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On the Stability of Periodic Travelling Waves with Large Spatial Period

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In many circumstances, a pulse to a partial differential equation (PDE) on the real line is accompanied by periodic wave trains that have arbitrarily large period. It is then interesting to investigate the PDE stability of the periodic wave trains given that the pulse is stable. Using the Evans function, Gardner has demonstrated that every isolated eigenvalue of the linearization about the pulse generates a small circle of eigenvalues for the linearization about the periodic waves. In this article, the precise location of these circles is determined. It is demonstrated that the stability properties of the periodic waves depend on certain decay and oscillation properties of the tails of the pulse. As a consequence, periodic waves with long wavelength typically destabilize at homoclinic bifurcation points at which multi-hump pulses are created. That is in contrast to the situation for the underlying pulses whose stability properties are not affected by these bifurcations. The proof uses Lyapunov–Schmidt reduction and relies on the existence of exponential dichotomies. The approach is also applicable to periodic waves with large spatial period of elliptic problems on $\mathbb{R}^n$ or on unbounded cylinders $\mathbb{R} \times \Omega$ with Ω bounded.

Key Words: wave trains; periodic travelling waves; pulses; stability.

1. INTRODUCTION

Spatially-periodic travelling waves arise in a wide variety of pattern-forming physical systems. Examples include reaction–diffusion equations such as the FitzHugh–Nagumo equations or the Gray–Scott model, systems

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from nonlinear optics like the nonlinear Schrödinger equation as well as hydrodynamical equations like the Euler equations for free surface waves. In these systems, spatially-periodic standing or travelling patterns typically exist for a continuum of wavelengths. Often, this continuum extends to infinite wavelength where the shape of the periodic pattern on a single periodicity interval approaches a pulse-like pattern. The limit of infinite wavelength is described by a homoclinic bifurcation for the spatial dynamics in the underlying partial differential equation (PDE). Two natural questions that one would like to answer are the existence of periodic patterns through such a homoclinic bifurcation and their stability with respect to the PDE.

Concerning existence, a Lyapunov–Schmidt type approach to homoclinic bifurcations has recently been initiated that allows for a fairly systematic study of the creation of periodic as well as multi-pulse patterns [24, 41, 31]. One advantage of this approach is its immediate generalization to PDEs that are posed on multi-dimensional domains such as reaction–diffusion equations on infinite cylinders [29, 25] or the aforementioned Euler equations for free surface waves.

It has been demonstrated that this approach, which is based on exponential dichotomies for the linearized problem and a careful Lyapunov–Schmidt type reduction procedure, is particularly well suited for analyzing the PDE-stability of bifurcating multi-pulse solutions [33].

In this paper, we show that the same approach allows us to systematically study the stability properties of long-wavelength periodic patterns that accompany pulses. The general idea is to use as much information as possible from the pulse to calculate the spectrum of the accompanying periodic patterns using a perturbation analysis. The arguments follow those given in [33] closely. A major difference, however, is that the spectrum of the periodic waves consists entirely of essential spectrum. This is in contrast to the situation for pulses and multi-pulses where stability is, in most applications, determined by point spectrum.

The spectral stability of the limiting pulse or soliton has been established in many of the aforementioned examples. An important tool for this kind of analysis is the Evans function that was introduced by Evans in [9] and has since then been further developed, in a more systematic way, in [1, 28, 13, 18]. Roughly speaking, the Evans function is an analytic complex-valued function of λ that is constructed in such a fashion that its zeros are in one-to-one correspondence with eigenvalues, counting multiplicity.

Gardner generalized the construction of the Evans function to arbitrary spatially-periodic patterns [11]. Later, he studied the limiting scenario when the wavelength of the periodic patterns tends to infinity [12]; as mentioned above, this is the case we are interested in. The main tool in Gardner's analysis was a topologically robust bundle construction which allowed him to pass to the limit of infinite wavelength. The topological nature of the construction did, however, not allow for a precise description of the spectra of the periodic patterns in the sense of an expansion in the wavelength. In particular, the methods in [12] do not allow for proving stability of periodic patterns even when the limiting pulse is exponentially stable.

In the remaining part of the introduction, we illustrate the problem at hand by means of an abstract reaction–diffusion system. We then summarize the results obtained in [11, 12] and briefly explain our main theorem.

Thus, consider a reaction–diffusion equation of the form

$$U_t = U_{xx} + F(U), \tag{1}$$

where $U(x, t) \in \mathbb{R}^N$ and $x \in \mathbb{R}$, $t \geqslant 0$ are space and time, respectively. If the nonlinearity F is smooth, this equation generates a smooth local nonlinear semiflow on the Banach space $X := \mathscr{C}^0_{\text{unif}}(\mathbb{R}, \mathbb{R}^N)$ of bounded and uniformly continuous functions on the real line.

A travelling-wave solution $U(x, t)$ of (1) is a pattern whose time evolution is a translation, with constant speed c, along the x-axis. In other words, we have $U(x, t) = Q(x - ct)$ for some function Q. Here, and in the following, we allow for $c = 0$ in which case the pattern is a standing wave. Transforming the reaction–diffusion system (1) into the moving coordinate frame $(\xi, t) = (x - ct, t)$, we obtain

$$U_t = U_{\xi\xi} + cU_\xi + F(U). \tag{2}$$

A travelling wave $Q(x - ct)$ with wave speed c of the original equation (1) corresponds to a steady-state $Q(\xi)$ of (2). The spectral stability properties of the equilibrium $Q(\xi)$ are determined by the linearized operator

$$\mathscr{L}(Q) = \partial_{\xi\xi} + c\partial_\xi + F_U(Q). \tag{3}$$

Both the steady-state equation associated with (2) and the eigenvalue problem associated with the operator $\mathscr{L}(Q)$ are ordinary differential equations (ODE). If we set $u = (U, U_\xi)$, then U is a bounded steady-state with

$$U_{\xi\xi} + cU_\xi + F(U) = 0, \qquad U \in \mathbb{R}^N$$

if, and only if, u is a bounded solution of the $2N$-dimensional system

$$u' = f(u), \qquad u \in \mathbb{R}^{2N}, \tag{4}$$

where $' = \frac{d}{d\xi}$ and

$$f(U, W) = \begin{pmatrix} W \\ -(cW + F(U)) \end{pmatrix}.$$

By the same argument, we see that V satisfies the eigenvalue problem

$$\mathscr{L}(Q)\, V = V_{\xi\xi} + cV_{\xi} + F_U(Q)\, V = \lambda V$$

for some eigenvalue $\lambda \in \mathbb{C}$ if, and only if, $v = (V, V_{\xi})$ is a bounded solution of

$$v' = (f_u(q(\xi)) + \lambda B)\, v, \tag{5}$$

where f_u is the Jacobian matrix of f, $q = (Q, Q_{\xi})$, and

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{6}$$

It turns out that the spectrum of periodic waves $Q(\xi)$ is determined entirely by (5) as we shall explain below.

Suppose that $Q(\xi)$ is periodic in ξ with minimal period $2L$. It is then true that $\lambda \in \mathbb{C}$ is in the spectrum of the linearization $\mathscr{L}(Q)$ if, and only if, there is a $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$ and a $V \in X$ such that

$$\mathscr{L}(Q)\, V = \lambda V, \qquad \text{for} \quad |\xi| < L \tag{7}$$

and

$$V(L) = e^{i\gamma} V(-L). \tag{8}$$

Indeed, solutions of the above two equations yield, by extension, bounded eigenfunctions of $\mathscr{L}(Q)$. On the other hand, by Floquet theory, bounded solutions of the "periodically-forced" linear ODE (5) are necessarily of the form (7–8) for some purely imaginary Floquet exponent $i\gamma$. It is then not hard to see that $\mathscr{L}(Q)$ is indeed invertible provided all Floquet exponents have non-zero real part; see, for instance, [11, Prop. 2.1] and the references therein. In terms of the first-order system (5) of ODEs, $\lambda \in \mathbb{C}$ is in the spectrum of the linearization $\mathscr{L}(Q)$ if, and only if, there is a $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$ and a bounded function v such that

$$\begin{aligned} v' &= (f_u(q(\xi)) + \lambda B)\, v, \qquad \text{for} \quad |\xi| < L \\ v(L) &= e^{i\gamma} v(-L). \end{aligned} \tag{9}$$

Thus, once a $2L$-periodic solution of the steady-state ODE (4) has been found, its spectrum as an equilibrium of the PDE (2) can be calculated by seeking solutions to the linearized ODE (9).

We focus on the situation where the steady-state equation (4) exhibits a homoclinic orbit $h(\xi)$ so that, for some constant vector $e \in \mathbb{R}^{2N}$, we have $h(\xi) \to e$ as $|\xi| \to \infty$. Such a homoclinic orbit corresponds to a pulse solution of the reaction–diffusion system (2). Typically, the pulse is accompanied by long-wavelength periodic waves as described above. In terms of the ODE (4), these periodic waves appear as a family of periodic solutions with large period close to the homoclinic orbit in the phase space $\mathbb{R}^{2N}$. More specifically, there is a number L_* such that, for every $L > L_*$, there is a unique $2L$-periodic solution $p_L(\xi)$ close to $h(\xi)$ for a wave speed $c = c_L$ that is close to the speed $c = c_\infty$ of the homoclinic orbit (in the case that $c_\infty = 0$, we expect that $c_L = 0$ for all L).

Given that the pulse $h(\xi)$ is stable with respect to the PDE (2), it is then of interest, and indeed the main focus of this article, to determine the stability properties of the accompanying periodic waves $p_L(\xi)$. Typically, the pulse $h(\xi)$ is stable provided $\lambda = 0$ is a simple eigenvalue and the rest of its spectrum is contained in the open left half-plane and bounded away from the imaginary axis. Note that the eigenvalue at zero is inevitable due to the translation symmetry. Gardner [12, Theorem 1.2] demonstrated that the spectrum of the periodic waves $p_L(\xi)$ that accompany the pulse $h(\xi)$ contains a circle of eigenvalues near $\lambda = 0$ that shrinks to $\lambda = 0$ as $L \to \infty$. He also showed that the rest of the spectrum is contained in the open left half-plane and bounded away from the imaginary axis uniformly in L. To establish spectral stability or instability of the $2L$-periodic wave trains $p_L(\xi)$, it is then necessary to locate the circle of eigenvalues near zero. Based upon the discussion above, this is equivalent to solving the boundary-value problem

$$\begin{aligned} v' &= (f_u(p_L(\xi)) + \lambda B)\, v, \qquad \text{for} \quad |\xi| < L \\ v(L) &= e^{i\gamma} v(-L) \end{aligned} \tag{10}$$

for $\lambda \in \mathbb{C}$ close to zero. Note that one solution to (10) is given by $v = p'_L$, $\gamma = 0$ and $\lambda = 0$. This eigenvalue is again enforced by the translation symmetry.

As announced above, the main result of this article is an expansion of the circle of critical eigenvalues of the long-wavelength periodic patterns in terms of their period. Under the assumption that the underlying pulse is stable, our main result demonstrates that the critical spectrum and the associated eigenfunctions about the $2L$-periodic wave are, to leading order, given by

$$\lambda = \frac{1}{M}((e^{i\gamma}-1)\langle \psi(L), h'(-L)\rangle + (1-e^{-i\gamma})\langle \psi(-L), h'(L)\rangle) \tag{11}$$
$$v(\xi) = e^{ik\gamma}h'(\xi), \qquad \text{for} \quad \xi \in ((2k-1)L, (2k+1)L), \ k \in \mathbb{Z}.$$

We shall explain the quantities that appear in these expressions. The number γ represents again the purely imaginary spatial Floquet exponent that appears in (10). The non-zero constant M is given explicitly by the Melnikov-type integral

$$M = \int_{-\infty}^{\infty} \langle \psi(x), Bh'(x)\rangle \, dx,$$

where B is determined by the type of the PDE; see (6). Here, and in the expansion (11), the function $\psi(\xi)$ is the non-trivial bounded solution of the adjoint variational equation

$$w' = -f_u(h(\xi))^* \, w \tag{12}$$

about the homoclinic orbit $h(\xi)$.

Thus, we see that the location of the circle of critical eigenvalues depends upon the decay properties of $h(\xi)$ and $\psi(\xi)$. Both of these are determined by the steady-state ODE (4).

The solution $\psi(\xi)$ to the adjoint variational equation (12) has the following analytical and geometric properties. First, recall that we had assumed that $\lambda = 0$ is a simple eigenvalue of the linearization $\mathscr{L}(H)$ where $h = (H, H_\xi)$ denotes the pulse. Therefore, the adjoint operator $\mathscr{L}(H)^*$ also has a simple eigenvalue at zero, and we denote its eigenfunction by $\Psi(\xi)$. A straightforward calculation shows that the solution $\psi(\xi)$ is related to this eigenfunction $\Psi(\xi)$ by

$$\psi(\xi) = \begin{pmatrix} F_U(H(\xi))\, \Psi(\xi) \\ \Psi(\xi) \end{pmatrix}.$$

Geometrically, in the phase space $\mathbb{R}^{2N}$ of the steady-state ODE (4), the solution $\psi(\xi)$ spans the orthogonal complement of the tangent spaces of stable and unstable manifolds of the asymptotic equilibrium e at the homoclinic point $h(\xi)$: We have

$$\text{span}\{\psi(\xi)\} = (T_{h(\xi)} W^s(e) + T_{h(\xi)} W^u(e))^\perp$$

for all ξ, where $e = \lim_{|\xi| \to \infty} h(\xi)$. Note that the equilibrium e of (4) is hyperbolic since, by assumption, the essential spectrum of the pulse does not touch zero. Finally, in the particular case where the steady-state equation (4) is Hamiltonian with Hamiltonian function $\mathscr{H}(u)$, we have $\psi(\xi)=\nabla\mathscr{H}(h(\xi))$.

We emphasize that it is not unexpected that the decay properties of both the homoclinic solution and the adjoint solution are important. It has been demonstrated in [24, 31] that scalar products of the form $\langle\psi(L), h'(-L)\rangle$ and $\langle\psi(-L), h'(L)\rangle$ enter the reduced bifurcation equations that determine the existence of periodic and multi-bump homoclinic orbits near $h(\xi)$. In fact, roughly speaking, whenever the sign of these scalar products oscillates in L or their magnitude decays faster in L than expected, then 2-homoclinic orbits bifurcate that follow the original orbit $h(\xi)$ twice in phase space; see [31]. In particular, this is the case near Shilnikov homoclinic orbits and near inclination or orbit-flip bifurcations; see [5, 16, 19, 31, 33, 34, 36]. In all these cases, the accompanying periodic orbits undergo saddle-node or period-doubling bifurcations. These bifurcations, however, should also lead to an instability with respect to the PDE. In the case of a period-doubling, for instance, (10) exhibits solutions for $\gamma=0$ and $\gamma=\pi$ at $\lambda=0$. While the solution with $\gamma=0$ is enforced by the translation symmetry, the other eigenvalue that has $\gamma=\pi$ should cross the imaginary axis upon unfolding the period-doubling bifurcation. In that respect, a change of the stability properties of the periodic waves that accompany a pulse indicates a homoclinic bifurcation for (4), and hence the possible appearance of other pulses.

