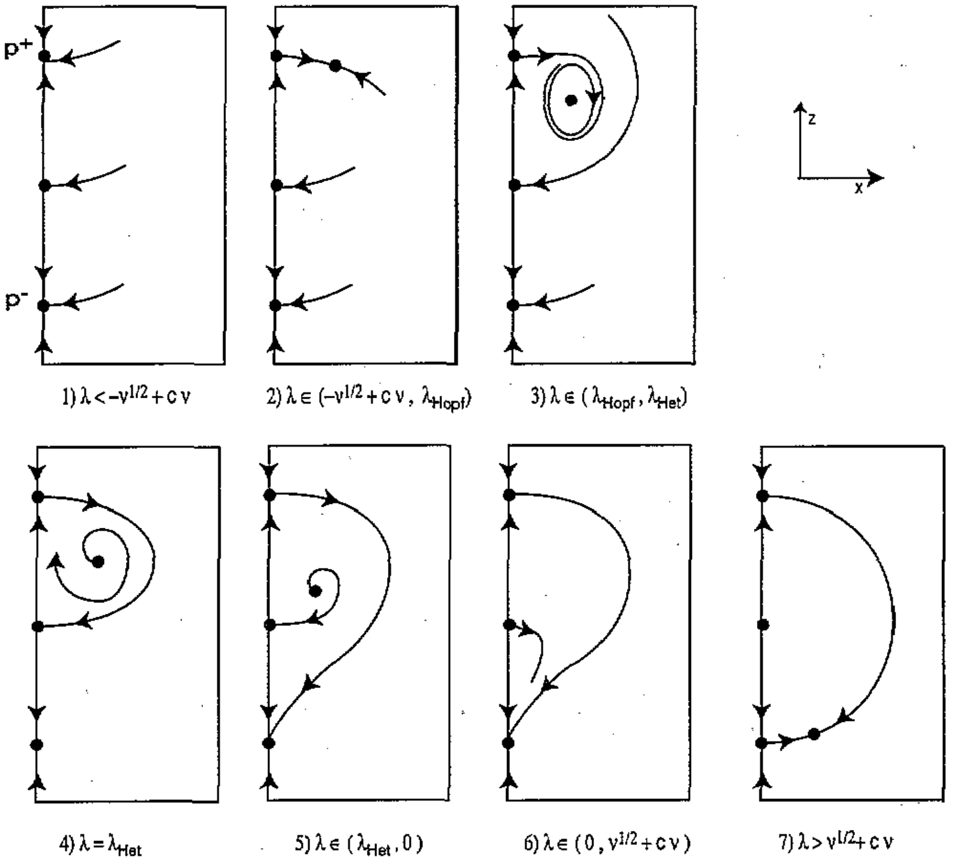

Figure 3. Bifurcation scenario with D_4^d-symmetry in the plane $\mathrm{Fix}(\mathbb{Z}_2)$.

heteroclinic orbits, connecting p^+ to p^- in the invariant planes. The linearization around p^+ yields the eigenvalues

$$\begin{aligned} \mu_x &= \lambda + \sqrt{\nu} - c \cdot \nu \\ \mu_y &= \lambda - \sqrt{\nu} - c \cdot \nu \\ \mu_z &= -2\nu\,. \end{aligned}$$

Lemma 3.2. *The planar system*

$$\begin{aligned} \dot{x} &= \lambda x + xz - x(ax^2 + cz^2) \\ \dot{z} &= \nu z - x^2 - z(z^2 + \beta x^2) \end{aligned}$$

possesses a mixed mode solution, bifurcating from the origin as $\lambda, \nu \sim 0, \nu > 0$, iff $\lambda > \sqrt{\nu} + c \cdot \nu$ or $0 > \lambda > -\sqrt{\nu} + c \cdot \nu$. At $\lambda = \sqrt{\nu} + c \cdot \nu$, it bifurcates from $p^- = (0, -\sqrt{\nu})$ via a supercritical pitchfork bifurcation. At $\lambda = 0$, it is created by a subcritical pitchfork bifurcation on the origin and disappears at $\lambda = -\sqrt{\nu} + c \cdot \nu$ in a supercritical pitchfork bifurcation. At $\lambda_{\mathrm{Hopf}} = -\frac{1}{\sqrt{3}}\sqrt{\nu} + \mathrm{O}(\nu)$ it undergoes a supercritical Hopf bifurcation. The periodic orbit disappears in a heteroclinic loop bifurcation $p^+ \to 0 \to p^+$ which is created at $\lambda_{\mathrm{het}} = -\frac{1}{2}\sqrt{\nu} + \mathrm{O}(\nu)$.

For all $\lambda \in (\lambda_{\mathrm{het}}(\nu), \sqrt{\nu} + c \cdot \nu)$, there exists a heteroclinic orbit, connecting p^+ to p^-. In this region of existence there is another curve $\lambda_{\mathrm{flip}} = \mathrm{O}(\nu)$ where the heteroclinic orbit is contained in the strong stable manifold of p^-.

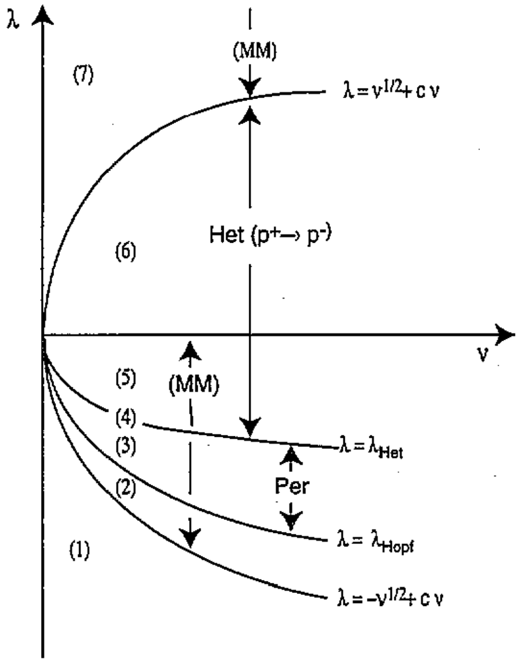

Figure 4. Existence of homoclinic cycles, periodic orbits and mixed modes in the D_4^d-symmetric bifurcation.

As these bifurcation scenarios are not directly connected to our main results, we postpone the proof until the appendix.

In the 4d-model, the same invariant planes exist and exhibit the same bifurcation behaviour. The important difference lies in the existence of a fourth dimension. The linearization in this fourth direction (eigenvalue μ_ξ) is essentially independent of the other three directions.

4. Results

In the following, we will list some statements on forced symmetry breaking bifurcations which appear in the neighbourhood of homoclinic cycles. We first determine—speaking in terms of section 2—which subgroups $H \leqslant \Gamma$ will preserve the cycle and which will generically break it. To single out the important direction of the perturbation, we consider the linear variational equation

$$\dot{v}(t) = Df(q(t)) \cdot v(t) \tag{4.1}$$

and its adjoint

$$\dot{w}(t) = -Df(q(t))^* \cdot w(t)\,. \tag{4.2}$$

The next assumption is a non-degeneracy condition.

(ND) Equation (4.1) possesses an up to scalar multiplication unique bounded solution $\dot{q}(t)$.

Note that this is always fulfilled in our three-dimensional examples. Moreover, it is satisfied in the four-dimensional tetrahedral example due to the decoupling of the fourth equation in the variational equation. Therefore, (ND) is only needed in proposition 4.1 and theorem 4.5 as an assumption. The hypothesis (ND) implies that (4.2) also possesses a unique bounded solution $\psi(t)$ which is in fact orthogonal to the sum of the tangent spaces to stable and unstable manifolds:

$$\psi(t) \perp (T_{q(t)}W^u(p_0) + T_{q(t)}W^s(p_1)).$$

Proposition 4.1. *Assume hypothesis (ND). Then a homoclinic cycle is structurally stable within the class of Γ-equivariant vector fields if and only if there exists* $\kappa \in G_{q(t)} \leqslant \Gamma$ *such that* $\kappa\psi(t) = -\psi(t)$ *or, equivalently, if* $W^u_{q(t)}(p_0) \overline{\pitchfork}^{\Sigma} W^s_{q(t)}(\sigma p_0)$ with $\Sigma = \mathrm{Fix}\ G_{q(t)}$.

Proof.

'$\Longleftarrow$' In Fix $(G_{q(t)})$, the heteroclinic orbit is transverse and therefore persists under Γ-equivariant perturbations. To see this just observe that $\psi \perp \mathrm{Fix}\ (G_{q(t)})$ and therefore $TW^u(p_0) + TW^s(p_1) \supseteq \mathrm{Fix}\ (G_{q(t)})$.

'$\Longrightarrow$' If no κ acts as $-id$ in the direction of ψ, then $G_{q(t)}|_{\langle\psi\rangle} = id$, because ψ is unique. Then we can continue the perturbation $\varepsilon g(q(t), \varepsilon) = \varepsilon \cdot \psi(t)$ Γ-equivariantly in a neighbourhood of the cycle and of course the cycle will break for $\varepsilon \neq 0$ because the Melnikov integral is non zero

$$\int_{\mathbb{R}} \psi(t) \cdot D_\varepsilon(f + \varepsilon g)(q(t))\mathrm{d}t \neq 0. \qquad \square$$

This proposition enables us to single out the subgroups of Γ which produce interesting bifurcation phenomena. Indeed, for $\Gamma = \mathbb{T} \oplus \mathbb{Z}_2$ and $\Gamma = D_4$ the subgroups $H = \mathbb{T}, \mathbb{Z}_3, \mathbb{Z}_6$ and $H = D_2, \mathbb{Z}_2, \mathbb{Z}_4$, respectively, are the only subgroups which allow for breaking the cycle but do, on the other hand, preserve the homoclinic structure, that is, the equilibria p_0 and p_1 are conjugated in H. Comparing with the proposition, we see that for these cycles, $\psi(t)$ is always orthogonal to the invariant planes and the element κ is just the reflection with respect to this plane.

We will now give precise statements on possible bifurcation scenarios in the neighbourhood of homoclinic cycles with $\mathbb{T} \oplus \mathbb{Z}_2$ or D_4-symmetry.

4.1. $\mathbb{T} \oplus \mathbb{Z}_2$-symmetry

In section 3, we described the unfolding of a $\mathbb{T} \oplus \mathbb{Z}_2$-symmetric vector field near the origin with the formation of a homoclinic cycle. Vector fields which are close to this equation can formally be described by

$$\dot{u} = f_\lambda(u, \varepsilon) \qquad \lambda \in \mathbb{R} \qquad \varepsilon \in \mathbb{R}^k \tag{4.3}$$

where $f_\lambda(\cdot, 0)$ is $\mathbb{T} \oplus \mathbb{Z}_2$-equivariant, $f_\lambda(\cdot, \varepsilon)$ is H-equivariant and $D_u f_0(0, 0) = 0$. The Taylor jet of $f_\lambda(u, 0)$ in the origin was given in (3.1) up to the third order. The dynamics near the homoclinic cycle in the perturbed vector field ($\varepsilon \neq 0$) depend on

- the eigenvalues of the linearization around p_0, which are

$$\mu_x = -2\lambda \qquad \mu_y = \lambda\left(1 - \frac{c}{a}\right) \qquad \mu_z = \lambda\left(1 - \frac{b}{a}\right)$$

  (see equation (3.2)) and
- the parameter ε of the perturbation.

In the case $\mu_z < \mu_x$, the strong stable manifold of p_0 does not lie on the x-axis and we will require the following assumption in theorems 4.2 and 4.3.

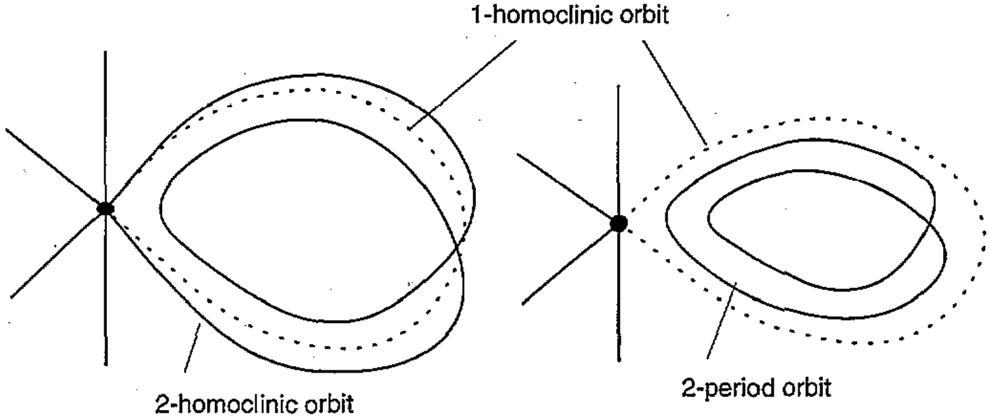

Figure 5. Definition of N-homoclinic and N-periodic orbits.

(W1) The heteroclinic orbit joining p_0 to σp_0 is not included in the strong stable manifold of σp_0.

This hypothesis is generic in the symmetric system and can be numerically tested. In the neighbourhood of the homoclinic cycle we will find for $\varepsilon \neq 0$ the following types of solutions:

- N-periodic solutions (N-per): these are solutions which pass N-times in the neighbourhood of the equilibrium in the quotient space $\mathbb{R}^3/H$ during one period.
- N-homoclinic solutions (N-hom): these are homoclinic orbits in the quotient space which pass $N-1$ times in a neighbourhood of the equilibrium before closing up. The homoclinic cycle for $\varepsilon = 0$ is 1-homoclinic.
- shift-dynamics: these are encoded—as in Shilnikov's homoclinic chaos—by the return time to a transverse cross section.

In real phase space $\mathbb{R}^3$, N-periodic solutions will explore scm $(O(\sigma), N)$ equilibria during one period, where $O(\sigma)$ is the order of σ in $H/G_{q(t)}$; in the $\mathbb{T} \oplus \mathbb{Z}_2$-symmetry, we have $O(\sigma) \in \{3, 6\}$ and in D_4-symmetry $O(\sigma) \in \{2, 4\}$. The interpretation of N-homoclinics is similar. Only $k \cdot O(\sigma)$-homoclinics for some k are homoclinic in $\mathbb{R}^3$, the others are heteroclinic cycles!

Theorem 4.2 ($\mathbb{T} \oplus \mathbb{Z}_2 \to \mathbb{T}$ symmetry breaking). *Consider the $\mathbb{T}$-equivariant steady-state bifurcation described by (3.1) and (4.3) and assume that, $f_\lambda(u, 0)$ satisfies (W1). Assume that – in the notation of section 3, $a < 0$, $b < a < c$ and $\lambda > 0$, such that a homoclinic cycle bifurcates from the origin for $\varepsilon = 0$. Denote by μ_x, μ_y and μ_z the eigenvalues of the equilibria bifurcating on the x-axis (for an expression of the eigenvalues in terms of the coefficients, see equation (3.2)). The regions with qualitatively different bifurcation behaviour for $\varepsilon \neq 0$ are described in figure 6.*

Then for a generic unfolding in the parameter ε, we have in region

SI, SII, U: a one-parameter unfolding in ε, which produces unique 1-periodic solutions which are either stable (SI, SII) or have one unstable Floquet multiplier (U); see figure 'trivial'.

KKO: a two-parameter unfolding with 1- and 2-periodic and -homoclinic solutions, compare figure 'doubling'; the stability of the bifurcating periodic solutions is given in the table below.

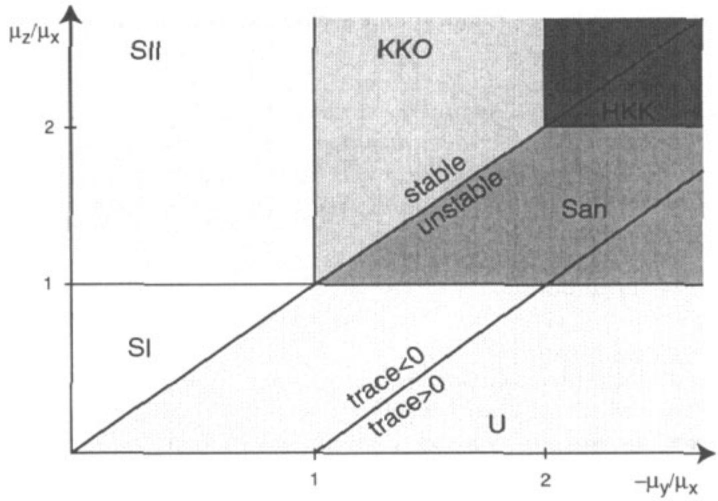

Figure 6. Regions with qualitatively different unfolding of a T⊕$\mathbb{Z}_2$-cycle under forced symmetry breaking.

	I	*II*	*III*	*IV*	*V*
1-per	*1 stable* *1 unstable*	*1 stable*	*1 stable*	*1 unstable*	–
2-per	–	–	*1 unstable*	–	–

San: a two-parameter unfolding with homoclinic doubling, homoclinic cascades and shift dynamics, see figure 'cascade'. Here stable as well as unstable periodic solutions bifurcate.

HKK: a three-parameter unfolding with horseshoes in open regions of parameter space.

All bifurcating periodic orbits are σ-symmetric. Reflecting the diagram with respect to $\varepsilon = 0$ yields $\kappa\sigma\kappa$-symmetric solutions.

Remarks.

(i) The names of the regions 'HKK', 'KKO', ..., are explained in the next section.

(ii) **Stability.** In regions 'U', 'SI' and 'SII', stability properties of the homoclinic cycle and the bifurcating periodic orbits correspond. In region 'KKO' the cycle is stable but there are stable and unstable periodic orbits bifurcating. In region 'San' there are stable periodic orbits if trace < 0, that is, if $\mu_z + \mu_y + \mu_x < 0$, generated by the period doubling, though the cycle is unstable; near the 'shift'-region, we also expect stable periodic orbits due to the Newhouse phenomenon if trace < 0.

(iii) If $a > 0$, we can reverse time and discover the same bifurcation phenomena for $\lambda < 0$, of course, stability properties also are reversed.

(iv) Note that the shift dynamics are encoded by the return times. The sequence of equilibria, explored by chaotic trajectories, is just $\sigma^i p_0$ or $\kappa\sigma^i p_0$ and is not chaotic at all.

(v) For the other symmetry groups $H = \mathbb{Z}_3$ or $H = \mathbb{Z}_6$, the bifurcation diagrams are the same. The bifurcating periodic orbits possess $\mathbb{Z}_3$- or $\mathbb{Z}_6$-symmetry. In the $\mathbb{Z}_6$-case, they explore all equilibria of the cycle! However, reflecting the diagram with respect to $\varepsilon = 0$ is meaningless in these two cases.

(vi) The genericity conditions needed for the ϵ-unfolding are implicitly given in lemmata 5.2 and 6.4.

Theorem 4.3 ($\mathbb{T} \oplus \mathbb{Z}_2 \to \mathbb{T}$ symmetry breaking in $\mathbb{R}^4$). *If under the assumptions of theorem 4.2, there is a fourth critical direction with eigenvalue ν in which the reflection $\kappa \in \mathbb{T}\oplus\mathbb{Z}_2$ acts as $(-id)$ and the rotation σ acts trivially, a generic $\mathbb{T}$-symmetric perturbation of the $\mathbb{T}\oplus\mathbb{Z}_2$-symmetric system (3.3) exhibits the following bifurcation phenomena when the perturbation parameter ε varies around 0:*

(i) in region SI, we have

- *(a) a unique periodic orbit if $-\nu > \mu_y$ or $\nu > \mu_y$ with 0 or 1 unstable Floquet multipliers respectively (cf figure 'trivial');*
- *(b) a two-parameter homoclinic doubling if $\nu \in (-\mu_y, 0)$ where stability is determined as in theorem 4.2 (cf figure 'doubling');*
- *(c) a homoclinic cascade (cf figure 'cascade') if $\nu \in (0, \mu_y)$;*

(ii) in region U, we have

- *(a) a unique periodic orbit if $-\nu > -\mu_z$ or $\nu > -\mu_z$ with 1 or 2 unstable Floquet multipliers;*
- *(b) homoclinic doubling if $\nu \in (0, -\mu_z)$ and*
- *(c) homoclinic cascades if $\nu \in (\mu_z, 0)$.*

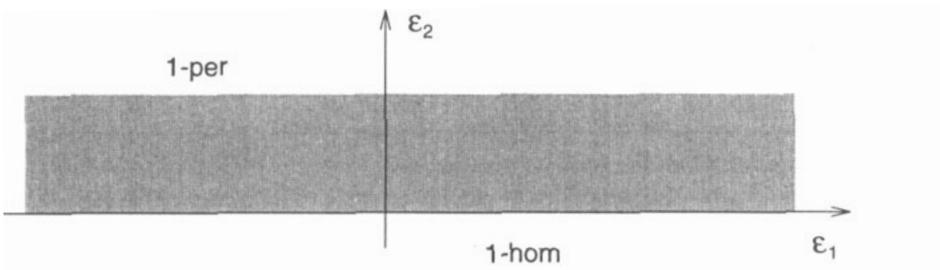

Figure 7. The trivial bifurcation.

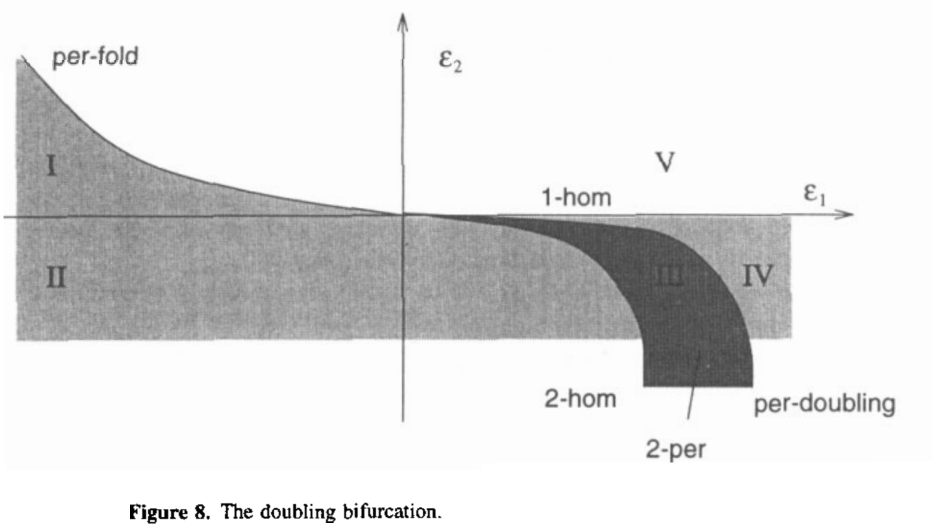

Figure 8. The doubling bifurcation.