This paper is organized as follows. We state the main reduction result in Section 2 and prove it in Section 3. In Section 4, we supply the necessary expansions and estimates for periodic waves that are needed to apply the reduction theorem. Section 5 contains various stability results for periodic waves under certain decay and oscillation assumptions on the pulse. These results are then applied in Section 6 to three PDEs on the real line. Finally, some comments and generalizations are collected in Section 7.

2. LOCATING THE SPECTRUM OF PERIODIC WAVES WITH LARGE SPATIAL PERIOD

Consider the ordinary differential equation

$$u'=f(u,\mu),\qquad (u,\mu)\in\mathbb{R}^n\times\mathbb{R}^p, \tag{13}$$

where f is at least $\mathscr{C}^2$ with $f(0,\mu)=0$ for all μ. A comparison with the example given in the introduction shows that, for reaction-diffusion systems, the dimension $n=2N$ of the phase space is twice the number N of species in the reaction-diffusion system, and the parameter $\mu = c$ is given by the wave speed c so that $p = 1$.

We assume that $f_u(0, 0)$ is hyperbolic. In other words, there are positive constants α^{s} and α^{u} such that, for every $\nu \in \operatorname{spec}(f_u(0, 0))$, either $\operatorname{Re} \nu < -\alpha^{\mathrm{s}}$ or $\operatorname{Re} \nu > \alpha^{\mathrm{u}}$. Let

$$\alpha = \min\{\alpha^{\mathrm{s}}, \alpha^{\mathrm{u}}\}.$$

The spectral projections associated with the stable and unstable eigenvalues of $f_u(0, 0)$ are denoted by P_0^{s} and P_0^{u}, respectively.

We assume that $h(x)$ is a homoclinic orbit to (13) at $\mu = 0$, that is, $h' = f(h, 0)$ and

$$\lim_{x \to \pm\infty} h(x) = 0.$$

We assume that the intersection of stable and unstable manifolds of the origin is as transverse as possible:

Hypothesis (N1). The only bounded solution to the variational equation

$$v' = f_u(h(x), 0)\, v, \qquad x \in \mathbb{R} \tag{14}$$

of (13) about $h(x)$ is given by $h'(x)$, up to constant scalar multiples.

For the underlying reaction-diffusion system, this assumption implies that $\lambda = 0$ has geometric multiplicity one as an eigenvalue of the PDE linearization.

Next, consider the adjoint variational equation

$$w' = -f_u(h(x), 0)^*\, w, \qquad x \in \mathbb{R} \tag{15}$$

with respect to the standard scalar product on $\mathbb{R}^n$. Hypothesis (N1) implies that (15) admits a unique, up to scalar multiples, bounded solution which we denote by $\psi(x)$. Indeed, a short calculation shows that the scalar product between solutions of the variational equation (14) and its adjoint (15) is independent of x. Therefore, bounded solutions of the adjoint variational equation have to be orthogonal to every solution of (14) that is bounded on either $\mathbb{R}^+$ or $\mathbb{R}^-$. In other words, any bounded solution of the adjoint variational equation lies in the orthogonal complement of the tangent spaces to stable and unstable manifolds at the homoclinic $h(x)$ which, by Hypothesis (N1), is one-dimensional. It is a consequence of hyperbolicity of the linearization $f_u(0, 0)$ that the solution $\psi(x)$ decays exponentially:

$$|\psi(x)| \leqslant C\mathrm{e}^{-\alpha|x|}, \qquad x \in \mathbb{R};$$

see, for instance, Lemma 3.1 in Section 3.1 and the remarks thereafter.

We are interested in periodic solutions to (13) that are close to the homoclinic orbit $h(x)$; in particular, their period is large. Thus, assume that $p_L(x)$ is a periodic solution of (13) close to $h(x)$ with large period $2L$ for $\mu = \mu_L$ close to $\mu = 0$. We shall see later in Sections 5 and 6 that, under quite general assumptions, such periodic solutions indeed exist. As explained in the introduction, the PDE spectrum associated with the periodic wave can be calculated as follows. A point $\lambda \in \mathbb{C}$ is in the PDE spectrum of the periodic wave if, and only if, there is a $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$ and a solution $v(x)$ to the boundary-value problem

$$v' = (f_u(p_L(x), \mu_L) + \lambda B)\, v, \qquad |x| < L \tag{16}$$

such that

$$v(L) = \mathrm{e}^{\mathrm{i}\gamma}\, v(-L). \tag{17}$$

Here, B is an $n \times n$ matrix that is related to the type of the partial differential equation. Recall that, for reaction-diffusion equations, $n = 2N$ and B is the block matrix given in (6). We concentrate on solutions of (16–17) for λ close to zero. The following theorem provides the crucial expansions that, together with expansions for the periodic waves, is used later to prove PDE stability or instability.

THEOREM 2.1. *Assume that Hypothesis* (N1) *is met. There are then positive numbers C and δ with the following property. Suppose that $p_L(x)$ is a periodic solution of* (13) *for $\mu = \mu_L$ with period $2L$ such that*

$$\sup_{|x| \leqslant L} |p_L(x) - h(x)| < \delta, \qquad |\mu_L| < \delta, \qquad 2L > \frac{1}{\delta}.$$

The boundary-value problem (16–17) *then has a solution (λ, γ, v) for $\lambda \in \mathbb{C}$ with $|\lambda| < \delta$ and $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$ if, and only if,*

$$E(\lambda, \gamma) = 0 \tag{18}$$

(*we omit the dependence of E on L*), *where*

$$E(\lambda, \gamma) = (\mathrm{e}^{\mathrm{i}\gamma} - 1)\langle \psi(L), h'(-L)\rangle + (1 - \mathrm{e}^{-\mathrm{i}\gamma})\langle \psi(-L), h'(L)\rangle \\ - \lambda \int_{-\infty}^{\infty} \langle \psi(x), Bh'(x)\rangle\, \mathrm{d}x + (\mathrm{e}^{\mathrm{i}\gamma} - 1)\, R(\lambda, \gamma) + \lambda\, \tilde{R}(\lambda, \gamma). \tag{19}$$

Here, $h'(x)$ is the derivative of the homoclinic orbit, $\psi(x)$ is a non-zero bounded solution of the adjoint variational equation (15), *B is an arbitrary matrix,* *and the eigenvalue λ and the purely imaginary Floquet exponent* iγ *appear as parameters in the boundary-value problem* (16–17). *The remainder terms* $R(\lambda, \gamma)$ *and* $\tilde{R}(\lambda, \gamma)$ *of the expansion are analytic in* (λ, γ) *and satisfy*

$$
\begin{aligned}
|\partial_\lambda^j \partial_\gamma^\ell R(\lambda, \gamma)| \leqslant C(&|p_L(L)|\,(|\lambda| + \mathrm{e}^{-\alpha L} + |\mu_L| + \sup_{|x| \leqslant L} |p_L(x) - h(x)|)^2 \\
&+ \mathrm{e}^{-\alpha L}\,(|P_0^{\mathrm{u}}(p_L(-L) - h(-L))| + |P_0^{\mathrm{s}}(p_L(L) - h(L))| \\
&+ |\mu_L| + |p_L(L)|^2 + \mathrm{e}^{-2\alpha L})) \\
|\partial_\gamma^\ell \tilde{R}(\lambda, \gamma)| \leqslant C(&|\lambda| + \mathrm{e}^{-\alpha L} + |\mu_L| + \sup_{|x| \leqslant L} |p_L(x) - h(x)|) \\
|\partial_\lambda^{j+1} \partial_\gamma^\ell \tilde{R}(\lambda, \gamma)| \leqslant C&
\end{aligned}
\tag{20}
$$

for $j, \ell \geqslant 0$. *Both* $R(\lambda, \gamma)$ *and* $\tilde{R}(\lambda, \gamma)$ *are real whenever* $(\lambda, \mathrm{e}^{\mathrm{i}\gamma})$ *is real.*

Finally, if (λ, γ) *is a zero of* $E(\lambda, \gamma)$ *with* $|\lambda| < \delta$ *and* $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$, *then the associated solution* $v(x)$ *of the boundary-value problem* (16–17) *satisfies*

$$
|v(x) - \mathrm{e}^{\mathrm{i}k\gamma} h'(x)| \leqslant C(|\lambda| + |p_L(L)| + \sup_{|y| \leqslant L} |p_L(y) - h(y)|) \tag{21}
$$

for $x \in ((2k-1)\,L, (2k+1)\,L)$ *with* $k \in \mathbb{Z}$.

Theorem 2.1 shows that it suffices to calculate solutions of the reduced system (18) to determine the spectrum of periodic waves with large period. To evaluate the terms that arise in the expansion (19), we have to derive expansions of the solutions $h'(x)$ and $\psi(x)$ for large $|x|$ and obtain estimates for the various terms that appear in (20). We address these issues in Section 5.

Theorem 2.1 applies to situations where the eigenvalue $\lambda = 0$ that is contained in the PDE spectrum of the underlying pulse is simple. In some applications, the PDE possesses a gauge symmetry that enforces a higher geometric multiplicity of the eigenvalue at zero. One example is the complex Ginzburg–Landau equation that admits a phase invariance. To include these PDEs, we give a more general theorem, Theorem 2.2, that is formulated in an equivariant set-up. We therefore assume equivariance of the underlying ODE with respect to a closed connected subgroup Σ of the orthogonal group $\mathrm{O}(n)$. Theorem 2.1 is then obtained by restricting to the special case $\Sigma = \{\mathrm{id}\}$.

Hypothesis (H1). There is a closed connected subgroup Σ of $\mathrm{O}(n)$ such that $f(\sigma u, \mu) = \sigma f(u, \mu)$ for every $(u, \mu) \in \mathbb{R}^n \times \mathbb{R}^p$ and $\sigma \in \Sigma$.

We can then no longer expect that the homoclinic orbit h satisfies the non-degeneracy condition stated in Hypothesis (N1) since h is part of the family Σh of homoclinic orbits. In fact, any element in the tangent space of the manifold Σh at $h(x)$ corresponds to a bounded solution of the variational equation (14) about $h(x)$. Thus, the appropriate non-degeneracy condition is that the only bounded solutions of the variational equation are those enforced by the symmetry. To be more specific, we choose a basis in the tangent space of Σh in the following fashion. Let $m = \dim \Sigma + 1$ and choose a basis $\{S_j\}_{j=1,\dots,m-1}$ of the Lie algebra $\operatorname{alg}(\Sigma)$ of Σ. It follows that the functions $h'(x)$ and $S_j h(x)$ for $j = 1, \dots, m-1$ are bounded solutions of (14). Indeed, using Hypothesis (H1), we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(\exp(S_j\tau)\,h(x)) = \exp(S_j\tau)\,h'(x) = f(\exp(S_j\tau)\,h(x), 0)$$

for every $\tau \in \mathbb{R}$ close to zero, where $\exp(S)$ is the exponential map from $\operatorname{alg}(\Sigma)$ into Σ; differentiating the aforementioned equation with respect to τ at $\tau = 0$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(S_j h(x)) = f_u(h(x), 0)\,S_j h(x).$$

Non-degeneracy of the homoclinic orbit is then expressed in the following hypothesis.

Hypothesis (H2) The solutions $h'(x)$ and $S_j h(x)$ for $j = 1, \dots, m-1$ are linearly independent, and every bounded solution to (14) is a linear combination of these functions.

In terms of the underlying PDE, this means that the eigenvalue $\lambda = 0$ of the PDE linearization about the pulse $h(x)$ has geometric multiplicity m, and the associated eigenspace is generated by the translation symmetry and the additional symmetry Σ. Note that Hypothesis (H2) implies that the vectors $S_j h(0)$ with $j = 1, \dots, m-1$ are linearly independent. In particular, the only element $\sigma \in \Sigma$ that is close to id and satisfies $\sigma h(0) = h(0)$ is $\sigma = \mathrm{id}$; thus, continuous isotropies of the pulse are excluded.

Arguing as in the non-symmetric case, we see that the adjoint variational equation (15) exhibits precisely m linearly independent bounded solutions which we denote by $\psi_j(x)$ for $j = 1, \dots, m$. All these solutions decay exponentially as $|x| \to \infty$. Once more, we consider the boundary-value problem (16–17).

THEOREM 2.2. *Assume that the equivariance assumption* (H1) *and the non-degeneracy condition* (H2) *on the homoclinic orbit $h(x)$ are met. There are then positive numbers C and δ with the following property. Suppose that $p_L(x)$ is a periodic solution of* (13) *for $\mu = \mu_L$ with period $2L$ such that*

$$\sup_{|x| \leqslant L} |p_L(x) - h(x)| < \delta, \qquad |\mu_L| < \delta, \qquad 2L > \frac{1}{\delta}.$$

The boundary-value problem (16–17) *has then a solution* (λ, γ, v) *for* $\lambda \in \mathbb{C}$ *with* $|\lambda| < \delta$ *and* $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$ *if, and only if,*

$$\det E(\lambda, \gamma) = 0 \tag{22}$$

(*we omit the dependence of E on L*), *where* $E(\lambda, \gamma)$ *is the* $m \times m$ *matrix with entries* $E_{kj}(\lambda, \gamma)$ *given by*

$$\begin{aligned} E_{kj}(\lambda, \gamma) = {} & (e^{i\gamma} - 1)\langle \psi_k(L), \varphi_j(-L)\rangle + (1 - e^{-i\gamma})\langle \psi_k(-L), \varphi_j(L)\rangle \\ & - \lambda \int_{-\infty}^{\infty} \langle \psi_k(x), B\varphi_j(x)\rangle \, dx + (e^{i\gamma} - 1)\, R_{kj}(\lambda, \gamma) + \lambda\, \tilde{R}_{kj}(\lambda, \gamma). \end{aligned} \tag{23}$$

Here, the functions $\varphi_j(x) := S_j h(x)$ *for* $j = 1, ..., m-1$ *and* $\varphi_m(x) = h'(x)$ *span the tangent space to the manifold* Σh *of homoclinic orbits at h. The remainder terms* $R_{kj}(\lambda, \gamma)$ *and* $\tilde{R}_{kj}(\lambda, \gamma)$ *in the expansion* (23) *are analytic in* (λ, γ) *and satisfy the same estimates* (20) *as R and* $\tilde{R}$ *in Theorem* 2.1. *Also,* $R_{kj}(\lambda, \gamma)$ *and* $\tilde{R}_{kj}(\lambda, \gamma)$ *are real whenever* $(\lambda, e^{i\gamma})$ *is real.*

This theorem is proved in Section 3 below.

3. PROOF OF THEOREM 2.2

We begin by outlining our proof which is carried out in several steps.

In Section 3.1, we provide some useful decay estimates for particular solutions of the variational equation (14). Due to hyperbolicity of the equilibrium, the space $\mathbb{R}^n$ can be written as a direct sum of two subspaces. These subspaces consists of all initial values with the property that the associated solutions of the variational equation about the homoclinic orbit decay exponentially in either forward or backward x-direction. This type of splitting is generally referred to as an exponential dichotomy and has been systematically exploited in [24, 31] to investigate homoclinic bifurcations; see also Remark 3.1.

After these preparations, we reformulate the relevant boundary-value problem (16–17)

$$\begin{aligned} v' &= (f_u(p_L(x), \mu_L) + \lambda B)\, v, \qquad x \in (-L, L) \\ v(L) &= e^{i\gamma}\, v(-L). \end{aligned}$$

First of all, we consider the linear ODE separately on the intervals $(-L, 0)$ and $(0, L)$ rather than on $(-L, L)$. We then have to match solutions at $x = 0$. It turns out that the resulting boundary-value problem can be solved in a unique fashion provided we relax the matching condition at $x=0$: we require only that solutions for $x>0$ and $x<0$ can be patched together, at $x=0$, on a certain $(n-m)$-dimensional subspace of $\mathbb{R}^n$ rather than on the entire space $\mathbb{R}^n$. The reduced equations (18) and (22) that arise in Theorem 2.1 and 2.2, respectively, are the remaining matching conditions on an m-dimensional complement of the aforementioned $(n-m)$-dimensional subspace of $\mathbb{R}^n$. In other words, using Lyapunov–Schmidt reduction, we invert a certain part of the relevant boundary-value problem and obtain reduced equations that describe the remaining unsolved part. To solve the invertible part of the boundary-value problem, we approximate the periodic solutions by the homoclinic orbit and then use perturbation arguments to obtain solutions to the correct problem.

3.1. *The Variational Equation about the Homoclinic Orbit*

In this section, we collect some useful properties of the variational equation

$$v' = f_u(h(x), 0)\, v \tag{24}$$

about $h(x)$. Let

$$Y^{\mathrm{c}} := \operatorname{span}\{S_1 h(0), ..., S_{m-1} h(0), h'(0)\},$$

$$Y^{\perp} := \operatorname{span}\{\psi_1(0), ..., \psi_m(0)\}.$$

Note that Y^{c} is the subspace of initial values that lead to bounded solutions of (24). Recall that we have denoted by P_0^{s} and P_0^{u} the spectral projections associated with the stable and unstable eigenvalues, respectively, of $f_u(0, 0)$. The constants α^{s} and α^{u} are lower bounds for the distance of the stable and the unstable eigenvalues of $f_u(0, 0)$ from the imaginary axis.

LEMMA 3.1. *The evolution $\Phi_+(x, y)$ of* (24) *for $x, y \geqslant 0$ can be written as $\Phi_+(x, y) = \Phi^{\mathrm{s}}_+(x, y) + \Phi^{\mathrm{u}}_+(x, y)$. The operators $\Phi^{\mathrm{s}}_+(x, y)$ and $\Phi^{\mathrm{u}}_+(y, x)$ satisfy*

$$|\Phi^{\mathrm{s}}_+(x, y)| \leqslant C e^{-\alpha^{\mathrm{s}} |x-y|}, \qquad |\Phi^{\mathrm{u}}_+(y, x)| \leqslant C e^{-\alpha^{\mathrm{u}} |x-y|}$$

for $x \geqslant y \geqslant 0$. Furthermore, the operator $P^{\mathrm{s}}_+(x) = \Phi^{\mathrm{s}}_+(x, x)$ is a projection with

$$|P^{\mathrm{s}}_+(x) - P^{\mathrm{s}}_0| \leqslant C e^{-\alpha^{\mathrm{s}} x}$$

for $x \geqslant 0$, *and we set* $P^{\mathrm{u}}_{+}(x) = \mathrm{id} - P^{\mathrm{s}}_{+}(x)$. *Analogously, the evolution* $\Phi_{-}(x, y)$ *of* (24) *for* $x, y \leqslant 0$ *can be written as* $\Phi_{-}(x, y) = \Phi^{\mathrm{s}}_{-}(x, y) + \Phi^{\mathrm{u}}_{-}(x, y)$. *The operators* $\Phi^{\mathrm{s}}_{-}(y, x)$ *and* $\Phi^{\mathrm{u}}_{-}(x, y)$ *satisfy*

$$|\Phi^{\mathrm{s}}_{-}(y, x)| \leqslant C\mathrm{e}^{-\alpha^{\mathrm{s}}|y-x|}, \qquad |\Phi^{\mathrm{u}}_{-}(x, y)| \leqslant C\mathrm{e}^{-\alpha^{\mathrm{u}}|y-x|}$$

for $x \leqslant y \leqslant 0$. *Furthermore, the operator* $P^{\mathrm{u}}_{-}(x) = \Phi^{\mathrm{u}}_{-}(x, x)$ *is a projection with*

$$|P^{\mathrm{u}}_{-}(x) - P^{\mathrm{u}}_{0}| \leqslant C\mathrm{e}^{-\alpha^{\mathrm{u}}|x|}$$

for $x \leqslant 0$ *and we set* $P^{\mathrm{s}}_{-}(x) = \mathrm{id} - P^{\mathrm{u}}_{-}(x)$. *In addition, there are spaces* Y^{s} *and* Y^{u} *such that* $Y^{\mathrm{c}} \oplus Y^{\mathrm{s}} \oplus Y^{\mathrm{u}} \oplus Y^{\perp} = \mathbb{R}^n$ *and*

$$\begin{aligned} \mathrm{R}(P^{\mathrm{s}}_{+}(0)) &= Y^{\mathrm{c}} \oplus Y^{\mathrm{s}}, \qquad & \mathrm{R}(P^{\mathrm{u}}_{+}(0)) &= Y^{\mathrm{u}} \oplus Y^{\perp}, \\ \mathrm{R}(P^{\mathrm{u}}_{-}(0)) &= Y^{\mathrm{c}} \oplus Y^{\mathrm{u}}, \qquad & \mathrm{R}(P^{\mathrm{s}}_{-}(0)) &= Y^{\mathrm{s}} \oplus Y^{\perp}. \end{aligned} \tag{25}$$

Proof. See, for instance, [31, Lemma 1.1 and Lemma 1.2(ii)]. ▮

In the phase space $\mathbb{R}^n$ of the ODE (13), the ranges of the projections $P^{\mathrm{s}}_{+}(0)$ and $P^{\mathrm{u}}_{-}(0)$ are the tangent spaces to stable and unstable manifolds of the origin at the homoclinic point $h(0)$:

$$\mathrm{R}(P^{\mathrm{s}}_{+}(0)) = T_{h(0)} W^{\mathrm{s}}(0), \qquad \mathrm{R}(P^{\mathrm{u}}_{-}(0)) = T_{h(0)} W^{\mathrm{u}}(0),$$
$$Y^{\mathrm{c}} = T_{h(0)} W^{\mathrm{s}}(0) \cap T_{h(0)} W^{\mathrm{u}}(0).$$

We emphasize that an analogous lemma is true for the adjoint variational equation. Since we shall not make use of the corresponding estimates, we omit a precise formulation. Basically, the evolution operators to the adjoint equation are the inverses of the adjoints of $\Phi_{\pm}$. The corresponding projections are the adjoints of the projections of the variational equation. In particular, the exponential decay estimates of the bounded solutions $\psi_k(x)$ of the adjoint variational equation (15) follow from this discussion.

Remark 3.1. Throughout Section 3, the operators $\Phi^{\mathrm{s}}_{+}(x, y)$ and $\Phi^{\mathrm{u}}_{+}(y, x)$ are used only for $x \geqslant y \geqslant 0$. Similarly, we use the operators $\Phi^{\mathrm{s}}_{-}(y, x)$ and $\Phi^{\mathrm{u}}_{-}(x, y)$ only for $x \leqslant y \leqslant 0$. This is important since the arguments given here for ODEs carry then over to elliptic PDEs; see [29].

Notation. The subscripts $+$ and $-$ correspond to $x > 0$ and $x < 0$, respectively. Different positive constants that are independent of L and the parameter μ are denoted by C. Also, $\delta > 0$ is a generic small constant that again does not depend upon L and μ. The indices j and k take integer values between 1 and m. Finally, for any direct sum $\mathbb{R}^n = Y_{\mathrm{rg}} \oplus Y_{\mathrm{ke}}$, we denote by $P(Y_{\mathrm{rg}}, Y_{\mathrm{ke}})$ the projection with range Y_{rg} and null space Y_{ke}.

3.2. *The Reformulation*

Recall that p_L is a periodic solution, with period $2L$, of

$$u' = f(u, \mu)$$

for $\mu = \mu_L$ so that $|\mu_L|$ is close to zero, $p_L(x)$ is close to $h(x)$ for $|x| \leqslant L$, and L is large. We seek bounded solutions v of

$$v' = (f_u(p_L(x), \mu_L) + \lambda B)\, v \tag{26}$$

for $|x| \leqslant L$ with $\lambda \in \mathbb{C}$ close to zero such that

$$v(L) = e^{i\gamma} v(-L) \tag{27}$$

for some $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$. We write (26–27) in the equivalent form

$$\begin{aligned} v'_- &= (f_u(p_L(x), \mu_L) + \lambda B)\, v_-, \qquad && x \in (-L, 0) \\ v'_+ &= (f_u(p_L(x), \mu_L) + \lambda B)\, v_+, \qquad && x \in (0, L) \\ v_-(0) &= v_+(0) && \\ v_+(L) &= e^{i\gamma} v_-(-L) && \end{aligned} \tag{28}$$

considered as equations over the complex field.

We shall exploit that the functions $\phi_j(x) = S_j p_L(x)$ for $j = 1, ..., m-1$ and $\phi_m(x) = p'_L(x)$ satisfy (28) for $(\lambda, \gamma) = 0$. Thus, we write

$$v_\pm(x) = \sum_{j=1}^{m} \phi_j(x)\, d_j + w_\pm(x), \tag{29}$$

where $d = (d_j) \in \mathbb{C}^m$ is an arbitrary vector. Define a projection Q: $\mathbb{R}^n \to \mathbb{R}^n$ by

$$\mathrm{R}(Q) = \operatorname{span}\{\phi_1(0), ..., \phi_m(0)\}, \qquad \mathrm{N}(Q) = Y^{s} \oplus Y^{u} \oplus Y^{\perp}.$$

Note that, if the periodic solution p_L is δ-close to the homoclinic orbit as assumed in Theorem 2.2, then $\mathrm{R}(Q)$ is close to the space Y^{c} which is a complement to $\mathrm{N}(Q)$ by the results of Section 3.1. Therefore, Q is well defined, and its norm depends only on the choice of δ but not on the periodic solution p_L itself.

Using the ansatz (29), we obtain the following equivalent formulation of the boundary-value problem (28):

$$\begin{aligned}
&\text{(i)}\quad w'_{\pm} = f_u(h(x),0)\, w_{\pm} + (f_u(p_L(x),\mu_L) - f_u(h(x),0) + \lambda B)\, w_{\pm} \\
&\qquad + \lambda \sum_{j=1}^{m} B\phi_j(x)\, d_j \\
&\text{(ii)}\quad Qw_{\pm}(0) = 0 \qquad\qquad (30) \\
&\text{(iii)}\quad w_+(0) - w_-(0) \in Y^{\perp} \\
&\text{(iv)}\quad w_+(L) - e^{i\gamma} w_-(-L) = (e^{i\gamma} - 1) \sum_{j=1}^{m} \phi_j(L)\, d_j
\end{aligned}$$

and

$$\langle \psi_k(0), w_+(0) - w_-(0) \rangle = 0 \qquad (31)$$

for $k=1, ..., m$. Indeed, we note that (30)(iii) and (31) are met if, and only if, $w_+(0) = w_-(0)$. Next, if $(w_{\pm}, d)$ satisfies (30–31), then $(v_{\pm})$ defined by (29) satisfies (28). On the other hand, suppose that $(v_{\pm})$ is a solution of (28). Define d_j by

$$v_-(0) = v_+(0) = \sum_{j=1}^{m} \phi_j(0)\, d_j + \tilde{v},$$

where $\tilde{v} \in \mathrm{N}(Q)$. Upon defining $(w_{\pm})$ by (29), it is easy to see that $(w_{\pm}, d)$ satisfies (30–31).

Thus, solving the boundary-value problem (26–27) reduces to solving (31) once (30) has been solved. Note that $w_{\pm}(x) = 0$ is a bounded solution to (30–31) with $\lambda = 0$ and $\gamma = 0$ for any d.

3.3. *The Reduction*

In this section, we solve an approximation of the linear eigenvalue problem (30) that is obtained by replacing the terms

$$(f_u(p_L(x),\mu_L) - f_u(h(x),0) + \lambda B)\, w_{\pm} + \lambda \sum_{j=1}^{m} B\phi_j(x)\, d_j$$

by a function $G_{\pm}(x)$ that does not depend upon λ or $w_{\pm}$. The resulting equation (32) below is referred to as the approximate linear eigenvalue problem. To solve (32), we set up a variation-of-constants formulation that captures all solutions to (32)(i)–(iii). Finally, using this formulation, we solve (32) and derive expansions of the reduced equations in terms of the period L and the right-hand side G.

Recall the definitions of Y^{u} and Y^{s} as complements of $T_{h(0)} W^{\mathrm{u}}(0) \cap T_{h(0)} W^{\mathrm{s}}(0) = Y^{\mathrm{c}}$ in $T_{h(0)} W^{\mathrm{u}}(0)$ and $T_{h(0)} W^{\mathrm{s}}(0)$, respectively; see (25). Define the spaces

$$\begin{aligned} V_w &:= \mathscr{C}^0([0, L], \mathbb{C}^n) \oplus \mathscr{C}^0([-L, 0], \mathbb{C}^n) \\ V_a &:= \mathrm{R}(P_0^{\mathrm{u}}) \oplus \mathrm{R}(P_0^{\mathrm{s}}) \\ V_b &:= Y^{\mathrm{s}} \oplus Y^{\mathrm{u}} \end{aligned}$$

considered over the complex field and let

$$w = (w_+, w_-) \in V_w, \qquad a = (a_+, a_-) \in V_a, \qquad b = (b_+, b_-) \in V_b.$$

Our approximation of the linear eigenvalue problem is given by

$$\begin{aligned} &\text{(i)} \quad w'_\pm = f_u(h(x), 0)\, w_\pm + G_\pm(x) \\ &\text{(ii)} \quad Q w_\pm(0) = 0 \\ &\text{(iii)} \quad w_+(0) - w_-(0) \in Y^\perp \\ &\text{(iv)} \quad w_+(L) - \mathrm{e}^{\mathrm{i}\gamma} w_-(-L) = D \end{aligned} \tag{32}$$

for elements $G = (G_+, G_-) \in V_w$ and $D \in \mathbb{C}^n$. We seek solutions to (32) in V_w for $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$. Lemma 3.2 below shows that solutions of (i)–(iii) are, in their general form, given by the following variation-of-constant formula:

$$\begin{aligned} w_-(x) &= \Phi^{\mathrm{s}}_-(x, -L)\, a_- + \Phi^{\mathrm{u}}_-(x, 0)\, b_- \\ &\quad + \int_0^x \Phi^{\mathrm{u}}_-(x, y)\, G_-(y)\, \mathrm{d}y + \int_{-L}^x \Phi^{\mathrm{s}}_-(x, y)\, G_-(y)\, \mathrm{d}y \\ w_+(x) &= \Phi^{\mathrm{u}}_+(x, L)\, a_+ + \Phi^{\mathrm{s}}_+(x, 0)\, b_+ \\ &\quad + \int_0^x \Phi^{\mathrm{s}}_+(x, y)\, G_+(y)\, \mathrm{d}y + \int_L^x \Phi^{\mathrm{u}}_+(x, y)\, G_+(y)\, \mathrm{d}y, \end{aligned} \tag{33}$$

where the elements $a \in V_a$ and $b \in V_b$ are arbitrary. We refer to Section 3.1 for the definition of the operators $\Phi^{\mathrm{s,u}}_\pm(x, y)$.