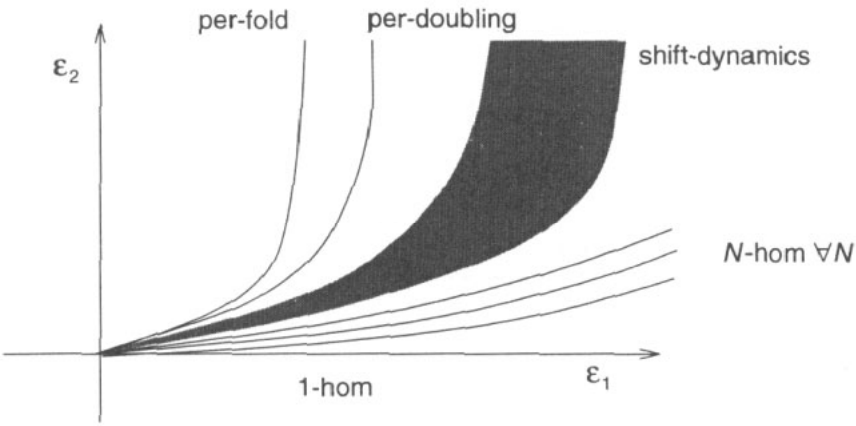

Figure 9. The cascade bifurcation.

Remarks.

(i) In the other regions a three- or four-parameter unfolding describes the dynamics for small ranges of ν. There are always values of ν which allow complicated dynamics in the ε-unfolding.

(ii) Interpretation of the bifurcation diagrams is the same as before. A detailed stability analysis in the cases 'doubling' and 'cascade' is similar in spirit as for the 3-dimensional bifurcations. Again, for weakly stable and unstable cycles, there are stable and unstable periodic orbits bifurcating (cf remark (ii) after theorem 4.2).

(iii) The genericity conditions of the perturbation are implicitly given in lemmata 5.1 and 6.4. In addition, the explicit perturbation

$$\varepsilon_1(0, 0, 0, x^2 + y^2 + z^2)^t + \varepsilon_2(yz, zx, xy, 0)^t$$

fulfills these conditions and is in fact the polynomial of lowest order satisfying it. This is proved in section 6.3.

4.2. D_4-symmetry

In our second example, we discuss a codimension-two steady-state bifurcation close to D_4-symmetry, described by

$$\dot{u} = f_{\lambda,\nu}(u, \varepsilon) \qquad \lambda, \nu \in \mathbb{R} \qquad \varepsilon \in \mathbb{R}^k \qquad u \in \mathbb{R}^3 \tag{4.4}$$

where $f_{\lambda,\nu}(\cdot, 0)$ is D_4^d-equivariant, $f_{\lambda,\nu}(\cdot, \varepsilon)$ is H-equivariant and $D_u f_{0,0}(0, 0) = 0$. The Taylor jet of $f_{\lambda,\nu}(u, 0)$ in the origin was given in (3.4) up to the third order. The eigenvalues in the linearization around the equilibria bifurcating on the z-axis are

$$\mu_x = \lambda + \sqrt{\nu} - c \cdot \nu$$
$$\mu_y = \lambda - \sqrt{\nu} - c \cdot \nu$$
$$\mu_z = -2\nu\,.$$

Similar to (W1), we need the next hypothesis in theorem 4.4.

(W2) The parameter (λ, ν) does not lie on the curve $\lambda = \lambda_{\text{flip}}$ with asymptotics $\lambda = O(\nu)$, where the heteroclinic orbit is included in the strong stable manifold (see lemma 3.2).

Theorem 4.4 ($D_4^d \to \mathbb{Z}_4^-$ symmetry breaking). *Consider the $\mathbb{Z}_4^-$-equivariant steady-state bifurcation described by equation (4.4) with $H = \mathbb{Z}_4^-$ and ε close to zero. Assume that in the third-order jet (3.4) for $\varepsilon = 0$ in the origin, we have $\alpha < 0$ and (W2) is satisfied.*

Then for a generic unfolding in $\varepsilon \neq 0$, we have the following different bifurcation behaviours with an interpretation similar to the $\mathbb{T}$-symmetric bifurcation:

Region	*Existence stability*	*Bifurcation diagram*	*Eigenvalues*
U	+ *Unstable*	*'trivial'*	$-\mu_y < -\mu_z < \mu_x$
San	+ *Unstable*	*'Cascade'*	$-\mu_z < -\mu_y < \min(\mu_x, -2\mu_z)$
HKK	+ *Unstable*	*Horseshoes occur*	$-2\mu_z < -\mu_y < \mu_x$
	+/− *Stable*		$-2\mu_z < \mu_x < -\mu_y$
KKO	− *Stable*	*'Doubling'*	$-\mu_z < \mu_x < \min(-2\mu_z, -\mu_y)$
SII	− *Stable*	*'Trivial'*	$\mu_x < -\mu_z < -\mu_y$

Then there are open regions in a generic three parameter unfolding where horseshoes occur. All orbits are $\kappa\sigma$-symmetric. Reflecting the diagram at $\varepsilon = 0$ yields $(\kappa\sigma)^{-1}$-symmetric orbits.

Remarks.

(i) Asymptotic stability again follows from [KM91]. For asymptotic stability we require $\mu_x < -\mu_y$. Note that this condition is weaker than the one in [AGH88] which only applies in region SII where the cycle does not exist near $\lambda, \nu = 0$! The other eigenvalue conditions can be easily verified by the reader.

(ii) Periodic orbits in regions 'U' and 'San' are unstable. In region HKK we expect Newhouse sinks when the horseshoe disappears, if only trace < 0, that is $\lambda < (c+1)\nu$.

(iii) Again, symmetry breakings to $\mathbb{Z}_2$ or D_2 symmetry are similar. In the D_2-case, we can reflect the bifurcation diagrams at $\varepsilon = 0$ and obtain $\kappa\sigma\kappa$-symmetric orbits from σ-symmetric ones.

(iv) The case $\alpha > 0$ follows by time-reversal. For existence of homoclinic cycles we then have to require $\gamma = +1(!)$ and reflect the (λ, ν)−diagram in the origin. The cycle will then be unstable in all regions of existence. However, in region 'San' there are stable periodic orbits bifurcating (trace is now positive and there are folds and period doublings!).

(v) The genericity conditions needed for the perturbation are implicitly given in lemmata 5.2 and 6.4.

Motivated by the work of [CH92], we will now discuss the case of $\mathbb{Z}_4$−symmetry in $\mathbb{R}^4$, which is close to D_4 and to $\mathcal{O}(2)$-symmetry where we consider the $l = 1$, $l = 2$ representation of $\mathcal{O}(2)$ on $\mathbb{R}^4$ (see [CH92], [AGH88]). It turns out that symmetry breaking from $\mathcal{O}(2)$ to D_4 yields a situation similar to the action of $\mathbb{T} \oplus \mathbb{Z}_2$-symmetry in $\mathbb{R}^4$, where the direction orthogonal to the invariant subspaces is the direction of the broken $\mathcal{O}(2)$-group orbit. As the equations in the invariant subspaces are the same as for D_4^d-symmetry in $\mathbb{R}^3$, we can refer to the same linear stability analysis and the conditions for existence as in the previous theorem, just adding a fourth weaker eigenvalue in the direction of the $\mathcal{O}(2)$-group orbit. A precise description of the possible bifurcation phenomena after having broken the reflection symmetry is only possible in region 'U'.

Theorem 4.5 ($D_4 \to \mathbb{Z}_4$ **symmetry breaking in** $\mathbb{R}^4$). *Consider a* $\mathbb{Z}_4$*-symmetric bifurcation in* $\mathbb{R}^4$*, which is close to a bifurcation in which* $\mathcal{O}(2)$*-symmetry with the* $l = 1$, $l = 2$ *representation is broken into* D_4*-symmetry in a generic way. Suppose that in one pair of invariant planes, there exists a homoclinic cycle* ($\nu > 0, \alpha < 0$) *and that the eigenvalue at the equilibrium of the cycle in the direction of the broken* $\mathcal{O}(2)$*-group orbit is the smallest one in modulus (small symmetry breaking!). If this eigenvalue is positive, assume in addition the generic hypothesis (ND). Moreover, suppose that the unfolding in* ε *describing the symmetry breaking from* D_4 *to* $\mathbb{Z}_4$ *is generic.*

Then, refering to figure 10, we have in region 'U' the following possible bifurcation scenarios:

'doubling': when the eigenvalue in the direction of the broken group-orbit is positive
and
'cascade': when it is negative.

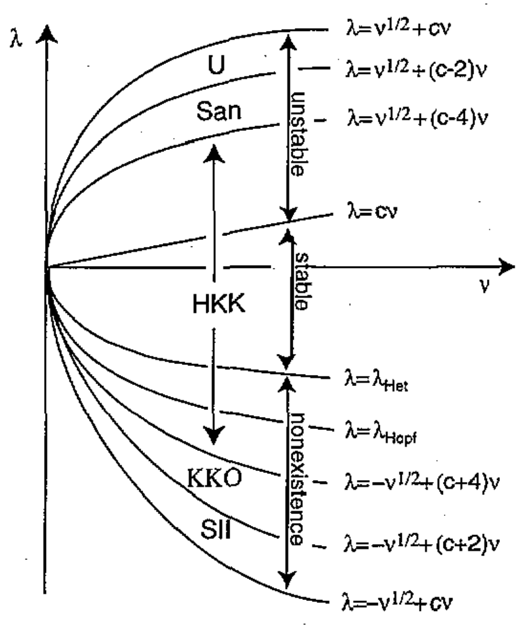

Figure 10. Regions with qualitatively different unfolding of a D_4-cycle under forced symmetry breaking.

Remarks.

(i) Again all periodic orbits are unstable. However, for $\alpha > 0$, we have trace < 0 in region U and although the cycle is unstable (the radial eigenvalue is positive!), for both signs of the eigenvalue in the direction of the broken $\mathcal{O}(2)$-group orbit, stable periodic orbits do bifurcate!

(ii) Unfortunately, in all other regions of (λ, ν)-parameter space, at least three parameters are necessary in order to describe the unfolding. Bifurcation diagrams are not known; we expect shift-dynamics to occur in most cases.

(iii) The $\mathbb{Z}_4$-symmetric N-periodic orbits and the N-homoclinic solutions (alias heteroclinic cycles) will explore all four heteroclinic orbits of the homoclinic cycle during one period or before closing up, respectively. In particular, the phase of the $l = 1$-mode (x–y coordinates in (3.4)) varies over the whole circle $[0, 2\pi)$ and the phase of the $l = 2$-mode oscillates between 0 and π. Now the two-periodic solutions will prefer one of the equilibria, when they approach the 2-homoclinics, that is, they will, while remaining intermittent, stay much longer time periods close to one of them.

(iv) The genericity condition of the perturbation is implicitly given in lemmata 5.1 and 6.4.

4.3. Making explicit the parameters ε_1 and ε_2

In order to localize the regions described in the ε-*bifurcation diagrams* 'generic', '*doubling*' and 'cascade' in actual $\mathbb{Z}_4^-$ or $\mathbb{T}$-symmetric bifurcations, one has to express ε in terms of the leading symmetry-breaking polynomials of the Taylor jet. The perturbation can in general be written as

$$f(u, \varepsilon) = f(u, 0) + \varepsilon_1\, g_1(u, \varepsilon) + \varepsilon_2\, g_2(u, \varepsilon)\,.$$

The perturbation g_2 should just cause a separation of stable and unstable manifolds, that is, the Melnikov integral $\int_{-\infty}^{+\infty} \psi(t) g_2(g(t))\mathrm{d}t$ should be non-zero. Lowest-order terms of g_2 are in the three-dimensional example

$$g_2(x, y, z) = (yz, xz, xy) \qquad \text{for } \mathbb{T}\text{–symmetry and}$$
$$g_2(x, y, z) = (y, -x, 0) \qquad \text{for } \mathbb{Z}_4\text{–symmetry}\,.$$

In both cases, all other symmetry breaking terms are of higher order. In $\mathbb{R}^4$, we have other second-order $\mathbb{T}$-equivariant polynomials, namely $\xi \cdot (x, y, z)$ and $(0, 0, 0, \xi^2)$ which, however, do not break the cycle, and $(0, 0, 0, x^2 + y^2 + z^2)$. Note that if the second polynomial is of higher order, the $(\varepsilon_1, \varepsilon_2)$-bifurcation diagram has to be deformed and 'most' of the parameter space will only exhibit bifurcation phenomena in a cusp region around the line in the $(\varepsilon_1, \varepsilon_2)$-diagram, determined by the leading polynomial.

In general, the second degeneracy condition is also given by integrals, which are nevertheless quite hard to evaluate. We declined at this stage of the work to give explicit expressions for the Taylor jet.

5. Generic bifurcations of codimension two

In this section we give a short review on two bifurcations of homoclinic orbits of codimension two for generic vector fields, namely the so-called orbit- and inclination-flip bifurcation. These results will be used to prove our theorems in the next sections. We will not give these results in full generality but will restrict ourselves to the situation needed for our proofs. Hence consider

$$\dot{u} = g(u, \varepsilon) \qquad (u, \varepsilon) \in \mathbb{R}^n \times \mathbb{R}^2\,. \tag{5.1}$$

Here g is not assumed to be equivariant. We assume $g(0,\varepsilon) = 0$ and $D_u g(0,0)$ has spectrum consisting of simple eigenvalues $\mu_u > 0 > -\mu_s > -\mu_{ss}$ and the remainder part of the spectrum has real part strictly less than $-\mu_{ss}$. Furthermore, let $q_0(t)$ be a homoclinic orbit converging to zero for t tending to $\pm\infty$ for $\varepsilon = 0$. Then there exists a unique bounded (up to scalar multiples) solution $\psi_0(t)$ of the adjoint variational equation

$$\dot{w} = -D_u g(q_0(t), 0)^* w \tag{5.2}$$

see section 4. As mentioned there, $\psi_0(t) \perp (T_{q_0(t)} W^s(0) + T_{q_0(t)} W^u(0))$ for all $t \in \mathbb{R}$. Indeed, $\langle \psi_0(t), v(t) \rangle$ is independent of time for any solution $v(t)$ of

$$\dot{v} = D_u g(q_0(t), 0) v\,. \tag{5.3}$$

We will denote by $\phi(t,s)$ the solution operator of equation (5.3). In theorem 5.1 and 5.2 we assume the following Melnikov condition.

(M) $\int_{-\infty}^{\infty} \langle \psi_0(t), D_\varepsilon g(q_0(t), 0) \rangle \mathrm{d}t \neq 0.$

It is well known that under this assumption there exists a unique branch $\varepsilon = \varepsilon^*(\tau)$ in parameter space and corresponding homoclinic solutions $q_\tau(t)$ of (5.1) for $\varepsilon = \varepsilon^*(\tau)$ such that $\varepsilon^*(0) = 0$. We formulate now further assumptions for both bifurcations separately.

5.1. The orbit-flip bifurcation

In this section we formulate the hypotheses which are needed in theorem 5.1 on the orbit-flip bifurcation stated in section 5.3. Firstly, we assume

(OF1) (i) $\lim_{t\to\infty} \mathrm{e}^{\mu_s t} q_0(t) = 0$,
(ii) $\lim_{t\to\infty} \mathrm{e}^{\mu_{ss} t} q_0(t) \neq 0$,
(iii) $\lim_{t\to-\infty} \mathrm{e}^{-\mu_s t} \psi_0(t) \neq 0$.

This hypothesis implies that $q_0(t) \in W^{ss}_{\mathrm{loc}}(0)$ is contained in the strong stable manifold of 0. Next we state an assumption about the dependence of g on the parameters.

Define $v^s(\tau) := \lim_{t\to\infty} \mathrm{e}^{\mu_s(\varepsilon^*(\tau))t} q_\tau(t)$. Here $-\mu_s(\varepsilon)$ denotes the eigenvalue of $D_u g(0,\varepsilon)$ continuing $-\mu_s$. We remark that $v^s(0) = 0$ due to (OF1)(i). We will assume

(OF2) $D_\tau v^s(\tau)|_{\tau=0} \neq 0$.

This assumption has the following geometric interpretation. The homoclinic solution $q_\tau(t)$ switches from one side of $W^{ss}(0)$ to the other while τ moves through 0, see figure 11.

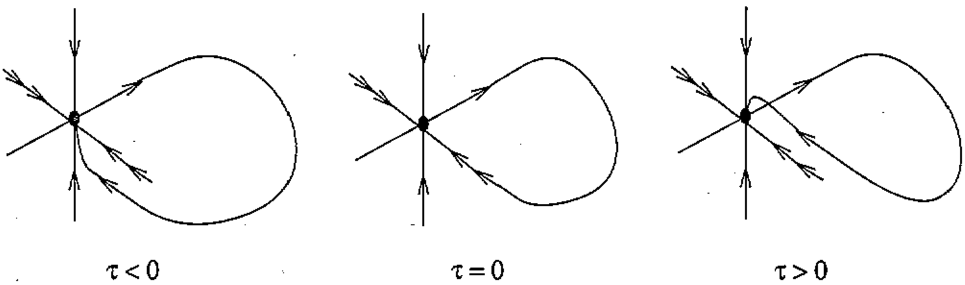

Figure 11. The orbit-flip unfolding.

Lemma 5.1. *The hypotheses (M) and (OF2) together are equivalent to linear independence of the vectors*

$$\int_{-\infty}^{\infty}\langle\psi_0(t), D_\varepsilon g(q_0(t),0)\rangle\,\mathrm{d}t$$
$$\int_{-\infty}^{\infty} Q_1(0)\ \phi(0,t)\ D_\varepsilon g(q_0(t),0)\,\mathrm{d}t$$

in parameter space. Here $Q_1(0)$ projects onto a complement of $T_{q_0(0)}W^{ss}(p_0)$ in $T_{q_0(0)}W^s(0)$, which will be identified with $\mathbb{R}$, along $\mathbb{R}\psi_0(0)$.

Proof. See [San93]. □

The next assumption is needed in theorem 5.1.

(N1) $\lim_{t\to\infty}\langle\psi_0(-t), q_0(t)\rangle e^{2\mu_{ss}t}\neq 0$.

5.2. The inclination-flip bifurcation

In this section we introduce the assumptions used in theorem 5.2 on the inclination–flip bifurcation in the next section. We request the following relations to hold.

(IF1) (i) $\lim_{t\to\infty} e^{\mu_s t} q_0(t)\neq 0$
(ii) $\lim_{t\to-\infty} e^{-\mu_s t}\psi_0(t)=0$
(iii) $\lim_{t\to-\infty} e^{-\mu_{ss} t}\psi_0(t)\neq 0$.

This assumption is equivalent to the fact that there does not exist a strong stable foliation along the homoclinic orbit $q_0(t)$, see figure 12. Now for each orbit $q_\tau(t)$ existing for $\varepsilon=\varepsilon^*(\tau)$ there is a corresponding bounded solution $\psi_\tau(t)$ of the equation

$$\dot{w}=-D_u g^*(q_\tau(t),\varepsilon^*(\tau))w\,.$$

We define $w^s(\tau)=\lim_{t\to\infty} e^{-\mu_s(\varepsilon^*(\tau))t}\psi_\tau(t)$ and assume

(IF2) $D_\tau w^s(\tau)\,|_{\tau=0}\neq 0$.

Geometrically, the strong stable foliation changes the topological type, see figure 12.

Lemma 5.2. *The hypotheses (M) and (IF2) together are equivalent to linear independence of the vectors*

$$\int_{-\infty}^{\infty}\langle\psi_0(t), D_\varepsilon g(q_0(t),0)\rangle\mathrm{d}t$$
$$\int_{-\infty}^{\infty}\langle\psi_0(t), (D_\varepsilon D_u g(q_0(t),0)+D_u^2 g(q_0(t),0)[v(t),\cdot])\phi(t,0)Q_2(0)\rangle\mathrm{d}t\,.$$

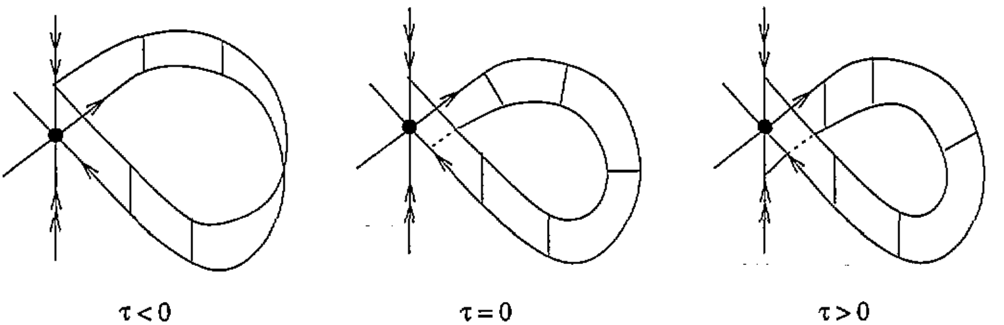

Figure 12. The inclination-flip unfolding.

Here

$$v(t) := \begin{cases} \int_0^t P^u(t)\phi(t,s)D_\varepsilon g(q_0(s),0)\,\mathrm{d}s + \int_{-\infty}^t (1 - P^u(t)\phi(t,s)g(q_0(s),0)\,\mathrm{d}s & \text{for } t \leqslant 0 \\ \int_0^t P^s(t)\phi(t,s)D_\varepsilon g(q_0(s),0)\,\mathrm{d}s + \int_{\infty}^t (1 - P^s(t)\phi(t,s)g(q_0(s),0)\,\mathrm{d}s & \text{for } t \geqslant 0 \end{cases}$$

and $P^u(t)$ projects onto $T_{q_0(t)}W^u(0)$, $P^s(t)$ onto $T_{q_0(t)}W^s(0)$. The projections of the exponential dichotomies $P^s(t)$ and $P^u(t)$ are defined for (5.3), see [San93]. Moreover, $Q_2(0)$ projects onto a complement of $T_{q_0(0)}W^r(0)$, i.e. the invariant manifold tangential to the eigenspace corresponding to the remainder part of the spectrum of $D_u g(0,0)$, in $T_{q_0(0)}W^{ss}(0)$ along $\mathbb{R}\psi_0(0)$. Again this one-dimensional complement is identified with $\mathbb{R}$.

Proof. See [DKO91] or [San93]. □

There exists an invariant not necessarily unique manifold $W^{su}_{\mathrm{loc}}(0)$, which is tangent at 0 to the eigenspace corresponding to the eigenvalues μ_u and $-\mu_s$. Moreover, $W^{su}(0)$ contains $q_0(t)$. By (IF1) $W^{su}(0)$ is tangent to $W^s(0)$ at $q_0(t)$, see figure 13.