Lemma 3.2. *There are constants C and L_* such that the following is true for every $L > L_*$. The right-hand side of* (33) *defines a linear operator*

$$W_1 \colon V_a \times V_b \times V_w \to V_w, \quad (a, b, G) \mapsto W_1(a, b, G).$$

Furthermore, there is a linear operator $B_1\colon V_a \times V_w \to V_b$ *such that* w *satisfies* (32)(i)–(iii) *if, and only if,*

$$w = W_1(a, B_1(a, G), G) \tag{34}$$

so that $b = B_1(a, G)$. *Finally, we have the estimates*

$$\begin{aligned} |B_1(a, G)| &\leqslant C(\mathrm{e}^{-\alpha L}\,|a| + |G|) \\ |W_1(a, b, G)| &\leqslant C(|a| + |b| + |G|) \\ |W_1(a, B_1(a, G), G)| &\leqslant C(|a| + |G|). \end{aligned} \tag{35}$$

Proof. Using the definition of the projection Q and the definitions of Y^{u} and Y^{s} in (25), it is straightforward to see that (33) is the general solution of (32)(i)–(ii). Indeed, the general solution to (i) can be represented in the form (33) with

$$(b_+, b_-) \in (Y^{\mathrm{s}} \oplus Y^{\mathrm{c}}) \oplus (Y^{\mathrm{u}} \oplus Y^{\mathrm{c}}).$$

The boundary condition (32)(iii) then restricts the allowed pairs (b_+, b_-) to $V_b = Y^{\mathrm{s}} \oplus Y^{\mathrm{u}}$.

The statements about W_1 follow from the exponential decay estimates for the evolution operators provided in Lemma 3.1. It remains to show that (32)(iii), $w_+(0) - w_-(0) \in Y^{\perp}$, is met for an appropriate choice of b. Evaluating (33) at $x = 0$, we get

$$\begin{aligned} w_+(0) - w_-(0) = {} & b_+ - b_- + \Phi^{\mathrm{u}}_+(0, L)\, a_+ - \Phi^{\mathrm{s}}_-(0, -L)\, a_- \\ & - \int_0^L \Phi^{\mathrm{u}}_+(0, y)\, G_+(y)\, \mathrm{d}y - \int_{-L}^0 \Phi^{\mathrm{s}}_-(0, y)\, G_-(y)\, \mathrm{d}y \end{aligned} \tag{36}$$

since $\Phi^{\mathrm{s}}_+(0, 0)\, b_+ = b_+$ and $\Phi^{\mathrm{u}}_-(0, 0)\, b_- = b_-$ due to (25) and $b \in V_b$.

We recall the notation $P(Y_{\mathrm{rg}}, Y_{\mathrm{ke}})$ for the projection onto Y_{rg} with null space Y_{ke} that we introduced in Section 3.1. By (32)(ii), we have that

$$P(Y^{\mathrm{c}}, Y^{\mathrm{s}} \oplus Y^{\mathrm{u}} \oplus Y^{\perp})\, w_\pm(0) = 0.$$

To solve (32)(iii), it therefore suffices to satisfy

$$\begin{aligned} P(Y^{\mathrm{u}}, Y^{\mathrm{c}} \oplus Y^{\mathrm{s}} \oplus Y^{\perp})(w_+ - w_-)(0) &= 0, \\ P(Y^{\mathrm{s}}, Y^{\mathrm{c}} \oplus Y^{\mathrm{u}} \oplus Y^{\perp})(w_+ - w_-)(0) &= 0. \end{aligned}$$

Thus, upon projecting (36) onto $Y^{\mathrm{s}} \oplus Y^{\mathrm{u}}$ and using the definition of Y^{s} and Y^{u} from (25), we see that $w_+(0) - w_-(0) \in Y^{\perp}$ if, and only if,

$$b_+ = P(Y^{\mathrm{s}}, Y^{\mathrm{c}} \oplus Y^{\mathrm{u}} \oplus Y^{\perp}) \left(\Phi^{\mathrm{s}}_-(0, -L)\, a_- + \int_{-L}^{0} \Phi^{\mathrm{s}}_-(0, y)\, G_-(y)\, \mathrm{d}y \right)$$

$$b_- = P(Y^{\mathrm{u}}, Y^{\mathrm{c}} \oplus Y^{\mathrm{s}} \oplus Y^{\perp}) \left(\Phi^{\mathrm{u}}_+(0, L)\, a_+ - \int_0^{L} \Phi^{\mathrm{u}}_+(0, y)\, G_+(y)\, \mathrm{d}y \right).$$

The right-hand sides of these equations define a bounded and linear operator $B_1 \colon V_a \times V_w \to V_b$ that satisfies $|B_1(a, G)| \leqslant C(\mathrm{e}^{-\alpha L}\, |a| + |G|)$ due to the uniform exponential decay properties of Φ^{s}_- and Φ^{u}_+ that we derived in Lemma 3.1. This completes the proof of the lemma. ∎

It remains to solve (32)(iv), that is, $w_+(L) - \mathrm{e}^{\mathrm{i}\gamma} w_-(-L) = D$.

Lemma 3.3. *There are constants C and L_* such that the following is true for any $L > L_*$. There exist analytic maps*

$$A_2 \colon \mathbb{R}/2\pi\mathbb{Z} \to \mathscr{L}(\mathbb{C}^n \times V_w, V_a), \qquad B_2 \colon \mathbb{R}/2\pi\mathbb{Z} \to \mathscr{L}(\mathbb{C}^n \times V_w, V_b),$$

$$W_2 \colon \mathbb{R}/2\pi\mathbb{Z} \to \mathscr{L}(\mathbb{C}^n \times V_w, V_w)$$

such that w satisfies (32) *if, and only if, w is given by the variation-of-constant formula* (33) *with*

$$a = A_2(\gamma)(D, G), \qquad b = B_2(\gamma)(D, G); \tag{37}$$

in other words,

$$w = W_2(\gamma)(D, G) = W_1(A_2(\gamma)(D, G), B_2(\gamma)(D, G), G), \tag{38}$$

where W_1 has been defined in Lemma 3.2. *For every $\ell \geqslant 0$, the operators A_2, B_2 and W_2 satisfy*

$$|\partial_\gamma^\ell A_2| + |\partial_\gamma^\ell B_2| + |\partial_\gamma^\ell W_2| \leqslant C. \tag{39}$$

Furthermore, we have the expansion

$$a = (a_+, a_-) = A_2(\gamma)(D, G) = (P^{\mathrm{u}}_0 D, -\mathrm{e}^{-\mathrm{i}\gamma} P^{\mathrm{s}}_0 D) + A_3(\gamma)(D, G) \tag{40}$$

for a bounded operator A_3 that satisfies

$$|A_3(\gamma)(D, G)| \leqslant C(\mathrm{e}^{-\alpha L}\, |D| + |G|). \tag{41}$$

Proof. Equation (33) evaluated at $x = \pm L$ is given by

$$w_+(L) = a_+ + (P^{\mathrm{u}}_+(L) - P^{\mathrm{u}}_0)\, a_+ + \Phi^{\mathrm{s}}_+(L, 0)\, b_+ + \int_0^L \Phi^{\mathrm{s}}_+(L, y)\, G_+(y)\, \mathrm{d}y$$

$$w_-(-L) = a_- + (P^{\mathrm{s}}_-(-L) - P^{\mathrm{s}}_0)\, a_- + \Phi^{\mathrm{u}}_-(-L, 0)\, b_- + \int_0^{-L} \Phi^{\mathrm{u}}_-(-L, y)\, G_-(y)\, \mathrm{d}y, \tag{42}$$

where the projections $P^{\mathrm{s}}_-(x) = \Phi^{\mathrm{s}}_-(x, x)$ and $P^{\mathrm{u}}_+(x) = \Phi^{\mathrm{u}}_+(x, x)$ have been defined in Lemma 3.1. Note that $P^{\mathrm{u}}_0 a_+ = a_+$ and $P^{\mathrm{s}}_0 a_- = a_-$ since $a \in V_a$. Substituting these expressions into (32)(iv), we obtain

$$\begin{aligned} D &= w_+(L) - \mathrm{e}^{\mathrm{i}\gamma} w_-(-L) \\ &= a_+ - \mathrm{e}^{\mathrm{i}\gamma} a_- + (P^{\mathrm{u}}_+(L) - P^{\mathrm{u}}_0)\, a_+ + \mathrm{e}^{\mathrm{i}\gamma}(P^{\mathrm{s}}_0 - P^{\mathrm{s}}_-(-L))\, a_- \\ &\quad + \Phi^{\mathrm{s}}_+(L, 0)\, b_+ - \mathrm{e}^{\mathrm{i}\gamma} \Phi^{\mathrm{u}}_-(-L, 0)\, b_- \\ &\quad + \int_0^L \Phi^{\mathrm{s}}_+(L, y)\, G_+(y)\, \mathrm{d}y - \mathrm{e}^{\mathrm{i}\gamma} \int_0^{-L} \Phi^{\mathrm{u}}_-(-L, y)\, G_-(y)\, \mathrm{d}y, \end{aligned} \tag{43}$$

where $b = B_1(a, G)$, as defined in Lemma 3.2, with $|b| \leqslant C(|a| + |G|)$. Equation (43) can then be written in the more compact form

$$D = a_+ - \mathrm{e}^{\mathrm{i}\gamma} a_- + A_1(\gamma)(a, G), \tag{44}$$

where $A_1 \colon \mathbb{R}/2\pi\mathbb{Z} \to \mathscr{L}(V_a \times V_w, V_a)$ is analytic. We shall solve this equation for a. Using the estimates in Lemma 3.1 and 3.2, it is straightforward to verify that

$$|\partial^{\ell}_{\gamma} A_1(\gamma)(a, G)| \leqslant C(\mathrm{e}^{-\alpha L}\, |a| + |G|) \tag{45}$$

for every $\ell \geqslant 0$ uniformly in γ. The principal part of (44) is given by the linear analytic map $J_1 \colon \mathbb{R}/2\pi\mathbb{Z} \to \mathscr{L}(V_a, \mathbb{C}^n)$ defined by

$$J_1(\gamma) \colon V_a \to \mathbb{C}^n, \qquad (a_+, a_-) \mapsto (a_+ - \mathrm{e}^{\mathrm{i}\gamma} a_-).$$

The linear operator $J_1(\gamma)$ is an isomorphism for every γ since $V_a = \mathrm{R}(P^{\mathrm{u}}_0) \oplus \mathrm{R}(P^{\mathrm{s}}_0) = \mathbb{C}^n$. Thus, there is an $L_* > 0$ such that, for every $L > L_*$, the operator

$$a \mapsto J_1(\gamma)\, a + A_1(\gamma)(a, 0)$$

is invertible. We can therefore solve (44) abstractly, and its solution is given by

$$a = (J_1(\gamma) + A_1(\gamma)\, I_1)^{-1}\, (D - A_1(\gamma)(0, G)) =: A_2(\gamma)(D, G), \tag{46}$$

where $I_1 a = (a, 0)$. Note that $A_2(\gamma)$ is analytic in γ, linear in (D, G) and bounded uniformly in $L > L_*$ and $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$. It remains to solve (32)(iv). If we set

$$B_2(\gamma)(D, G) := B_1(A_2(\gamma)(D, G), G)$$

$$W_2(\gamma)(D, G) := W_1(A_2(\gamma)(D, G), B_2(\gamma)(D, G), G),$$

then $w = W_2(\gamma)(D, G)$ satisfies the integral equation (33) for every (D, G) by definition of B_1 and W_1, see Lemma 3.2, and the above construction of A_2.

Having solved the entire boundary-value problem (32), it remains to verify the uniform estimates (39) and the expansion (40) together with the error estimate (41). We notice that, due to (45),

$$|\partial_\gamma^\ell A_2| \leqslant C. \tag{47}$$

Using this estimate together with (35) and the definitions of B_2 and W_2, we get

$$|\partial_\gamma^\ell B_2| + |\partial_\gamma^\ell W_2| \leqslant C.$$

This proves the uniform estimate (39). The expansion (40) is obtained from the expansion of (32)(iv) that we gave in (44) together with the estimates (45) for A_2 and (47) for $a = A_2(D, G)$. We omit the details. ∎

Remark 3.2. Note that, under the assumptions of Lemma 3.3, we have

$$(w_+(L), w_-(-L)) = (P_0^{\mathrm{u}} D, -\mathrm{e}^{-\mathrm{i}\gamma} P_0^{\mathrm{s}} D) + \tilde{W}_2(\gamma)(D, G), \tag{48}$$

where $\tilde{W}_2 \colon \mathbb{R}/2\pi\mathbb{Z} \to \mathscr{L}(\mathbb{C}^n \times V_w, \mathbb{C}^n)$ is analytic and, for every $\ell \geqslant 0$, we have

$$|\partial_\gamma^\ell \tilde{W}_2(\gamma)(D, G)| \leqslant C(\mathrm{e}^{-\alpha L}\, |D| + |G|); \tag{49}$$

see (42), (39) and (40).

The next remark follows from the variation-of-constant formula (33) upon using the uniform estimates (39) for (a, b) and the exponential decay estimates from Lemma 3.1.

Remark 3.3. We have $w = W_2(\gamma)(D, 0) + W_2(\gamma)(0, G)$ and

$$\begin{aligned} |\partial_\gamma^\ell W_2(\gamma)(D,0)(x)| &\leqslant C\mathrm{e}^{-\alpha^{\mathrm{s}}(L+x)}\,|D|, \qquad \text{for} \quad x \in [-L, 0] \\ |\partial_\gamma^\ell W_2(\gamma)(D,0)(x)| &\leqslant C\mathrm{e}^{-\alpha^{\mathrm{u}}(L-x)}\,|D|, \qquad \text{for} \quad x \in [0, L] \end{aligned} \tag{50}$$

for every $\ell \geqslant 0$.