In theorem 5.2 the following assumption occurs.

(N2) $W^{su}(0)$ and $W^s(0)$ have a quadratic tangency at $q_0(t)$.

5.3. The bifurcation diagrams

Depending on further conditions on the eigenvalues $\mu_u, -\mu_s$ and $-\mu_{ss}$ there are three different types of bifurcation phenomena occuring in the orbit-flip and inclination-flip bifurcation. As in section 4, we define N-homoclinics (and N-periodic) solutions as homoclinic (periodic) orbits, which wind N-times in a small neighbourhood of $q_0(t)$.

The following results have been obtained so far for both of the bifurcations mentioned above.

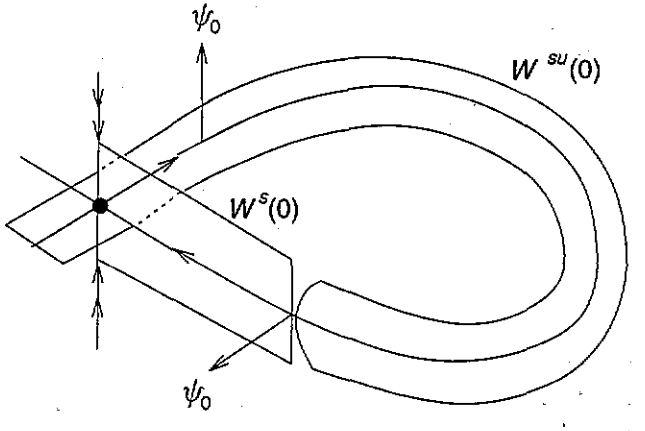

Figure 13. Quadratic tangency of $W^{su}(0)$ and $W^s(0)$.

Theorem 5.1. *Orbit-flip bifurcation. Assume (M), (OF1), (OF2).*

Further assumptions	*Eigenvalue conditions*	*Diagram*
	$\mu_u < \mu_s$	*Trivial*
	$\mu_s < \mu_u < \mu_{ss}$	*Doubling*
(N1)	$\mu_s < \mu_{ss} < \mu_u$	*Cascade*

See [San93] for the proof.

Theorem 5.2. *Inclination-flip bifurcation. Assume (M), (IF1), (IF2).*

Further assumptions	*Eigenvalue conditions*	*Diagram*	*Reference*
	$\mu_u < \mu_s$	*Trivial*	*[KKO93]*
	$\mu_s < \mu_u < \min(2\mu_s, \mu_{ss})$	*Doubling*	*[KKO93]*
(N2)	$\mu_s < 2\mu_s < \min(\mu_{ss}, \mu_u)$	*Cascade*	*[HKK93]*
(N1)	$\mu_s < \mu_{ss} < \min(2\mu_s, \mu_u)$	*Cascade*	*[San93]*

See figures 7–9 for the corresponding bifurcation diagrams. The results on stability are proved in [San94].

Remark. The results on the orbit-flip bifurcation are still valid in the case of higher-dimensional unstable manifolds. Then we have to assume that $D_u g(0, 0)$ has simple eigenvalues $\mu_u > 0 > -\mu_s > -\mu_{ss}$ and the remaining eigenvalues have real part strictly less than $-\mu_{ss}$ or strictly larger than μ_u. In addition, the hypotheses (ND) and

(OF3) (i) $\lim_{t\to-\infty} \mathrm{e}^{-\mu_u t} q_0(t) \neq 0$,
(ii) $\lim_{t\to\infty} \mathrm{e}^{\mu_u t} \psi_0(t) \neq 0$

have to be fulfilled. The conclusions of theorem 5.1 and lemma 5.1 still hold under these additional assumptions. For a proof see again [San93].

6. The proofs

The idea for proving the theorems is the following. We factor out the remaining symmetry by identifying Poincaré sections at the different heteroclinic connections using the symmetry σ. This induces a new dynamical system now possessing one homoclinic orbit instead of a homoclinic cycle. No symmetry will be left after the identification.

The next step consists of determining whether the assumptions of the theorems in section 5 are satisfied for a nonlinearity coming from a generic equivariant vector field for the original system. Then we just need to apply these results in order to obtain the theorems. In fact, symmetry will help us a lot in verifying the assumptions, though in one case symmetry will prevent one hypothesis from being fulfilled.

6.1. Reduction to the orbit space

In our examples equilibria and heteroclinic orbits have different isotropy groups. Thus the flow on the orbit space, which is a manifold, possesses a degenerate stationary point. Therefore instead of looking at the flow (or one of its extensions) on the orbit space we construct an equivalent system directly from the equivariant equations. Choose a heteroclinic connection q_0 as well as a section Σ_0 at $q_0(0)$ transverse to the flow. Next take any cyclic group $\mathbb{Z}_k$ with generator σ in the remaining symmetry group H and define $\Sigma_j := \sigma^j \cdot \Sigma_0$ and $q_j = \sigma^j \cdot q_0$ for $j \in 0, \dots, k-1$. Now we identify the section Σ_0 and Σ_{n-1} by σ^{n-1}. Thus we obtain a vector field possessing a homoclinic solution consisting of pieces of q_0 and q_{n-1} to the equilibrium p_0. We consider the resulting equations only locally near the new homoclinic orbit forgetting about the other heteroclinic connections. Any solution of the new vector field corresponds to a solution of the original equation, which follow the homoclinic cycle $q_0, \dots, q_{n-1}$. Due to the fact that we identify two sections it is sufficient to verify the assumptions of the theorems in section 4 for one heteroclinic orbit in the original equation. By passing to the reduced equation it is clear that the resulting homoclinic orbit will fulfil these conditions, too.

6.2. *The inclination-flip*

In this section we consider the three-dimensional versions of our examples, i.e. equations (3.1) and (3.4) with a particular perturbation $\varepsilon_1 h_1(x, y, z) + \varepsilon_2 h_2(x, y, z)$.

We assume that these equations are equivariant with respect to $H = \mathbb{T}$ resp. $H = \mathbb{Z}_4$ for $\varepsilon \neq 0$. In order to verify the assumptions of theorem 5.2 we fix the heteroclinic connection $q_0(t) \subset \mathbb{R} \times \{0\} \times \mathbb{R}$ in the xz-plane, i.e. $q_0(t) = (u_1(t), 0, u_3(t))$, for both (3.1) and (3.4). We now have to check the assumptions (IF1), (IF2), (M) and (N1), (N2).

Let us first verify the assumptions concerning the equations for $\varepsilon = 0$.

Lemma 6.1. *(IF1) is satisfied in regions SII, KKO, San and HKK.*

Proof. The condition (IF1) (i), i.e. $\lim_{t\to\infty} e^{\mu_s t} q_0(t) \neq 0$, is fulfilled by assumptions (W1) and (W2) of theorems 4.2 and 4.4. The bounded solution $\psi_0(t)$ of the adjoint variational equation is perpendicular to $T_{q(t)}W^s$, which coincides with the invariant xz-plane. Now the radial stable eigenvalue μ_z has modulus less than the other stable eigenvalue μ_x by assumption. Thus the asymptotic one-dimensional equation, which is fulfilled by $\psi_0(t) = (w_1(t), w_2(t), w_3(t))$ for $t \to -\infty$, is given by

$$\dot{w}_2 = \mu_{ss} w_2 = -\mu_x w_2 .$$

Hence (IF1) (ii) and (iii) are satisfied. □

Let us consider hypothesis (N1) next. In fact, as we will see, this assumption is forced to hold by symmetry. The main observation is the following. By symmetry the z-axis is invariant in both cases and the heteroclinic connection $q_0(t)$ is not contained therein. Moreover,

$$\langle \psi_0(-t), q_0(t) \rangle = \langle (w_1(-t), 0, 0), (u_1(t), 0, u_3(t)) \rangle = w_1(-t) \cdot u_1(t) .$$

Hence it is sufficient to prove $\lim_{t\to\infty} e^{\mu_{ss} t} u_1(t) \neq 0$. This is the content of the following lemma applied to $q_0(t)$ near p_1.

Lemma 6.2. *Consider the equation*

$$\dot{x} = \mu_{ss} x + x \cdot f_1(x, z) \qquad \dot{z} = \mu_s z + f_2(x, z)$$

in $\mathbb{R}^2$, which satisfies $0 < \mu_s < \mu_{ss}$ and $f_1 = O(|x| + |z|)$, $f_2 = O(|x|^2 + |z|^2)$. Take an initial point (x_0, z_0) sufficiently close to zero such that $x_0 \neq 0$. Then

$$\lim_{t\to\infty} e^{\mu_{ss} t} x(t) \neq 0 .$$

Proof. The solution $(x(t), z(t))$ satisfies the integral equation

$$x(t) = \mathrm{e}^{-\mu_{ss}t}x_0 + \int_0^t \mathrm{e}^{-\mu_{ss}(t-\tau)}x(\tau)f_1(x(\tau), z(\tau))\,\mathrm{d}\tau$$
$$z(t) = \mathrm{e}^{-\mu_s t}z_0 + \int_0^t \mathrm{e}^{-\mu_s(t-\tau)}f_2(x(\tau), z(\tau))\,\mathrm{d}\tau\,.$$

Thus

$$\mathrm{e}^{\mu_{ss}t}x(t) = x_0 + \int_0^t \mathrm{e}^{\mu_{ss}\tau}x(\tau)f_1(x(\tau), z(\tau))\,\mathrm{d}\tau\,.$$

By using $|x(t)| + |z(t)| \leqslant K(|x_0| + |z_0|)\mathrm{e}^{-\mu_s t}$ and Gronwall's inequality we obtain for $|x_0| + |z_0|$ small

$$\mathrm{e}^{\mu_{ss}t}|x(t)| \leqslant K|x_0|\mathrm{e}^{\delta t}$$

for $\delta \leqslant K(|x_0| + |z_0|)$ arbitrarily small. Hence

$$|x(t)|\mathrm{e}^{-(\mu_{ss}-\delta)t} \leqslant K|x_0|\,.$$

For $\delta < \mu_s$, i.e. $|x_0| + |z_0|$ sufficiently small, the following integral exists

$$\left|\int_0^\infty \mathrm{e}^{\mu_{ss}\tau}x(\tau)f_1(x(\tau), z(\tau))\mathrm{d}\tau\right| \leqslant K|x_0|(|x_0| + |z_0|)\,.$$

Hence

$$x(t)\mathrm{e}^{\mu_{ss}t} = x_0 + \int_0^t \mathrm{e}^{\mu_{ss}\tau}x(\tau)f_1(x(\tau), z(\tau))\,\mathrm{d}\tau$$
$$\to x_0 + \int_0^\infty \mathrm{e}^{\mu_{ss}\tau}x(\tau)f_1(x(\tau), z(\tau))\,\mathrm{d}\tau$$

as t tends to infinity. Moreover, we obtain for $x_0 \neq 0$ and $|x_0| + |z_0|$ small enough

$$\left|x_0 + \int_0^\infty \mathrm{e}^{\mu_{ss}\tau}x(\tau)f_1(x(\tau), z(\tau))\mathrm{d}\tau\right| \geqslant x_0 - O(|x_0|(|x_0| + |z_0|)) > 0$$

which proves the lemma. □

Remark. The hypothesis (N2) can never be satisfied in our cases. Indeed, the manifold $W^{su}(p_0)$ is a neighbourhood of p_0 in the xz-space, while $W^s(p_1)$ is a neighbourhood of p_1 in the same plane. Hence these manifolds coincide preventing a quadratic tangency. By including a third parameter it is possible to get a curve of inclination-flip points in a three-dimensional parameter space, all of which except for $\varepsilon = 0$ satisfy (N2), see lemma 6.3 below. □

Before we verify the assumptions about the unfolding of the flip, we state a lemma about equivariant extensions of small perturbations of the vector field near the heteroclinic orbit $q_0(t)$.

Lemma 6.3.

(i) Any small perturbation of the vector fields (3.1) or (3.4) inside the xz-plane near $q_0(0)$ can be extended equivariantly in $\mathbb{R}^3$.

(ii) Any small parameter dependent perturbation of the form $\varepsilon \cdot h(u, \varepsilon)$ of (3.1) or (3.4) with support close to $q_0(0)$ can be extended equivariantly with respect to the subgroups H, mentioned below proposition 4.1, in $\mathbb{R}^3$.

Proof.

(i) Here points inside the xz-plane close to $q_0(0)$ possess isotropy $\mathbb{Z}_2$ for $\varepsilon = 0$, while points close to $q_0(0)$ but not contained in the coordinate plane have trivial isotropy. Hence we can extend the perturbation to a neighbourhood of $q_0(0)$ by using a $\mathbb{Z}_2$-equivariant cutoff-function. By applying the (discrete) symmetry group the perturbation extends equivariantly to $\mathbb{R}^3$.

(ii) is proved similarly. □

By lemma 6.3 (i) and due to invariance of the xz-plane it is clear that assumption (IF1)(i), i.e. $q_0(t) \notin W^{ss}_{\text{loc}}(p_1)$, is generically fulfilled. Otherwise change the vector field in that plane a little bit and extend this perturbation. We consider (M) and (IF2) next.

Lemma 6.4. *The assumptions (M) and (IF2) are fulfilled for generic H-equivariant perturbations for $H = \mathbb{T}$ or $\mathbb{Z}_4$, respectively.*

Proof. By lemma 5.2 we have to show that for generic H-equivariant perturbations the following two vectors are linearly independent in parameter space

$$J_1 = \int_{-\infty}^{\infty} \langle \psi_0(t), h(q_0(t), 0) \rangle \, dt$$

$$J_2 = \int_{-\infty}^{\infty} \langle \psi_0(t), \big(D_u h(q_0(t), 0) + D_u^2 f(q_0(t), 0)[v(t), \cdot]\big)\phi(t, 0) Q_2(0) \rangle \, dt$$

with

$$v(t) = \begin{cases} \displaystyle\int_0^t P^u(t)\phi(t,s)h(q_0(s),0)ds + \int_{-\infty}^t (1 - P^u(t)\phi(t,s)h(q_0(s),0)\, ds \\ \hfill \text{for } t \leqslant 0 \\ \displaystyle\int_0^t P^s(t)\phi(t,s)h(q_0(s),0) + \int_{-\infty}^t (1 - P^s(t)\phi(t,s)h(q_0(s),0)\, ds \\ \hfill \text{for } t \geqslant 0\,. \end{cases}$$

Moreover, $Q_2(0)$ projects onto $T_{q_0(0)}W^{ss}(p_1)$ in $T_{q_0(0)}W^s(p_1)$ along $\mathbb{R}\psi_0(0)$. The image of $\phi(t, 0)Q_2(0)$ is a line transverse to span $\dot{q}_0(0)$ in the invariant xz-plane. We prove the genericity by claiming that the set

$$M = \{h(u, \varepsilon) \mid J_1 \text{ and } J_2 \text{ are linearly independent}\}$$

is open and dense in the space of H-equivariant C^1-vector fields. Clearly, M is open, because J_1 and J_2 are linear continuous functionals on C^1. In order to prove our claim it is therefore sufficient to show the existence of two arbitrarily small H-equivariant vector fields with the following properties:

(i) The first vector field makes J_1 non-zero but may change J_2, too.
(ii) The second vector field changes J_2 arbitrarily without changing J_1.

Moreover, due to lemma 6.3 (ii), it is sufficient to construct these vector fields near $q_0(0)$. The first property (i) is easily obtained by choosing $h(u, 0) = \psi_0(0)$ for u close to $q_0(0)$ and multiplication with a cut-off function. For (ii) we choose $h(u, 0)$ in such a way that it vanishes at the heteroclinic orbit, i.e.

$$h(q_0(t), 0) = 0 \qquad \forall t\,.$$

Hence $J_1 = 0$ and

$$J_2 = \int_{-\infty}^{\infty} \langle \psi_0(t), D_u h(q_0(t), 0)\phi(t, 0)Q_2(0) \rangle \, dt$$

because $v(t)$ vanishes, too. The remaining integral can be changed arbitrarily by taking $D_u h(q_0(t), 0)$ as a small rotation with axis $\dot{q}_0(t)$. Indeed, $\psi_0(t)$ is perpendicular to the xz-plane, whereas the range of $\phi(t, 0)Q_2(0)$ is transverse to $\dot{q}(t)$ in the xz-coordinate plane. This proves the lemma. □

Lemma 6.5. *J_1 is non-zero (and therefore (M) is satisfied) for the lowest-order polynomials, which break the symmetry G to H. These polynomials are given by*

$$G = \mathbb{T} \oplus \mathbb{Z}_2, H = \mathbb{T} : h(u, 0) = (yz, zx, xy)$$
$$G = D_4, H = \mathbb{Z}_4 : h(u, 0) = (y, -x, 0).$$

Proof. Here $\psi_0(t) = (0, w_2(t), 0)$ and $q_0(t) = (u_1(t), 0, u_3(t))$, see lemma 5.1. First consider $G = \mathbb{T} \oplus \mathbb{Z}_2$. We obtain the formula

$$\int_{-\infty}^{\infty} \langle \psi_0(t), h(q_0(t), 0)\rangle \, dt = \int_{-\infty}^{\infty} \langle \psi_0(t), (yz, zx, xy)|_{q_0(t)}\rangle \, dt = \int_{-\infty}^{\infty} w_2(t) u_1(t) u_3(t) \, dt .$$

This integral is non-zero, because $u_1(t)$, $u_3(t)$ and $w_2(t)$ do not change sign. Next we look at $G = D_4$ and obtain

$$\int_{-\infty}^{\infty} \langle \psi_0(t), h(q_0(t), 0)\rangle dt = \int_{-\infty}^{\infty} \langle \psi_0(t), (y, -x, 0)|_{q_0(t)}\rangle dt = -\int_{-\infty}^{\infty} w_2(t) u_1(t) dt .$$

Again $u_1(t)$ and $w_2(t)$ do not change sign, which implies that J_1 is non-zero. It is straightforward to show that the given polynomials are of lowest order among the H-equivariant ones. □

With this series of lemmata the proofs of theorems 4.1 and 4.2 are completed. Indeed, the hypothesis (IF1) was proved in lemma 6.1, while (IF2) and (M) are shown in lemma 6.4. Moreover, we proved assumption (N1) in lemma 6.2. In the remark following lemma 6.2 the necessity of a three-parameter unfolding is explained due to the failure of (N2) caused by symmetry. The proofs of bifurcation diagrams in the cases SI and U can be done similarly, using the standard results on homoclinic bifurcation (for references see [Lin90]). □

6.3. The orbit-flip

Let us now consider the four-dimensional systems with tetrahedral symmetry and D_4-symmetry with H-equivariant perturbations $\varepsilon_1 h_1(x, y, z, \xi, \varepsilon) + \varepsilon_2 h_2(x, y, z, \xi, \varepsilon)$. In this paragraph we will prove the results (theorem 4.3 and 4.5) concerning the existence of an orbit-flip bifurcation for these systems. The proofs follow the same lines as those given for the inclination-flip in the previous section.

We will first consider the equations with tetrahedral symmetry (3.3) in the case where the eigenvalue ν in the fourth direction is negative. For this equation we are able to give an explicit expression for a perturbation which satisfies the genericity conditions (M) and (OF2). In fact, this perturbation is the polynomial of lowest possible degree which breaks the symmetry.

Hence we consider only (3.3) with a $\mathbb{T}$-equivariant perturbation

$$g(x, y, z, \varepsilon) = \varepsilon_1(0, 0, 0, x^2 + y^2 + z^2)^t + \varepsilon_2(yz, zx, xy, 0)^t$$

from now on. We have to prove that (ND), (M), (OF1), (OF2) and (N1) are satisfied.

We fix the heteroclinic connection $q(t)$ in the xy-coordinate plane, thus $q_0(t) = (x(t), y(t), 0, 0)$. The variational equation along q_0 is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{\xi} \end{pmatrix} = \begin{pmatrix} A(t) & 0 & 0 \\ 0 & \lambda + bx^2(t) + cy^2(t) & dx(t)y(t) \\ 0 & 0 & -\nu \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{\xi} \end{pmatrix}.$$

Therefore, the bounded solution ψ of the adjoint equation solves

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi = \begin{pmatrix} -A^T(t) & 0 & 0 \\ 0 & -(\lambda + bx^2 + cy^2) & 0 \\ 0 & -dxy & \nu^{\circ} \end{pmatrix} \psi .$$

On the other hand, we have

$$T_{q(t)} W^{ss}(p_1) = \{z = \xi = 0\}.$$

Hence $\psi(t) = (0, 0, \psi_3(t), \psi_4(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} -(\lambda + bx^2 + cy^2) & 0 \\ -dxy & \nu \end{pmatrix}\begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} =: \begin{pmatrix} -\lambda(t) & 0 \\ -dxy & \nu \end{pmatrix}\begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}.$$

Thus

$$\psi_3(t) = \mathrm{e}^{-\int_0^t \lambda(\tau)\,\mathrm{d}\tau}\psi_3(0)$$
$$\psi_4(t) = -d\int_{\infty}^{t} \mathrm{e}^{\nu(t-\tau)} x(\tau)y(\tau)\psi_3(\tau)\,\mathrm{d}\tau .$$

We first compute the Melnikov integrals

$$M = \int_{-\infty}^{\infty} \langle \psi(t), D_{\varepsilon} g(q_0(t), 0)\rangle\,\mathrm{d}t$$

$D_{\varepsilon} g$ is given by

$$D_{\varepsilon_1} g(q_0(t), 0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ x^2(t) + y^2(t) \end{pmatrix} \qquad D_{\varepsilon_2} g(q(t), 0) = \begin{pmatrix} 0 \\ 0 \\ x(t)y(t) \\ 0 \end{pmatrix}$$

and we obtain

$$M_2 = \int_{-\infty}^{\infty} \langle \psi(t), D_{\varepsilon_2} g(q_0(t), 0)\rangle\,\mathrm{d}t = \int_{-\infty}^{\infty} \psi_3(t)x(t)y(t)\,\mathrm{d}t$$
$$= \int_{-\infty}^{\infty} \psi_3(0)\mathrm{e}^{-\int_0^t \lambda(\tau)\,\mathrm{d}\tau} x(\tau)y(\tau)\,\mathrm{d}\tau \neq 0$$

because $x(t)$ and $y(t)$ do not change sign.

Next we consider the assumption (OF1). Again (OF1) (i) and (ii) are fulfilled due to the assumptions on the eigenvalues of $Df(p_0)$.