In summary, we have demonstrated that w is a bounded solution of (32) if, and only if, w is given by the variation-of-constant formula (33) where (a, b) are defined by the bounded linear operators A_2 and B_2 that appear in (37); see Lemma 3.3. The solutions w_+ and w_- are continuous at $x = 0$, that is, $w_+(0) = w_-(0)$, if, and only if, the jumps

$$\xi_k := \langle \psi_k(0), w_+(0) - w_-(0) \rangle \tag{51}$$

vanish identically for $k = 1, ..., m$. In the following lemma, we derive a concrete expression for the jumps ξ_k.

LEMMA 3.4. *There are constants C and L_* such that the following is true for any $L > L_*$. Let w be given by (38). The jumps ξ_k defined above are then given by*

$$\begin{aligned} \xi_k = {} & \langle \psi_k(L), P_0^{\mathrm{u}} D \rangle + \mathrm{e}^{-\mathrm{i}\gamma} \langle \psi_k(-L), P_0^{\mathrm{s}} D \rangle \\ & - \int_0^L \langle \psi_k(x), G_+(x) \rangle \,\mathrm{d}x \\ & - \int_{-L}^0 \langle \psi_k(x), G_-(x) \rangle \,\mathrm{d}x + \hat{R}_{1,k}(\gamma)(D, G) \end{aligned} \tag{52}$$

for a certain analytic remainder term $\hat{R}_1 \colon \mathbb{R}/2\pi\mathbb{Z} \to \mathscr{L}(\mathbb{C}^n \times V_w, \mathbb{C}^m)$ that satisfies

$$|\partial_\gamma^\ell \hat{R}_1(\gamma)(D, G)| \leqslant C\mathrm{e}^{-\alpha L}(\mathrm{e}^{-\alpha L}\,|D| + |G|) \tag{53}$$

for every $\ell \geqslant 0$.

Proof. Throughout the proof, we choose (a, b, w) according to (37) and (38) from Lemma 3.3 so that (32) is solved by the variation-of-constant formula (33). Substituting the expressions for $w_\pm$ from (33) into the equation (51) for the jumps, we see that the jumps ξ_k are linear in (D, G) and analytic in γ. More precisely, taking the scalar product of the equation (36) for $w_+(0) - w_-(0)$ with $\psi_k(0)$ and using that $\langle \psi_k(0), b_\pm \rangle = 0$ by definition of V_b, we obtain

$$
\begin{aligned}
&\langle \psi_k(0), w_+(0) - w_-(0) \rangle \\
&\quad = \langle \psi_k(0), \Phi^{u}_{+}(0, L)\, a_+ \rangle - \langle \psi_k(0), \Phi^{s}_{-}(0, -L)\, a_- \rangle \\
&\qquad - \int_0^L \langle \psi_k(0), \Phi^{u}_{+}(0, x)\, G_+(x) \rangle \,\mathrm{d}x \\
&\qquad - \int_{-L}^0 \langle \psi_k(0), \Phi^{s}_{-}(0, x)\, G_-(x) \rangle \,\mathrm{d}x \\
&\quad = \langle \psi_k(L), a_+ \rangle - \langle \psi_k(-L), a_- \rangle \\
&\qquad - \int_0^L \langle \psi_k(x), G_+(x) \rangle \,\mathrm{d}x - \int_{-L}^0 \langle \psi_k(x), G_-(x) \rangle \,\mathrm{d}x. \qquad (54)
\end{aligned}
$$

In the second identity, we used that the evolution operators of the adjoint variational equation are given by $((\Phi^{u,s}_{\pm}(x, y))^{-1})^*$, that is, by the inverses of the adjoints of the solution operators to the variational equation. Upon substituting the expansion (40) from Lemma 3.3 for a, and using the estimate (41) for the remainder term, we get

$$
\begin{aligned}
&\langle \psi_k(L), a_+ \rangle - \langle \psi_k(-L), a_- \rangle \\
&\quad = \langle \psi_k(L), P^{u}_{0} D \rangle + \mathrm{e}^{-\mathrm{i}\gamma} \langle \psi_k(-L), P^{s}_{0} D \rangle + \mathrm{O}(\mathrm{e}^{-\alpha L}(\mathrm{e}^{-\alpha L}\,|D| + |G|))
\end{aligned} \tag{55}
$$

since

$$
|\psi_k(x)| \leqslant C\mathrm{e}^{-\alpha\,|x|} \tag{56}
$$

for $k = 1, ..., m$. Finally, all of the aforementioned estimates are also true for derivatives with respect to γ. This completes the proof of the lemma. ∎

3.4. *Small Linear Perturbations*

We return to the original boundary-value problem (30–31) which we consider as a perturbation of (32). In this section, we allow for arbitrary small linear perturbations of the first equation (30)(i) that are of the form $H(x)\, w_\pm$. More precisely, we consider the linear boundary-value problem

$$
\begin{aligned}
&\text{(i)} && w'_\pm = f_u(h(x), 0)\, w_\pm + H(x)\, w_\pm + g(x) \\
&\text{(ii)} && Q w_\pm(0) = 0 \\
&\text{(iii)} && w_+(0) - w_-(0) \in Y^\perp \\
&\text{(iv)} && w_+(L) - \mathrm{e}^{\mathrm{i}\gamma} w_-(-L) = D
\end{aligned} \tag{57}
$$

for $H \in V_H := \mathscr{C}^0([-L, L], \mathscr{L}(\mathbb{C}^n))$, $g = (g_+, g_-) \in V_w$ and $D \in \mathbb{C}^n$. We seek solutions to (57) in V_w for $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$.

We denote by U_δ the δ-neighborhood of $H = 0$ in V_H.

LEMMA 3.5. *There are positive constants C, L_* and δ such that the following is true for $L > L_*$. There is a solution operator*

$$W\colon \mathbb{R}/2\pi\mathbb{Z} \times U_\delta \to \mathscr{L}(\mathbb{C}^n \times V_w, V_w), \qquad (\gamma, H) \mapsto W(\gamma, H)$$

that depends analytically on the parameter γ and the perturbation H, such that w satisfies the boundary-value problem (57) if, and only if, $w = W(\gamma, H)(D, g)$. Furthermore, we have

$$|\partial^{\ell}_{(\gamma, H)} W| \leqslant C \tag{58}$$

for every $\ell \geqslant 0$, and

$$\begin{aligned}(w_+(L), w_-(-L)) = {} & (P_0^{\mathrm{u}} D, -\mathrm{e}^{-\mathrm{i}\gamma} P_0^{\mathrm{s}} D) \\ & + \tilde{W}_2(\gamma)(D, g) + \tilde{W}_2(\gamma)(0, HW(\gamma, H)(D, g));\end{aligned} \tag{59}$$

see Remark 3.2 for the definition of $\tilde{W}_2$. Moreover, the components of the jump of w at $x = 0$ are given by

$$\begin{aligned}\xi_k = {} & \langle \psi_k(L), P_0^{\mathrm{u}} D\rangle + \mathrm{e}^{-\mathrm{i}\gamma} \langle \psi_k(-L), P_0^{\mathrm{s}} D\rangle \\ & - \int_{-L}^{L} \langle \psi_k(x), g(x)\rangle \, \mathrm{d}x + \hat{R}_{2,k}(\gamma, H)(D, g).\end{aligned} \tag{60}$$

The function $\hat{R}_2\colon \mathbb{R}/2\pi\mathbb{Z} \times U_\delta \to \mathscr{L}(\mathbb{C}^n \times V_w, \mathbb{C}^m)$ is analytic, and its Taylor expansion in H is given by

$$\hat{R}_2(\gamma, H)(D, g) = T_0(\gamma)(D, g) + T_1(\gamma)(D, g)[H] + T_2(\gamma, H)(D, g)[H, H],$$

where

$$\begin{aligned}|\partial^{\ell}_{\gamma} T_0(\gamma)(D, g)| &\leqslant C\mathrm{e}^{-\alpha L}(\mathrm{e}^{-\alpha L}|D| + |g|) \\ |\partial^{\ell}_{\gamma} T_1(\gamma)(D, g)| &\leqslant C(\mathrm{e}^{-\alpha L}|D| + |g|) \\ |\partial^{\ell}_{\gamma, H} T_2(\gamma, H)| &\leqslant C\end{aligned} \tag{61}$$

for every $\ell \geqslant 0$.

Proof. A comparison of the boundary-value problem (57) and the unperturbed equation (32) shows that we have to set

$$G = Hw + g \tag{62}$$

in the unperturbed problem. Hence, w satisfies (57) if, and only if,

$$w = W_2(\gamma)(D, Hw + g),$$

where $W_2(\gamma)$ is the solution operator for equation (32) that we defined in (38); see Lemma 3.3. We write the above equation as

$$w = W_3(\gamma, H)\, w + W_2(\gamma)(D, g), \tag{63}$$

where $W_3(\gamma, H)\, w = W_2(\gamma)(0, Hw)$ is analytic in (γ, H) and

$$|W_3(\gamma, H)\, w| \leqslant C\, |H|\, |w| \leqslant C\delta\, |w|.$$

Thus, for $\delta > 0$ sufficiently small but independent of γ and $L > L_*$, we can invert the operator $\mathrm{id} - W_3(\gamma, H)$ and obtain a unique solution of (63) that is represented by the operator

$$w = W(\gamma, H)(D, g) = (\mathrm{id} - W_3(\gamma, H))^{-1}\, W_2(\gamma)(D, g). \tag{64}$$

This operator is analytic in (γ, H). The expansion (59) is a consequence of Remark 3.2 and the estimates obtained above. This proves the first part of the lemma.

Next, we compute the jumps of the perturbed boundary-value problem. Upon substituting $G = Hw + g$ into the expansion (52) for the jumps, see Lemma 3.4, we obtain

$$\begin{aligned}\xi_k = {} & \langle \psi_k(L), P_0^{\mathrm{u}} D\rangle + \mathrm{e}^{-\mathrm{i}\gamma}\langle \psi_k(-L), P_0^{\mathrm{s}} D\rangle \\ & - \int_0^L \langle \psi_k(x), H(x)\, w_+(x)\rangle\, \mathrm{d}x - \int_{-L}^0 \langle \psi_k(x), H(x)\, w_-(x)\rangle\, \mathrm{d}x \\ & - \int_{-L}^L \langle \psi_k(x), g(x)\rangle\, \mathrm{d}x + \hat{R}_{3,k}(\gamma, H)(D, g),\end{aligned} \tag{65}$$

where $w = W(\gamma, H)(D, g)$ and

$$\begin{aligned}&\hat{R}_3\colon \mathbb{R}/2\pi\mathbb{Z} \times U_\delta \to \mathscr{L}(\mathbb{C}^n \times V_w, \mathbb{C}^m), \\ &\hat{R}_3(\gamma, H)(D, g) := \hat{R}_1(\gamma)(D, Hw + g)\end{aligned} \tag{66}$$

is analytic. Recall from (63) that

$$w = W_2(\gamma)(D, g) + W_3(\gamma, H)\, W(\gamma, H)(D, g).$$

We set

$$\hat{R}_{4,k}(\gamma, H)(D, G) = \int_0^L \langle \psi_k(x), H(x)\, w_+(x) \rangle \,\mathrm{d}x + \int_{-L}^0 \langle \psi_k(x), H(x)\, w_-(x) \rangle \,\mathrm{d}x;$$

then $\hat{R}_4$ is analytic in (γ, H) and

$$\hat{R}_4(\gamma, H)(D, G) = \hat{T}_1(\gamma)(D, g)[H] + \hat{T}_2(\gamma, H)(D, g)[H, H] \tag{67}$$

with

$$\hat{T}_1(\gamma)(D, g)[H] = \int_0^L \langle \psi_k(x), H(x)\, W_2(\gamma)(D, g)(x) \rangle \,\mathrm{d}x + \int_{-L}^0 \langle \psi_k(x), H(x)\, W_2(\gamma)(D, g)(x) \rangle \,\mathrm{d}x$$

$$\hat{T}_2(\gamma, H)(D, g)[H, H] = \int_0^L \langle \psi_k(x), H(x)\, W_3(\gamma, H)\, W(\gamma, H)(D, g)(x) \rangle \,\mathrm{d}x + \int_{-L}^0 \langle \psi_k(x), H(x) \times W_3(\gamma, H)\, W(\gamma, H)(D, g)(x) \rangle \,\mathrm{d}x.$$

It follows from the estimates for W_3 and W that

$$|\partial^\ell_{\gamma, H} \hat{T}_2(\gamma, H)| \leqslant C$$

for every $\ell \geqslant 0$. Furthermore, using the estimate (50), we have

$$|\partial^\ell_\gamma W_2(\gamma)(D, 0)(x)| \leqslant C\mathrm{e}^{-\alpha^{\mathrm{s}}(L+x)}\, |D|,$$
$$|\partial^\ell_\gamma W_2(\gamma)(D, 0)(x)| \leqslant C\mathrm{e}^{-\alpha^{\mathrm{u}}(L-x)}\, |D|$$

for $x \leqslant 0$ and $x \geqslant 0$, respectively. Using these estimates, we obtain

$$\left| \int_0^L \langle \psi_k(x), H(x)\, W_2(\gamma)(D, 0)(x) \rangle \,\mathrm{d}x \right| \leqslant \int_0^L C\, |\psi_k(x)|\, |H|\, \mathrm{e}^{-\alpha^{\mathrm{s}}(L+x)}\, |D| \,\mathrm{d}x \leqslant C\mathrm{e}^{-\alpha L}\, |H|\, |D|$$

and similarly

$$\left| \int_{-L}^{0} \langle \psi_k(x), H(x)\, W_2(\gamma)(D, 0)(x) \rangle \, \mathrm{d}x \right| \leqslant C \mathrm{e}^{-\alpha L} \, |H| \, |D|.$$

Here, we made the constant α that appears in (56) a bit larger than $\min\{\alpha^{\mathrm{s}}, \alpha^{\mathrm{u}}\}$. We conclude that

$$|\partial_\gamma^\ell \hat{T}_1(\gamma)(D, g)| \leqslant C(\mathrm{e}^{-\alpha L} \, |D| + |g|)$$

for every $\ell \geqslant 0$. In summary, we have

$$\xi_k = \langle \psi_k(L), P_0^{\mathrm{u}} D \rangle + \mathrm{e}^{-\mathrm{i}\gamma} \langle \psi_k(-L), P_0^{\mathrm{s}} D \rangle - \int_{-L}^{L} \langle \psi_k(x), g(x) \rangle \, \mathrm{d}x$$
$$+ \hat{R}_{2,k}(\gamma, H)(D, g),$$

where

$$\hat{R}_2(\gamma, H) = \hat{R}_3(\gamma, H) + \hat{R}_4(\gamma, H).$$

This last expression can be written as

$$\hat{R}_2(\gamma, H) = T_0(\gamma)(D, g) + T_1(\gamma)(D, g)[H] + T_2(\gamma, H)(D, g)[H, H],$$

where

$$\begin{aligned} T_0(\gamma)(D, g) &= \hat{R}_1(\gamma)(D, g) \\ T_1(\gamma)(D, g)[H] &= \hat{R}_1(\gamma)(0, HW(\gamma, 0)(D, g)) + \hat{T}_1(\gamma)(D, g)[H] \\ T_2(\gamma, H)(D, g)[H, H] &= \hat{R}_1(\gamma)(0, H(W(\gamma, H) - W(\gamma, 0))(D, g)) \\ &\quad + \hat{T}_2(\gamma)(D, g)[H, H] \end{aligned}$$

are analytic; see (66) and (67). It follows from the estimates for $\hat{T}_1$ and $\hat{T}_2$ as well as (53) that

$$\begin{aligned} |\partial_\gamma^\ell T_0(\gamma)(D, g)| &\leqslant C \mathrm{e}^{-\alpha L} (\mathrm{e}^{-\alpha L} \, |D| + |g|) \\ |\partial_\gamma^\ell T_1(\gamma)(D, g)| &\leqslant C(\mathrm{e}^{-\alpha L} \, |D| + |g|) \\ |\partial_{\gamma, H}^\ell T_2(\gamma, H)| &\leqslant C. \end{aligned}$$