We have to show (OF1) (iii). Because $-\nu$ is the stable eigenvalue close to zero, we have to consider

$$\lim_{t\to-\infty} \mathrm{e}^{-\nu t}\psi(t).$$

For the second component we obtain

$$\begin{aligned}\lim_{t\to-\infty} e^{-\nu t}\psi_4(t) &= \lim_{t\to-\infty} e^{-\nu t}\left(-d\int_\infty^t e^{\nu(t-\tau)}x(\tau)y(\tau)\psi_3(\tau)\,d\tau\right)\\ &= -\psi_3(0)d\int_{-\infty}^{\infty} e^{-\nu\tau}x(\tau)y(\tau)e^{\int_0^\tau\lambda(s)\,ds}\,d\tau\\ &\neq 0.\end{aligned}$$

due to $d \neq 0$. Hence (OF1) (iii) is fulfilled.

By lemma 5.1 (OF2) is equivalent to linear independence of M and

$$N = \int_{-\infty}^{\infty} Q_1(0)\phi(0,t)D_\varepsilon g(q_0(t),0)\,dt\,.$$

Here $Q_1(0)$ projects onto a complement of $T_{q_0(0)}W^{ss}(p_0)$ in $T_{q_0(0)}W^s(p_0)$ along $\mathbb{R}\psi(0)$. In particular, $\{\tilde{z} = \tilde{\xi} = 0\} \subset \operatorname{Ker} Q_1(0)$ and we are concerned only with the equation

$$\frac{d}{dt}\begin{pmatrix}\tilde{z}\\ \tilde{\xi}\end{pmatrix} = \begin{pmatrix}\lambda(t) & dx(t)y(t)\\ 0 & -\nu\end{pmatrix}\begin{pmatrix}\tilde{z}\\ \tilde{\xi}\end{pmatrix}$$

in order to compute $Q_1(0)$. The flow of this linear system is given by

$$\tilde{\phi}(t,s) = \begin{pmatrix} e^{\int_s^t\lambda(\tau)\,d\tau} & d\int_s^t e^{\int_\tau^t\lambda(\sigma)\,d\sigma}x(\tau)y(\tau)e^{-\nu(\tau-s)}\,d\tau\\ 0 & e^{-\nu(t-s)}\end{pmatrix}.$$

Hence the kernel of $Q_1(0)$ is given by $\{\tilde{\xi} = 0\}$. Moreover, the kernel is invariant under the flow $\phi(0,t)$ and we can write $Q_1(t)$ as

$$Q_1(t) = (0,0,a(t),0)^t \cdot \langle(0,0,0,1)^t,\cdot\rangle$$

for some function $a(\cdot)$ Now, $D_{\varepsilon_2}g(q_0(t),0) \subset \{\tilde{\xi} = 0\}$ and therefore $N_2 = 0$. N_1 is given by

$$\begin{aligned}N_1 &= \int_{-\infty}^{\infty}\phi(0,t)\;Q_1(t)\;D_{\varepsilon_1}g(q_0(t),0)\,dt\\ &= \int_{-\infty}^{\infty}\phi(0,t)\;(x^2(t)+y^2(t))\begin{pmatrix}a(t)\\ 1\end{pmatrix}dt\\ &= \int_{-\infty}^{\infty}(x^2(t)+y^2(t))\begin{pmatrix}a(0)\\ 1\end{pmatrix}e^{\nu t}\,dt\\ &= \begin{pmatrix}a(0)\\ 1\end{pmatrix}\underbrace{\int_{-\infty}^{\infty} e^{\nu t}(x^2(t)+y^2(t))dt}_{>0}\,.\end{aligned}$$

Therefore we finally obtain

$$M = (*, M_2) \qquad N = (N_1, 0)$$

and $N_1 \neq 0$, $M_2 \neq 0$. Hence, M and N are linearly independent, which in turn yields (OF2). (N1) follows from the fact that

$$\begin{aligned}e^{\lambda t}\psi_3(t) &= e^{\int_0^t(\lambda-\lambda(\tau))\,d\tau}\psi_3(0)\\ &= e^{-\int_0^t(bx^2(\tau)+cy^2(\tau))\,d\tau}\psi_3(0)\end{aligned}$$

which cannot converge to zero, because

$$\int_0^\infty (bx^2(\tau)+cy^2(\tau))\,d\tau < \infty\,.$$

This proves theorem 4.3 in the case $\nu < 0$. □

In the tetrahedral case, when the fourth eigenvalue ν is positive, the unstable and stable manifolds of the equilibria are two-dimensional. Hence we have to apply the remark following theorem 5.2 to the time-reversed system. The hypotheses (OF2) and (M) on the unfolding are again satisfied for the explicit polynomial perturbation given above. Checking (OF1) and (N1) for the time-reversed system follows the same lines as for ν negative. The additional hypothesis (ND) is satisfied for equation (3.3), because the fourth equation of the linearization along a heteroclinic orbit decouples. The last assumption (OF3) is fulfilled, because the coordinate axes are invariant. As these are the strong stable directions in case U, the solutions $q(t)$ and $\psi(t)$ cannot converge with these strong contraction rates, see section 6.2 on the inclination-flip for more details.

The proofs of the D_4-symmetric case are again very similar to the ones for the inclination-flip, whence we will not go into details here. Let us just note that (ND) is fulfilled by assumption. Moreover, (OF1), (OF3) and (N1) are fullfilled due to genericity of symmetry breaking from $\mathcal{O}(2)$ to D_4. The unfolding conditions are again satisfied for generic perturbations breaking symmetry from D_4 to $\mathbb{Z}_4$.

7. Existence of strange attractors

We consider again the three-dimensional equations (3.1) and (3.4), see section 3. In the unperturbed case, we know by [KM91] that the cycle is asymptotically stable provided the stable non-radial eigenvalue is in modulus larger than the unstable one. Then the stability of the heteroclinic cycle 'persists' for non-zero ε.

Lemma 7.1. *Assume that an asymptotically stable homoclinic cycle exists for (3.1) or (3.4). Then for any small ε there exists an attractor A_ε for the ε-perturbed equation, which is upper semicontinuous in ε. Moreover, A_ε is the maximal invariant set in some ε-independent neighbourhood of the cycle.*

Proof. In order to apply standard results, e.g. [Hal88, theorem 3.5.2], we have to show, that there exists a set B near the cycle which attracts points, i.e.

$$\lim_{t\to\infty} \operatorname{dist}(B, u(t)) = 0$$

for any solution of $(Th)_\varepsilon$ or $(D_4)_\varepsilon$ such that

$$u(0) \in U_\delta = \{u|\operatorname{dist}(u, Q) < \delta\}$$

for some small $\delta > 0$. Here $Q = \bigcup_{t\in\mathbb{R}} \Gamma q_0(t)$ denotes the heteroclinic cycle.

Due to the stability of Q, there exists an open neighbourhood V of Q such that $V \subset\subset U_\delta$ and $\phi_0(t)V \subset V$ for $t \geqslant 0$, see [Hal88, lemma 3.3.1]. Here $\phi_\varepsilon(t)$ denotes the flow of the ε-perturbed equation. Due to the compactness of *cl* V and the asymptotic stability of Γ there exists a $t_0 > 0$ such that

$$\phi_0(t_0)V \subset\subset V.$$

Now choose an open set W such that

$$\phi_0(t_0)V \subset\subset W \subset\subset V.$$

For the same reasons as above, there exists a $t_1 > 0$ such that $\phi_0(t_1)U_\delta \subset\subset W$.

We define $B = \bigcup_{|\varepsilon|<\varepsilon_0} \bigcup_{t\in[0,t_0]} \phi_\varepsilon(t)W$ and claim that B attracts points in U_δ for $|\varepsilon| < \varepsilon_0$ sufficiently small. Firstly, B is contained in U_δ for ε_0 small. Indeed,

$$\bigcup_{t\in[0,t_0]} \phi_0(t)W \subset \bigcup_{t\in[0,t_0]} \phi_0(t)V \subset V \subset\subset U_\delta .$$

For ε_0 small we still have

$$\phi_\varepsilon(t_1)U_\delta \subset W \text{ and } \phi_\varepsilon(t_0)W \subset \phi_\varepsilon(t_0)V \subset W .$$

Therefore

$$\bigcup_{t\geqslant 0}\phi_\varepsilon(t)W = \bigcup_{t\in[0,t_0]} \phi_\varepsilon(t_0)W$$

is forward invariant. Moreover,

$$\phi_\varepsilon(t_1)U_\delta \subset W \subset \bigcup_{t\in[0,t_0]} \phi_\varepsilon(t)W \subset B$$

which proves our claim and the lemma. □

By using this lemma together with the theorems 4.1 and 4.2, we obtain the following corollary.

Corollary 7.2. *Consider $(T_h)_\varepsilon$ and $(D_4)_\varepsilon$ in the regions HKK. Then there exist parameter values for which the attractor A_ε contains horseshoes.*

Of course, we have not shown the existence of dense orbits in this case. But at least for equation with tetrahedral symmetry we are able to show the existence of a strange attractor for a generic unfolding.

Theorem 7.3. *Consider (3.1) in the parameter region KKO of theorem 4.2 with the assumptions of that theorem being fulfilled. Then there exist values of ε, such that the attractor A_ε of the ε-perturbed equation near Q contains a geometric Lorenz attractor as an asymptotically stable set.*

Proof. The existence of A_ε is guaranteed by lemma 7.1. We still show that the equation (3.1) fulfils the conditions of [Ryc90, theorem 1.2] together with the simplifications in [KKO93, section 3c]. Note that in the statement of that theorem the requirement

$$\mu_s < \mu_u < \min(2\mu_s, \mu_{ss})$$

is missing, but occurs in [Ryc90, (2.4) and lemma 3.4].

Firstly we will show that we can reduce (3.1) to a system possessing two homoclinic orbits conjugated by a $\mathbb{Z}_2$-symmetry. This will again be done by a suitable identification, see figure 15.

We choose two generators of $\mathbb{T}$

$$\sigma : (x, y, z) \mapsto (y, z, x) \qquad (\mathbb{Z}_3) \tag{7.1}$$

$$\kappa : (x, y, z) \mapsto (-x, -y-z) \qquad (\mathbb{Z}_2) . \tag{7.2}$$

Now we identify q_0 and q_2 by using $\kappa\sigma$, and q_1 and q_3 by $\sigma^2\kappa\sigma^2$. Hence we obtain a system with two different homoclinic solutions $\tilde{q}_0$ and $\tilde{q}_1$. The group element $\sigma^2\kappa\sigma$ induces a $\mathbb{Z}_2$-symmetry on the reduced system, which maps $\tilde{q}_0$ onto $\tilde{q}_1$. Furthermore, it acts like

$$\sigma^2\kappa\sigma : (x, y, z) \mapsto (x, -y, -z)$$

near p_0.

Together with the proof of theorem 4.2, we have therefore proved that the assumptions of [Ryc90, theorem 1.2] are fulfilled. This proves the theorem. □

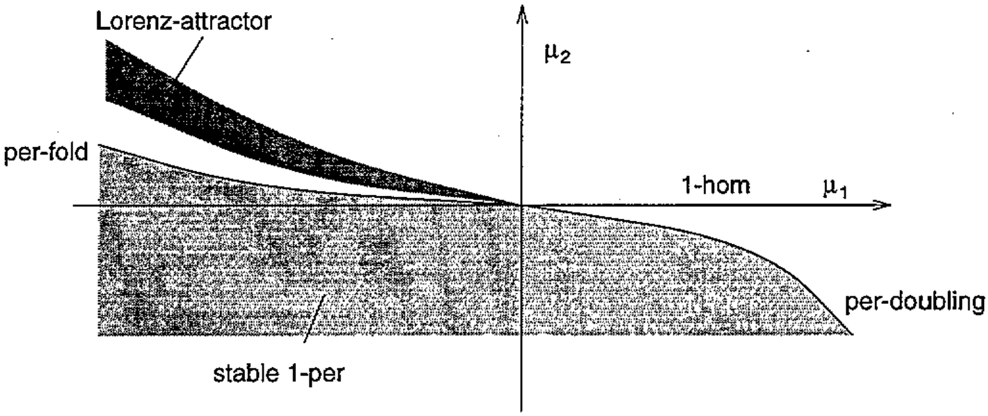

Figure 14. The domain of existence of a Lorenz attractor in parameter space.

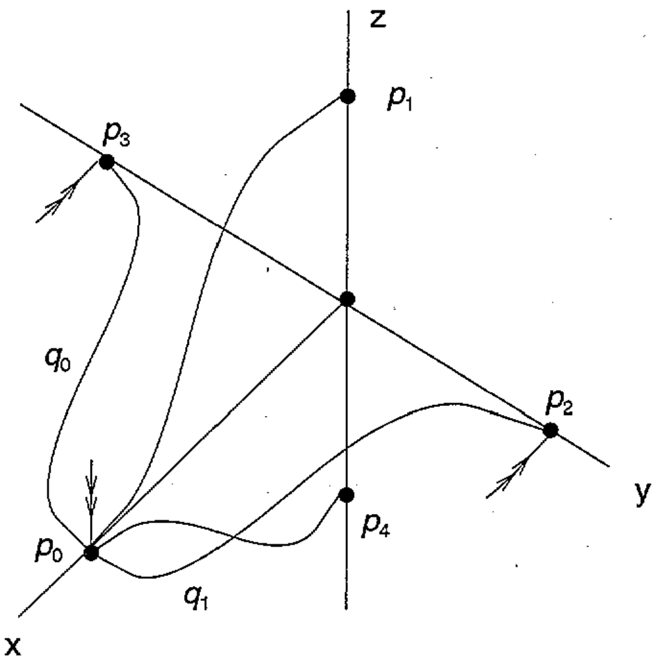

Figure 15. The $\mathbb{Z}_2$-symmetric homoclinic orbits.

8. Discussion

The present work can be understood as a first attempt to give a detailed description of dynamical phenomena created by symmetry breaking effects on homoclinic cycles. The principal observations are that the unfolding of a codimension zero situation in the equivariant context will require one, two or even more parameters even in simple situations with large residual symmetry groups. Unfortunately, there does not seem to be an easy connection between the stability properties of the heteroclinic cycles and bifurcating cycles and periodic orbits. Another important complication arises when one tries to determine *all* orbits in a neighbourhood of the cycle. Our method of looking for symmetric orbits then fails and the complications involved are not fully explained by our results in the last section on Lorenz attractors. We hope to be able to describe the dynamics on the attractor in such situations more comprehensively in a forthcoming paper.

In space dimension four, there is actually another way of forcing an inclination flip situation by symmetry. Krupa and Melbourne [KM91] observed that in $\mathbb{R}^4$, there are basically three different possibilities for a homoclinic cycle, captured by the following three examples:

(i) the group $\mathbb{T} \oplus \mathbb{Z}_2$ acting as in section 3;
(ii) the group $\mathbb{T} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, where $\mathbb{T} \oplus \mathbb{Z}_2$ acts in $\mathbb{R}^3$ as usually and the second copy of $\mathbb{Z}_2$ is a reflection with this copy of $\mathbb{R}^3$ as a fixed point space; the cycle is the same as in (i);
(iii) the group $\mathbb{Z}_4 \cdot \mathbb{Z}_2^4$ (see [FS91]) generated by a cyclic permutation of the coordinate axes and a reflection with respect to $\mathbb{R}^3$. There may exist homoclinic cycles lying in two-dimensional fixed point spaces of $\mathbb{Z}_2^2$.

In case (iii), an unfolding will require at least two parameters as can be seen as follows. In order to avoid the orbit-flip situation, we assume that in the equilibria with isotropy $\mathbb{Z}_2^3$, the radial direction (full isotropy!) is the strongest. The picture in the remaining directions then simplifies and is shown in the following diagram:

We can now easily observe that the strong stable fibres in $W^s(p_1)$ converge to the weakly stable direction in p_0 as $t \to -\infty$: a non-generic situation occurs which is captured precisely by hypothesis (IF1).

We would also like to point out that the bifurcation diagrams we have obtained are robust in the sense that they persist with respect to symmetry breaking with less residual symmetry.

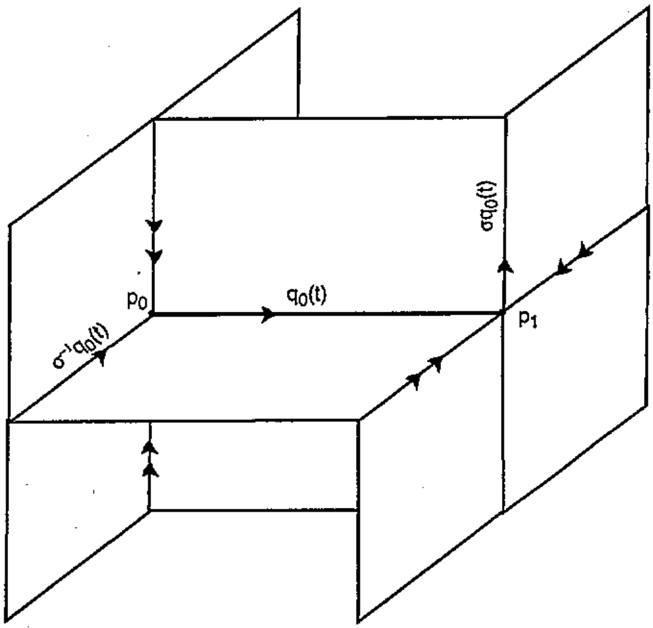

Figure 16. Homoclinic cycle with $\mathbb{Z}_4 \cdot \mathbb{Z}_2^4$ symmetry.

In the case of equivariance with respect to continous symmetry groups, we would guess that our results can be obtained whenever the continuous symmetry is preserved (see also [Cho92] and [CF92]). We were not able to describe symmetry breaking bifurcations when continuous symmetries are broken.

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Appendix A. Bifurcation scenario for D_4-symmetry in $y = 0$

Let us first determine the mixed modes. For $x \neq 0$, the equations for equilibria simplify to

$$\lambda + z + cz^2 + ax^2 = 0 \qquad \nu z - z^3 - (1 - \beta z)x^2 = 0\,.$$

In $x^2 = z = \lambda = \nu = 0$, the linearization with respect to the variables z and x^2 is $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and therefore invertible, which yields a unique solution $z(\lambda, \nu)$ and $x^2(\lambda, \nu)$. It remains to ensure that x^2 is positive, which is equivalent to the conditions

$$\lambda + z + cz^2 > 0 \qquad \text{and} \qquad \nu z - z^3 > 0\,.$$

So the only bifurcation points are at $x^2 = 0, z \in \{\pm\sqrt{\nu}, 0\}$ and it is now easy to see the pitchfork bifurcations that describe appearing and vanishing of the mixed modes.

The Hopf curve is determined by trace $(Df) = 0$ on the curve of bifurcating equilibria

$$z(\lambda, \nu) = -\lambda + a\lambda\nu - c\lambda^2 + \lambda O((\nu + |\lambda|)^2)$$
$$x^2(\lambda, \nu) = -\lambda\nu + \lambda O((\nu + |\lambda|)^2)\,.$$

The trace condition gives

$$\lambda + z + cz^2 + 3ax^2 = \nu - 3z^2 + \beta x^2$$

or

$$\nu = 3\lambda^2 + \lambda O(|\lambda|^2)$$

which gives the desired curve. One can easily verify that the determinant is non-zero. The only possibility for the periodic orbit to disappear is a blue-sky catastrophy near a heteroclinic cycle $p^+ \to 0 \to p^+$.

Therefore the Hopf bifurcation has to be supercritical. The curve for the codimension-one global heteroclinic loop bifurcation will now be obtained by a particular scaling. We set

$$\tilde{x} = \nu^{-\frac{1}{2}}x \qquad \tilde{\lambda} = \nu^{-\frac{1}{2}}\lambda$$
$$\tilde{z} = \nu^{-\frac{1}{2}}z \qquad \frac{\mathrm{d}}{\mathrm{d}\tau} = \nu^{-\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}t}\,.$$

and we obtain the scaled equations

$$\tilde{x}' = \tilde{\lambda}\tilde{x} + \tilde{x}\tilde{z} + \nu^{\frac{1}{2}}\tilde{x}(a\tilde{x}^2 + c\tilde{z}^2)$$
$$\tilde{z}' = -\tilde{x}^2 + \nu^{\frac{1}{2}}\tilde{z}(\alpha\tilde{x}^2 + \beta\tilde{z}^2 + 1)$$

where $' = \frac{\mathrm{d}}{\mathrm{d}\tau}$. We study this system as a perturbation of the limit in $\nu = 0$

$$\tilde{x}' = (\tilde{\lambda} + \tilde{z})\tilde{x} \qquad \tilde{z}' = -\tilde{x}^2\,.$$

Dividing by the Euler multiplicator $\tilde{x}$, the right-hand side is just a linear vector field, turning around $(0, -\tilde{\lambda})$. The $\tilde{z}$-axis consists entirely of equilibria. The strong unstable manifold of the equilibrium $(0, \zeta)$ with $\zeta > -\tilde{\lambda}$ is identical with the strong stable manifold of the equilibrium $(0, -2\tilde{\lambda} - \zeta)$. Fixing two equilibria and varying the parameter $\tilde{\lambda}$, the stable and unstable fibres cross transversely in $\tilde{\lambda}$. The Melnikov function can easily be calculated to be non-degenerate. In particular, for $\tilde{\lambda} = -\frac{1}{2}$, the stable and unstable fibres of $\zeta = 1$ and $\zeta = 0$ (which correspond to p^+ and 0) intersect. As the stable and unstable fibres depend smoothly on the parameter $\nu^{\frac{1}{2}}$ (cf [Fen79]), this intersection persists for some nearby parameter value $\tilde{\lambda}_{\text{het}} = -\frac{1}{2} + O(\nu^{\frac{1}{2}})$. For $\tilde{\lambda} > \tilde{\lambda}_{\text{het}}$, the unstable fibre of the equilibrium $\zeta = 1$ intersects a stable fibre of a point on the ζ-axis, which converges to $\zeta = -1$ and therefore yields a heteroclinic orbit $p^+ \to p^-$. For $\tilde{\lambda} < \tilde{\lambda}_{\text{het}}$, the unstable fibre approaches the singular point $\zeta = -\tilde{\lambda}$, where the mixed mode is created for $\nu > 0$. In the original coordinates this yields a heteroclinic bifurcation at $\lambda_{\text{het}} = -\frac{1}{2}\sqrt{\nu} + O(\nu)$ and a heteroclinic orbit $p^+ \to p^-$ for $\lambda > \lambda_{\text{het}}$. The upper bound for the existence of $p^+ \to p^-$ connections can, also from this picture, be seen to be precisely the bifurcation point, when p^- loses stability.

Let us remark that near $\lambda = \lambda_{\text{flip}} = O(\nu)$, the heteroclinic orbit switches side: it approaches the equilibrium p^- from below for $\lambda > \lambda_{\text{flip}}$. At $\lambda = \lambda_{\text{flip}}$, a symmetry breaking unfolding should exhibit phenomena of the orbit-flip bifurcations in the four-dimensional examples. Only here this bifurcation is of codimension one already in the symmetric context.

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