This completes the proof of the lemma. ∎

3.5. *The Substitution*

To complete the proof of Theorem 2.2, we return to the original boundary-value problem (30). Define the linear operators

$$K_1 : \mathbb{C}^m \to \mathbb{C}^n, \qquad d = (d_j) \mapsto \sum_{j=1}^{m} \phi_j(L)\, d_j$$
$$K_2 : \mathbb{C}^m \to V_w, \qquad d = (d_j) \mapsto \sum_{j=1}^{m} B\phi_j(x)\, d_j, \tag{68}$$

so that

$$|K_1| \leqslant C\, |p_L(L)|, \qquad |K_2| \leqslant C, \tag{69}$$

for a certain constant C that is independent of λ and L. A comparison of the original boundary-value problem (30) and the boundary-value problem (57) that we have studied in the previous section shows that they are identical once we set

$$\begin{aligned} D &= (\mathrm{e}^{\mathrm{i}\gamma} - 1)\, K_1 d \\ g &= \lambda K_2 d \\ H &= f_u(p_L(\cdot), \mu_L) - f_u(h(\cdot), 0) + \lambda B, \end{aligned} \tag{70}$$

where, by assumption,

$$|H| \leqslant C(|\lambda| + |\mu_L| + \sup_{|x| \leqslant L} |p_L(x) - h(x)|) \leqslant C\delta. \tag{71}$$

Thus, we can apply Lemma 3.5. We obtain a solution w that is analytic in (λ, γ), and the jumps are given by

$$\begin{aligned} \xi_k = {} & (\mathrm{e}^{\mathrm{i}\gamma} - 1) \sum_{j=1}^{m} \langle \psi_k(L), P_0^{\mathrm{u}} \phi_j(L) \rangle\, d_j \\ & + (1 - \mathrm{e}^{-\mathrm{i}\gamma}) \sum_{j=1}^{m} \langle \psi_k(-L), P_0^{\mathrm{s}} \phi_j(L) \rangle\, d_j \\ & - \lambda \sum_{j=1}^{m} \int_{-L}^{L} \langle \psi_k(x), B\phi_j(x) \rangle\, d_j\, \mathrm{d}x + \hat{R}_{2,k}(\gamma, H)(D, g). \end{aligned} \tag{72}$$

First, we replace the functions $\phi_j(x)$ that appear in the scalar products by the functions $\varphi_j(x)$ for $j = 1, ..., m$, and change the interval of integration from $(-L, L)$ to $\mathbb{R}$. Afterwards, we verify the estimates that appear in Theorem 2.2.

Recall that $\phi_j = S_j p_L$ and $\varphi_j = S_j h$ for $j = 1, ..., m-1$, while $\phi_m = p'_L$ and $\varphi_m = h'$. On account of equivariance (H1), the operators $S_j \in \mathrm{alg}(\Sigma)$ commute with the spectral projections P^{s}_0 and P^{u}_0. Thus, using (56), we obtain

$$\begin{aligned} &|\langle \psi_k(L), P^{\mathrm{u}}_0 S_j(p_L(L) - h(-L))\rangle| \leqslant C\mathrm{e}^{-\alpha L}\, |P^{\mathrm{u}}_0(p_L(L) - h(-L))| \\ &|\langle \psi_k(-L), P^{\mathrm{s}}_0 S_j(p_L(L) - h(L))\rangle| \leqslant C\mathrm{e}^{-\alpha L}\, |P^{\mathrm{s}}_0(p_L(L) - h(L))| \end{aligned} \tag{73}$$

for $j = 1, ..., m-1$ and some constant C that does not depend upon L. On the other hand, exploiting the Taylor expansion of f, we get

$$\begin{aligned} &|\langle \psi_k(L), P^{\mathrm{u}}_0(p'_L(L) - h'(-L))\rangle| + |\langle \psi_k(-L), P^{\mathrm{s}}_0(p'_L(L) - h'(L))\rangle| \\ &\quad \leqslant C\mathrm{e}^{-\alpha L}(|P^{\mathrm{u}}_0(p_L(L) - h(-L))| + |P^{\mathrm{s}}_0(p_L(L) - h(L))| + |\mu_L| \\ &\quad + |p_L(L)|^2 + \mathrm{e}^{-2\alpha L}). \end{aligned} \tag{74}$$

Analogously, we see that

$$\begin{aligned} &\left| \int_{-L}^{L} \langle \psi_k(x), B\phi_j(x)\rangle \,\mathrm{d}x - \int_{-\infty}^{\infty} \langle \psi_k(x), B\varphi_j(x)\rangle \,\mathrm{d}x \right| \\ &\quad \leqslant C(\mathrm{e}^{-2\alpha L} + \sup_{|x| \leqslant L} |p_L(x) - h(x)|) \end{aligned} \tag{75}$$

for $j = 1, ..., m$. In summary, (72) is given by

$$\begin{aligned} \xi_k = {} & (\mathrm{e}^{\mathrm{i}\gamma} - 1) \sum_{j=1}^{m} \langle \psi_k(L), P^{\mathrm{u}}_0 \varphi_j(-L)\rangle\, d_j \\ & + (1 - \mathrm{e}^{-\mathrm{i}\gamma}) \sum_{j=1}^{m} \langle \psi_k(-L), P^{\mathrm{s}}_0 \varphi_j(L)\rangle\, d_j \\ & - \lambda \sum_{j=1}^{m} \int_{-\infty}^{\infty} \langle \psi_k(x), B\varphi_j(x)\rangle\, d_j \,\mathrm{d}x + \hat{R}_{2,k}(\gamma, H)(D, g) \\ & + \mathrm{O}(\mathrm{e}^{-\alpha L}(|P^{\mathrm{u}}_0(p_L(L) - h(-L))| + |P^{\mathrm{s}}_0(p_L(L) - h(L))| + |\mu_L| \\ & + |p_L(L)|^2 + \mathrm{e}^{-2\alpha L}) + |\lambda|\, (\mathrm{e}^{-\alpha L} + \sup_{|x| \leqslant L} |p_L(x) - h(x)|))\, |d|. \end{aligned} \tag{76}$$

It remains to substitute the expressions for D, g and H into the remainder term $\hat{R}_2$ and to verify that the estimates that appear in Theorem 2.2 are true. Since $\hat{R}_2$ is linear in (D, g), we have

$$\begin{aligned} \hat{R}_2(\gamma, H)(D, g) &= \hat{R}_2(\gamma, H)(D, 0) + \hat{R}_2(\gamma, H)(0, g) \\ &= (\mathrm{e}^{\mathrm{i}\gamma} - 1)\, \hat{R}_2(\gamma, H)(K_1, 0)\, d + \lambda\, \hat{R}_2(\gamma, H)(0, K_2)\, d, \end{aligned}$$

where we replaced D, g and H by the expressions in (70). Substituting the estimates (69) and (71) for $K_{1,2}$ and H, respectively, into the estimates (61) for $\hat{R}_2$ in Lemma 3.5, we obtain

$$\begin{aligned}|\hat{R}_2(\gamma, H)(K_1, 0)| &\leqslant C\,|K_1|(\mathrm{e}^{-\alpha L} + |H|)^2 \\ &\leqslant C\,|p_L(L)|(\mathrm{e}^{-\alpha L} + |\lambda| + |\mu_L| + \sup_{|x| \leqslant L} |p_L(x) - h(x)|)^2 \qquad (77)\\ |\hat{R}_2(\gamma, H)(0, K_2)| &\leqslant C\,|K_2|(\mathrm{e}^{-\alpha L} + |H|) \\ &\leqslant C(\mathrm{e}^{-\alpha L} + |\lambda| + |\mu_L| + \sup_{|x| \leqslant L} |p_L(x) - h(x)|).\end{aligned}$$

Analogous estimates can be derived for the derivatives with respect to (λ, γ).

It is then not difficult to check that the overall remainder term that consists of $\hat{R}_2$ and the additional term $\mathrm{O}(\cdots)$ that appears in (76) is of the form

$$(\mathrm{e}^{\mathrm{i}\gamma} - 1)\, R(\lambda, \gamma)\, d + \lambda\, \tilde{R}(\lambda, \gamma)\, d.$$

Furthermore, upon collecting the estimates (76) and (77), the estimates (20) in Theorem 2.1 for the overall remainder term and its derivatives follow.

Finally, we argue that $R(\lambda, \gamma)$ and $\tilde{R}(\lambda, \gamma)$ are real whenever $(\lambda, \mathrm{e}^{\mathrm{i}\gamma})$ is real. The reason is that, if $(\lambda, \mathrm{e}^{\mathrm{i}\gamma})$ is real, the boundary-value problem (30) involves only real quantities and can be solved over the field of real numbers.

This completes the proof of Theorem 2.2.

4. EXISTENCE OF PERIODIC WAVES WITH LARGE PERIOD

In order to use Theorem 2.2 to determine the spectrum of the linearization about a periodic wave with large spatial period, we have to estimate, in particular, the terms

$$|p_L(L)| + \sup_{|x| \leqslant L} |p_L(x) - h(x)|.$$

This is accomplished in the next theorem.

Theorem 4.1. *Assume that the equivariance hypothesis* (H1) *and the non-degeneracy condition* (H2) *are met. There are positive constants* C, δ

and L_* with the following property. Let $L > L_*$, then (13) has a periodic solution $p_L(x)$ with period $2L$ at $\mu = \mu_L$ such that

$$\sup_{|x| \leqslant L} |p_L(x) - h(x)| < \delta, \qquad |\mu_L| < \delta$$

if, and only if,

$$\langle \psi_k(L), h(-L) \rangle - \langle \psi_k(-L), h(L) \rangle - \int_{-\infty}^{\infty} \langle \psi_k(x), f_\mu(h(x, 0) \rangle \, \mathrm{d}x \, \mu + R_k(\mu) = 0 \tag{78}$$

at $\mu = \mu_L$ for $k = 1, ..., m$, where $R(\mu)$ is differentiable in μ and

$$R(\mu) \leqslant C(\mathrm{e}^{-\alpha L} + |\mu|)(|\mu| + \mathrm{e}^{-2\alpha L}), \qquad \partial_\mu R(\mu) \leqslant C(\mathrm{e}^{-\alpha L} + |\mu|). \tag{79}$$

Furthermore, any such periodic solution $p_L(x)$ satisfies

$$\begin{aligned} \sup_{|x| \leqslant L} |p_L(x) - h(x)| &\leqslant C(|\mu_L| + \mathrm{e}^{-\alpha L}) \\ |P_0^{\mathrm{u}}(p_L(-L) - h(-L))| + |P_0^{\mathrm{s}}(p_L(L) - h(L))| &\leqslant C(|\mu_L| + \mathrm{e}^{-2\alpha L}). \end{aligned} \tag{80}$$

Proof. It has been demonstrated in [41, Theorem 1] that $p_L(x) = h(x) + w(x)$ is a periodic solution with period $2L$ close to $h(x)$ if, and only if, w satisfies

$$\begin{aligned} &\text{(i)} \quad w'_\pm = f(h(x) + w_\pm, \mu) - f(h(x), 0) \\ &\text{(ii)} \quad Qw_\pm(0) = 0 \\ &\text{(iii)} \quad w_+(0) - w_-(0) \in Y^\perp \\ &\text{(iv)} \quad w_+(L) - w_-(-L) = h(-L) - h(L) \end{aligned} \tag{81}$$

and

$$\langle \psi_k(0), w_+(0) - w_-(0) \rangle = 0 \tag{82}$$

for $k = 1, ..., m$. Furthermore, [41, Lemma 11] asserts that (81) has a unique solution $w(\mu)$ for every $L > L_*$ and every μ with $|\mu| < \delta$, and $w(\mu)$ is differentiable in μ as a function into V_w such that

$$|w| \leqslant C(\mathrm{e}^{-\alpha L} + |\mu|), \qquad |\partial_\mu^\ell w| \leqslant C \tag{83}$$

for $\ell \geqslant 1$. It remains to compute the expansion of the bifurcation equation (82). We would like to make use of the estimates in Lemma 3.5 and write

$$\begin{aligned} f(h(x)+w,\mu)-f(h(x),0) &= f_u(h(x),0)\,w + f_\mu(h(x),0)\,\mu \\ &\quad + \int_0^1 [f_u(h(x)+\tau w,0)-f_u(h(x),0)]\,\mathrm{d}\tau\, w \\ &\quad + \int_0^1 [f_\mu(h(x)+w,\tau\mu)-f_\mu(h(x),0)]\,\mathrm{d}\tau\,\mu. \end{aligned}$$

The statement of the theorem is then a consequence of Lemma 3.5 applied with $\gamma = 0$ and

$$D = h(-L)-h(L)$$

$$H(x) = \int_0^1 [f_u(h(x)+\tau w(\mu),0)-f_u(h(x),0)]\,\mathrm{d}\tau$$

$$g(x) = f_\mu(h(x),0)\,\mu + \int_0^1 [f_\mu(h(x)+w(\mu),\tau\mu)-f_\mu(h(x),0)]\,\mathrm{d}\tau\,\mu.$$

It is tedious but straightforward to establish the estimates (79) and (80) using the estimates

$$|D| \leqslant C\mathrm{e}^{-\alpha L}, \qquad |H| \leqslant C(\mathrm{e}^{-\alpha L}+|\mu|),$$
$$|g(x)-f_\mu(h(x),0)\,\mu| \leqslant C\,|\mu|\,(\mathrm{e}^{-\alpha L}+|\mu|).$$

We omit the details. ∎

5. SOLVING THE REDUCED EIGENVALUE PROBLEM

In this section, we use Theorem 2.1 to determine the spectrum of the linearization about a periodic wave with large spatial period. We invoke Theorem 4.1 to express the reduced equation

$$E(\lambda,\gamma)=0$$

in terms of the homoclinic orbit $h(x)$ and the associated adjoint solution $\psi(x)$.

In Section 5.1, we consider generic vector fields that have no symmetries at all. In Section 5.2, we assume that the vector field is reversible; this situation often arises for standing pulses that have wave speed zero. In either case, the bifurcating periodic waves can be stable or unstable. The location of their spectrum depends crucially on whether the tails of the pulse converge to zero monotonically or in an oscillatory fashion. In the case of monotone tails, we have that the periodic waves are either stable for all large L or else unstable for all L. In the case of oscillatory tails, the periodic waves are alternately stable and unstable as a function of their spatial period L; the circle of critical eigenvalues crosses the imaginary axis periodically in L. Physically, we may interpret this phenomenon as a locking phenomenon between the decaying oscillatory tails of neighboring pulses in the periodic wave trains.

5.1. *Generic Vector Fields*

Consider the ordinary differential equation

$$u' = f(u, \mu), \qquad (u, \mu) \in \mathbb{R}^n \times \mathbb{R} \tag{84}$$

so that $u = 0$ is a hyperbolic equilibria for all μ. Assume that (84) admits a homoclinic solution $h(x)$ for $\mu = 0$ with $h(x) \to 0$ as $|x| \to \infty$. Typically, the wave speed c of the pulse would supply the one-dimensional parameter μ. We recall that the variational equation about $h(x)$ is given by

$$v' = f_u(h(x), 0)\, v, \qquad x \in \mathbb{R}. \tag{85}$$

As in Hypothesis (N1), we assume that the stable and unstable manifolds of the equilibrium $u = 0$ intersect as transversely as possible for $\mu = 0$.

Hypothesis (G1). Assume that $h'(x)$ is the only bounded solution, up to constant scalar multiples, of the variational equation (85).

The associated non-zero bounded solution of the adjoint variational equation

$$w' = -f_u(h(x), 0)^*\, w, \qquad x \in \mathbb{R}$$

is denoted by $\psi(x)$. Finally, we assume that the stable and unstable manifolds of $u = 0$ cross transversely with respect to the one-dimensional parameter μ at $\mu = 0$.

Hypothesis (G2). $\int_{-\infty}^{\infty} \langle \psi(x), f_\mu(h(x), 0) \rangle \, dx \neq 0$.

Choose $\alpha > 0$ so that $\alpha < \min\{|\mathrm{Re}\, \nu|;\ \nu \in \mathrm{spec}(f_u(0, 0))\} < 3\alpha/2$. We then have the following existence result.

PROPOSITION 5.1. *Assume that the non-degeneracy assumption* (G1) *and the Melnikov condition* (G2) *are met. There are then positive constants* C

and δ with the following property. For every $L > 1/\delta$, there is a unique periodic solution $p_L(x)$ of period $2L$ of (84) *for a unique $\mu = \mu_L$ such that*

$$\sup_{|x| \leqslant L} |p_L(x) - h(x)| < \delta, \qquad |\mu_L| < \delta.$$

Furthermore, we have

$$\begin{aligned} |p_L(L)| + \sup_{|x| \leqslant L} |p_L(x) - h(x)| &\leqslant C\mathrm{e}^{-\alpha L} \\ |\mu_L| &\leqslant C\mathrm{e}^{-2\alpha L} \\ |P_0^{\mathrm{u}}(p_L(-L) - h(-L))| + |P_0^{\mathrm{s}}(p_L(L) - h(L))| &\leqslant C\mathrm{e}^{-2\alpha L}. \end{aligned}$$

Proof. The existence part and the first two estimates had been established in [3, 24]; see also Theorem 4.1. The third estimate is a consequence of Theorem 4.1; see also [32, Section 5]. ▮

Substituting the estimates obtained in Proposition 5.1 into the reduced equations (18–19), see Theorem 2.1, we obtain the following result.

THEOREM 5.1. *Assume that* (G1) *and* (G2) *are met. There are positive constants C and δ with the following property. The boundary-value problem* (16–17) *that describes eigenvalues λ with spatial Floquet exponents* $\mathrm{i}\gamma$ *of the periodic wave $p_L(x)$ described in Proposition* 5.1 *has a solution* (λ, γ, v) *for $\lambda \in \mathbb{C}$ with $|\lambda| < \delta$, $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$ and $L > 1/\delta$ if, and only if,*

$$\begin{aligned} 0 = (\mathrm{e}^{\mathrm{i}\gamma} - 1)\langle \psi(L), h'(-L)\rangle + (1 - \mathrm{e}^{-\mathrm{i}\gamma})\langle \psi(-L), h'(L)\rangle \\ - \lambda \int_{-\infty}^{\infty} \langle \psi(x), Bh'(x)\rangle \, \mathrm{d}x + (\mathrm{e}^{\mathrm{i}\gamma} - 1)\, R(\lambda, \gamma) + \lambda\, \tilde{R}(\lambda, \gamma). \end{aligned} \tag{86}$$

The remainder terms are real whenever $(\lambda, \mathrm{e}^{\mathrm{i}\gamma})$ *is real, and we have*

$$\begin{aligned} |\partial_\lambda^j \partial_\gamma^\ell R(\lambda, \gamma)| &\leqslant C\mathrm{e}^{-3\alpha L} \\ |\partial_\gamma^\ell \tilde{R}(\lambda, \gamma)| &\leqslant C(|\lambda| + \mathrm{e}^{-\alpha L}) \\ |\partial_\lambda^{j+1} \partial_\gamma^\ell \tilde{R}(\lambda, \gamma)| &\leqslant C \end{aligned} \tag{87}$$

uniformly in L for $j, \ell \geqslant 0$.

The theorem shows that the spectrum of the periodic waves that accompany the pulse $h(x)$ depends upon the tails of the pulse, that is, on the behavior of $h(x)$ for large $|x|$.

Before we proceed and calculate the spectrum depending on properties of the tails of the pulse, we assume that the first-order term in λ is non-zero.

Hypothesis (G3). $M := \int_{-\infty}^{\infty} \langle \psi(x), Bh'(x) \rangle \, \mathrm{d}x \neq 0$.

It has been shown in [33, Lemma 5.5] that M is equal to the derivative of the Evans function of the pulse at $\lambda = 0$ for an appropriate normalization of the Evans function. Hence, (G3) states that $\lambda = 0$ is a simple eigenvalue for the pulse.

5.1.1. *Real leading eigenvalues* (*saddle equilibria*). We assume that the eigenvalue of $f_u(0, 0)$ that is closest to the imaginary axis is real and simple. Without loss of generality, we may also assume that its real part is positive; otherwise, we change x to $-x$.

Hypothesis (G4). There is a simple real eigenvalue $\nu^{\mathrm{u}} \in \operatorname{spec}(f_u(0, 0))$ such that $|\mathrm{Re}\, \nu| > \nu^{\mathrm{u}} > 0$ for every $\nu \in \operatorname{spec}(f_u(0, 0))$ with $\nu \neq \nu^{\mathrm{u}}$.

It is a consequence of [6, Chapter 3.8] that there are eigenvectors v_0 and w_0 of $f_u(0, 0)$ and $f_u(0, 0)^*$, respectively, belonging to the eigenvalue ν^{u} such that

$$\begin{aligned} h'(x) &= \mathrm{e}^{\nu^{\mathrm{u}} x} v_0 + \mathrm{O}(\mathrm{e}^{(\nu^{\mathrm{u}} + \delta) x}) && \text{as} \quad x \to -\infty \\ \psi(x) &= \mathrm{e}^{-\nu^{\mathrm{u}} x} w_0 + \mathrm{O}(\mathrm{e}^{-(\nu^{\mathrm{u}} + \delta) x}) && \text{as} \quad x \to \infty. \end{aligned} \tag{88}$$

Generically, the vectors v_0 and w_0 are non-zero. We can then compute the spectrum of the periodic waves $p_L(x)$ that accompany the pulse $h(x)$; see Proposition 5.1.

THEOREM 5.2. *Assume that the homoclinic orbit is non-degenerate* (G1), *transversely unfolded* (G2), *that $\lambda = 0$ is a simple eigenvalue of the pulse* (G3), *and that the leading eigenvalue of $f_u(0, 0)$ is real and simple* (G4). *There are positive constants C and δ and a function $\lambda(\gamma)$ that is analytic in $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$ such that the boundary-value problem* (16–17) *has a solution (λ, γ, v) for $|\lambda| < \delta$, $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$ and $L > 1/\delta$ if, and only if, $\lambda = \lambda(\gamma)$. Furthermore, we have the expansion*

$$\lambda(\gamma) = (\mathrm{e}^{\mathrm{i}\gamma} - 1)\, \mathrm{e}^{-2\nu^{\mathrm{u}} L} \left(\frac{\langle v_0, w_0 \rangle}{M} + R(\gamma) \right) \tag{89}$$

for $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$. The remainder term is analytic in γ with

$$|\partial_\gamma^\ell R(\gamma)| \leqslant C \mathrm{e}^{-\delta L} \tag{90}$$

for $\ell \geqslant 0$ and real-valued for $\gamma \in \{0, \pi\}$.

Proof. It follows from [6, Chapter 3.8] and simplicity of the leading eigenvalue (G4) that

$$|h'(x)| \leqslant C\mathrm{e}^{-(\nu^{\mathrm{u}}+\delta)x} \qquad \text{as} \quad x \to \infty$$

$$|\psi(x)| \leqslant C\mathrm{e}^{(\nu^{\mathrm{u}}+\delta)x} \qquad \text{as} \quad x \to -\infty.$$

Recall that (86) is the reduced equation for solutions of the eigenvalue problem that we derived in from Theorem 5.1. Upon substituting the above inequalities and the expansions (88) into (86), and using the properties (87) of the remainder terms, we obtain the reduced equation

$$\lambda(M + \mathrm{O}(|\lambda| + \mathrm{e}^{-\delta L})) = (\mathrm{e}^{\mathrm{i}\gamma} - 1)\,\mathrm{e}^{-2\nu^{\mathrm{u}}L}(\langle v_0, w_0\rangle + \mathrm{O}(\mathrm{e}^{-\delta L})), \qquad (91)$$

where the remainder terms are analytic in (λ, γ) and real whenever $(\lambda, \mathrm{e}^{\mathrm{i}\gamma})$ is real. The statement of the theorem then follows from the Implicit Function Theorem. ∎

COROLLARY 5.1. *Assume, in addition to the assumptions of Theorem* 5.2, *that* $\langle v_0, w_0\rangle \neq 0$. *The circle of critical eigenvalues close to* $\lambda = 0$ *has then a quadratic tangency at zero with the imaginary axis, and the periodic waves are spectrally stable if, and only if,* $M\langle v_0, w_0\rangle > 0$.

Proof. We expand the real part of

$$\lambda(\gamma) = (\mathrm{e}^{\mathrm{i}\gamma} - 1)\,\mathrm{e}^{-2\nu^{\mathrm{u}}L}\left(\frac{\langle v_0, w_0\rangle}{M} + R(\gamma)\right)$$

in a Taylor series. The derivatives of this expression evaluated at $\gamma = 0$ are given by

$$\lambda'(0) = \mathrm{i}\mathrm{e}^{-2\nu^{\mathrm{u}}L}\left(\frac{\langle v_0, w_0\rangle}{M} + R(0)\right)$$

$$\lambda''(0) = -\mathrm{e}^{-2\nu^{\mathrm{u}}L}\left(\frac{\langle v_0, w_0\rangle}{M} + R(0)\right) + 2\mathrm{i}\mathrm{e}^{-2\nu^{\mathrm{u}}L}R'(0).$$

Using the fact that $R(0)$ is real, we get

$$\begin{aligned}\mathrm{Re}\,\lambda(\gamma) &= \mathrm{Re}\,\lambda(0) + \gamma\,\mathrm{Re}\,\lambda'(0) + \frac{\gamma^2}{2}\,\mathrm{Re}\,\lambda''(0) + \mathrm{O}(\gamma^3)\\ &= -\frac{1}{2}\,\mathrm{e}^{-2\nu^{\mathrm{u}}L}\left(\frac{\langle v_0, w_0\rangle}{M} + R(0)\right)\gamma^2 + \mathrm{e}^{-2\nu^{\mathrm{u}}L}\,\mathrm{Re}(\mathrm{i}R'(0))\,\gamma^2 + \mathrm{O}(\gamma^3).\end{aligned}$$

Using the estimate (90), we obtain

$$\mathrm{Re}\,\lambda(\gamma) = -\frac{1}{2}\,\mathrm{e}^{-2\nu^{\mathrm{u}}L}\left(\frac{\langle v_0, w_0\rangle}{M} + \mathrm{O}(\mathrm{e}^{-\delta L}) + \mathrm{O}(\gamma)\right)\gamma^2.$$

This completes the proof of the corollary. ∎

5.1.2. *Non-real leading eigenvalues* (*saddle-focus equilibria*). The remaining generic case is that the eigenvalues of $f_u(0, 0)$ that are closest to the imaginary axis are complex conjugate and simple. Again, we may assume that their real part is positive.

Hypothesis (G5) There is a pair of simple complex-conjugate eigenvalues $\nu^{\mathrm{u}} \pm \mathrm{i}\beta^{\mathrm{u}} \in \mathrm{spec}(f_u(0, 0))$ such that $|\mathrm{Re}\,\nu| > \nu^{\mathrm{u}} > 0$ for every $\nu \in \mathrm{spec}(f_u(0, 0))$ with $\nu \neq \nu^{\mathrm{u}} \pm \mathrm{i}\beta^{\mathrm{u}}$. We assume that $\beta^{\mathrm{u}} \neq 0$.

It is a consequence of [6, Chapter 3.8] that

$$\langle \psi(L), h'(-L)\rangle = a\,\sin(2\beta^{\mathrm{u}}L + b)\,\mathrm{e}^{-2\nu^{\mathrm{u}}L} + \mathrm{O}(\mathrm{e}^{-(2\nu^{\mathrm{u}}+\delta)\,L}) \qquad (92)$$

for certain constants a and b; see also [33, Lemma 6.1]. Generically, a is non-zero.

THEOREM 5.3. *Assume that the homoclinic orbit is non-degenerate* (G1), *transversely unfolded* (G2), *that* $\lambda = 0$ *is a simple eigenvalue of the pulse* (G3), *and that the leading eigenvalues of* $f_u(0, 0)$ *are simple and complex conjugate* (G5). *There are positive constants* C *and* δ *and a function* $\lambda(\gamma)$ *that is analytic in* $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$ *such that the boundary-value problem* (16–17) *has a solution* (λ, γ, v) *for* $|\lambda| < \delta$, $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$ *and* $L > 1/\delta$ *if, and only if,* $\lambda = \lambda(\gamma)$. *Furthermore, we have the expansion*

$$\lambda(\gamma) = (\mathrm{e}^{\mathrm{i}\gamma} - 1)\,\mathrm{e}^{-2\nu^{\mathrm{u}}L}\left(\frac{a}{M}\sin(2\beta^{\mathrm{u}}L + b) + R(\gamma)\right) \qquad (93)$$

for $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$. *The remainder term* $R(\gamma)$ *is real-valued for* $\gamma \in \{0, \pi\}$ *and satisfies* (90).

We omit the proof as it is analogous to the proof of Theorem 5.2.

Arguing as in Corollary 5.1, we then see that

$$\mathrm{Re}\,\lambda(\gamma) = -\frac{1}{2}\,\mathrm{e}^{-2\nu^{\mathrm{u}}L}\left(\frac{a}{M}\sin(2\beta^{\mathrm{u}}L + b) + \mathrm{O}(\mathrm{e}^{-\delta L}) + \mathrm{O}(\gamma)\right)\gamma^2.$$

Hence, if $a \neq 0$, then the periodic waves change their stability periodically in L regardless of the sign of a and M.

5.2. *Reversible Vector Fields*

In many applications, the underlying partial differential equation is invariant under the reflection $x \mapsto -x$ of the spatial variable x. In this situation, a particularly interesting class of pulses consists of even standing waves that have zero wave speed and are symmetric with respect to reflections of the x-variable. We refer to the theme issue [30] for examples and more background.

The reflection symmetry of the PDE translates into a reversibility of the associated ordinary differential equation that describes steady-states; see Hypothesis (R1) below. Consider the ODE

$$u' = f(u), \qquad u \in \mathbb{R}^{2n} \tag{94}$$

with $f(0) = 0$, and assume that the ODE admits a homoclinic solution $h(x)$ with $h(x) \to 0$ as $|x| \to \infty$. Again, the associated variational equation about $h(x)$ is given by

$$v' = f_u(h(x))\, v, \qquad x \in \mathbb{R}. \tag{95}$$

We assume that (94) is reversible.

Hypothesis (R1). There is a linear involution $\mathscr{R}$: $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with $\mathscr{R}^2 = \mathrm{id}$ so that f anti-commutes with $\mathscr{R}$, that is, $f(\mathscr{R}u) = -\mathscr{R}f(u)$ for every $u \in \mathbb{R}^{2n}$. Furthermore, we assume that the fixed-point space $\mathrm{Fix}(\mathscr{R})$ of $\mathscr{R}$ is n-dimensional and that $h(0) \in \mathrm{Fix}(\mathscr{R})$.

Since $h(0) \in \mathrm{Fix}(\mathscr{R})$, we have $h(-x) = \mathscr{R}h(x)$, and the homoclinic orbit is invariant as a set under $\mathscr{R}$. Any orbit of (94) that is invariant as a set under $\mathscr{R}$ is called symmetric or reversible.

We again assume non-degeneracy of the intersection of stable and unstable manifolds. We emphasize that we do not need any parameters since reversible homoclinic orbits in reversible systems are a robust phenomenon. In addition, the long-wavelength periodic orbits that accompany the homoclinic orbit $h(x)$ typically coexist with $h(x)$; see [41] or [30]. With respect to the underlying PDE, this means that the pulse as well as the periodic waves are stationary and symmetric in x.

Hypothesis (R2). Assume that $h'(x)$ is the only bounded solution, up to constant scalar multiples, of the variational equation (95).

The associated non-zero bounded solution of the adjoint variational equation

$$w' = -f_u(h(x))^*\, w, \qquad x \in \mathbb{R} \tag{96}$$

is denoted by $\psi(x)$. As before, choose α so that $\alpha < \min\{|\mathrm{Re}\, \nu|; \nu \in \mathrm{spec}(f_u(0,0))\} < 3\alpha/2$. We then have the following existence result.

PROPOSITION 5.2. *Assume reversibility* (R1) *and non-degeneracy* (R2). *There are then positive constants* C *and* δ *with the following property. For every* $L > 1/\delta$, *there is a unique periodic solution* $p_L(x)$ *of period* $2L$ *of* (94) *such that*

$$\sup_{|x| \leqslant L} |p_L(x) - h(x)| < \delta.$$

Furthermore, we have

$$\begin{aligned} |p_L(L)| + \sup_{|x| \leqslant L} |p_L(x) - h(x)| &\leqslant C e^{-\alpha L} \\ |P_0^{\mathrm{u}}(p_L(-L) - h(-L))| + |P_0^{\mathrm{s}}(p_L(L) - h(L))| &\leqslant C e^{-2\alpha L}. \end{aligned} \tag{97}$$

Proof. The existence part has been established in [41]; note that their existence condition

$$T_{h(0)} W^{\mathrm{s}}(0) \oplus \mathrm{Fix}(\mathscr{R}) = \mathbb{R}^{2n}$$

is satisfied on account of (R2). Indeed, if

$$v \in T_{h(0)} W^{\mathrm{s}}(0) \oplus \mathrm{Fix}(\mathscr{R}),$$

then $v = \mathscr{R} v \in T_{h(0)} W^{\mathrm{u}}(0)$ by reversibility, and v is therefore the initial value of a bounded solution of the variational equation. We shall show that $v = 0$. Since $\mathscr{R} h'(0) = -h'(0)$, we have that $h'(0) \notin \mathrm{Fix}(\mathscr{R})$. Hence, we conclude from (R2) that $v = 0$. Having established existence, the estimates (97) are again a consequence of Theorem 4.1. ∎

Thus, it is the transverse intersection of $W^{\mathrm{s}}(0)$ with $\mathrm{Fix}(\mathscr{R})$ that is responsible for the structural stability of reversible homoclinic orbits. Regarding the spectral stability of the periodic waves, we have the following theorem.

THEOREM 5.4. *Assume that the reversibility assumption* (R1) *and the non-degeneracy hypothesis* (R2) *are met. There are positive constants* C *and* δ *such that the boundary-value problem* (16–17) *for eigenvalues* λ *with spatial*

Floquet exponent $\mathrm{i}\gamma$ *has a solution* (λ, γ, v) *for* $|\lambda| < \delta$, $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$ *and* $L > 1/\delta$ *if, and only if,*

$$0 = 2(\cos(\gamma) - 1)\langle \psi(L), h'(-L)\rangle - \lambda \int_{-\infty}^{\infty} \langle \psi(x), Bh'(x)\rangle \,\mathrm{d}x$$
$$+ (\mathrm{e}^{\mathrm{i}\gamma} - 1)\, R(\lambda, \gamma) + \lambda\, \tilde{R}(\lambda, \gamma), \qquad (98)$$

where the remainder terms satisfy the estimates (87), *and* R *and* $\tilde{R}$ *are real-valued whenever* $(\lambda, \mathrm{e}^{\mathrm{i}\gamma})$ *is real.*

Proof. It has been proved in [41, Lemma 4(ii)–(iii)] that $\mathcal{R}^*\psi(0) = \psi(0)$. Thus, $\mathcal{R}^*\psi(L) = \psi(-L)$. We also have $\mathcal{R}h'(L) = -h'(-L)$ since $h(x) = \mathcal{R}h(-x)$. Substituting these expressions together with the estimates obtained in Proposition 5.2 into the expansion (18–19) that we derived in Theorem 2.1 completes the proof of the theorem. ∎

Note that the theorem shows that the spectrum is real to leading order. We would expect this since reversibility and self-adjointness of the linearized operator are related (though not the same). We again assume that the eigenvalue $\lambda = 0$ is simple.

Hypothesis (R3). $M := \int_{-\infty}^{\infty} \langle \psi(x), Bh'(x)\rangle \,\mathrm{d}x \neq 0$.

Note that the spectrum of $f_u(0)$ is symmetric with respect to reflections across the imaginary axis. More precisely, it is invariant under multiplication with -1 which can be seen by exploiting the identity $\mathcal{R}f_u(0) = -f_u(0)\,\mathcal{R}$ that is an immediate consequence of the reversibility assumption (R1).

5.2.1. *Real leading eigenvalues* (*saddle equilibria*). First, we consider the case that the eigenvalues of $f_u(0)$ that are closest to the imaginary axis are real and simple. Recall that, by reversibility, the spectrum of $f_u(0)$ is symmetric with respect to the imaginary axis.

Hypothesis (R4). There is a simple real eigenvalue $\nu^{\mathrm{u}} \in \operatorname{spec}(f_u(0))$ such that $|\operatorname{Re} \nu| > \nu^{\mathrm{u}} > 0$ for every $\nu \in \operatorname{spec}(f_u(0))$ with $\nu \neq \pm\nu^{\mathrm{u}}$.

Again, there are eigenvectors v_0 and w_0 of $f_u(0)$ and $f_u(0)^*$, respectively, belonging to the eigenvalue ν^{u} such that

$$\begin{aligned} h'(x) &= \mathrm{e}^{\nu^{\mathrm{u}}x} v_0 + \mathrm{O}(\mathrm{e}^{(\nu^{\mathrm{u}}+\delta)\,x}) \qquad &&\text{as} \quad x \to -\infty \\ \psi(x) &= \mathrm{e}^{-\nu^{\mathrm{u}}x} w_0 + \mathrm{O}(\mathrm{e}^{-(\nu^{\mathrm{u}}+\delta)\,x}) \qquad &&\text{as} \quad x \to \infty. \end{aligned} \qquad (99)$$

THEOREM 5.5. *Assume that the reversibility assumption* (R1) *and the non-degeneracy hypothesis* (R2) *are met, that the eigenvalue* $\lambda = 0$ *is simple*

(R3) *and that the leading eigenvalues of* $f_u(0)$ *are simple and real* (R4). *Under these conditions, there are positive constants* C *and* δ *and a function* $\lambda(\gamma)$ *that is analytic in* $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$ *such that the boundary-value problem* (16–17) *has a solution* (λ, γ, v) *for* $|\lambda| < \delta$, $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$ *and* $L > 1/\delta$ *if, and only if,* $\lambda = \lambda(\gamma)$. *Furthermore, we have the expansion*

$$\lambda(\gamma) = 2(\cos(\gamma) - 1)\, \mathrm{e}^{-2\nu^{\mathrm{u}} L} \frac{\langle v_0, w_0 \rangle}{M} + (\mathrm{e}^{\mathrm{i}\gamma} - 1)\, \mathrm{e}^{-2\nu^{\mathrm{u}} L} R(\gamma) \tag{100}$$

for $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$. *The remainder term* $R(\gamma)$ *is real-valued for* $\gamma \in \{0, \pi\}$ *and satisfies* (90).

We omit the proofs of the theorem and the next corollary since they are analogous to the proofs in Section 5.1.

COROLLARY 5.2. *Assume that, in addition to the assumptions of Theorem* 5.5, $\langle v_0, w_0 \rangle \neq 0$. *The circle of critical eigenvalues then has a quadratic tangency at zero with the imaginary axis, and the periodic waves are spectrally stable if, and only if,* $M \langle v_0, w_0 \rangle > 0$.

5.2.2. *Non-real leading eigenvalues* (*saddle-focus equilibria*). Finally, we assume that the eigenvalues of $f_u(0)$ that are closest to the imaginary axis are simple and complex conjugate.

Hypothesis (R5). There is a pair of simple, complex conjugate eigenvalues $\nu^{\mathrm{u}} \pm \mathrm{i}\beta^{\mathrm{u}} \in \operatorname{spec}(f_u(0))$ such that $|\mathrm{Re}\, \nu| > \nu^{\mathrm{u}} > 0$ for every $\nu \in \operatorname{spec}(f_u(0))$ with $\nu \neq \pm \nu^{\mathrm{u}} \pm \mathrm{i}\beta^{\mathrm{u}}$. We assume that $\beta^{\mathrm{u}} \neq 0$.

THEOREM 5.6. *Assume that the hypotheses* (R1)–(R3) *and* (R5) *are met. Under these conditions, there are positive constants* C *and* δ *and a function* $\lambda(\gamma)$ *that is analytic in* $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$ *such that the boundary-value problem* (16–17) *has a solution* (λ, γ, v) *for* $|\lambda| < \delta$, $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$ *and* $L > 1/\delta$ *if, and only if,* $\lambda = \lambda(\gamma)$. *Furthermore, we have the expansion*

$$\lambda(\gamma) = 2(\cos(\gamma) - 1)\, \mathrm{e}^{-2\nu^{\mathrm{u}} L} \frac{a}{M} \sin(2\beta^{\mathrm{u}} L + b) + (\mathrm{e}^{\mathrm{i}\gamma} - 1)\, \mathrm{O}(\mathrm{e}^{-(2\nu^{\mathrm{u}} + \delta) L}) \tag{101}$$

for $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$, *where the remainder term* $R(\gamma)$ *is real-valued for* $\gamma \in \{0, \pi\}$ *and satisfies* (90).

We omit its proof since it is analogous to the proof of Theorem 117. It follows as in Section that, if $a \neq 0$, then the periodic waves change their stability periodically in L regardless of the sign of a and M.

6. APPLICATIONS

In this section, we apply the results on the stability of periodic wave trains with large period from the previous section to the FitzHugh–Nagumo equation and a fourth-order equation that models the propagation of signals in optical fibers with phase-sensitive amplifiers. Both equations support stable pulses, and we demonstrate that the accompanying periodic waves are also spectrally stable.

To illustrate the kind of arguments that are needed to invoke our stability results, we start with a straightforward application of these results to the real Ginzburg–Landau equation.

6.1. *The Real Ginzburg–Landau Equation*

We consider the scalar parabolic equation

$$u_t = u_{xx} + u(1-u^2), \qquad u \in \mathbb{R}, \quad x \in \mathbb{R}, \quad t > 0.$$

This equation is symmetric under reflections of x and in u. It admits two stable front or layer solutions that are given explicitly by

$$u_+(x) = \tanh(x/\sqrt{2}), \qquad u_-(x) = \tanh(-x/\sqrt{2}).$$

These two layer solutions are accompanied by a family of stationary periodic patterns that are periodic solutions of the ODE

$$u' = v, \qquad v' = -u(1-u^2). \tag{102}$$

The ODE is Hamiltonian with energy $H(u, v) = \frac{1}{2}(v^2 + u^2 - \frac{1}{2}u^4)$. The two layers form a heteroclinic cycle that connects the equilibrium $(1, 0)$ with $(-1, 0)$ and vice versa.

On account of Sturm–Liouville theory, the layer solutions are stable waves of the PDE; the linearized operator $\partial_{xx} + 1 - 3u_\pm^2$ has a simple eigenvalue at zero and the rest of its spectrum is strictly contained in the open left half-plane.

The periodic waves that accompany the layers are unstable. This is most easily seen by Sturm–Liouville theory. The eigenvalue problem is again a Sturm–Liouville problem, and the eigenfunction to the zero-eigenvalue, induced by translation, possesses two zeros. The leading eigenvalue is therefore unstable, and its eigenfunction is strictly positive. We give an independent proof that does not make use of the maximum principle and that, in addition, gives more detailed information on the leading unstable eigenvalue of the periodic waves.

Our goal is to apply Corollary 5.2 which is concerned with reversible homoclinic orbits that have monotone tails. We argue that the signs of the constants that appear in the statements of Corollary 5.2 can be readily calculated by merely inspecting the phase portrait to the ODE (102).

In order to view the two heteroclinic solutions as a homoclinic orbit, we consider the quotient space of the phase space $\mathbb{R}^2$ under the symmetry $(u, v) \mapsto (-u, -v)$, and represent the quotient space by the upper half-plane. Thus, the homoclinic orbit is represented by the upper layer solution $h(x) = (u_+(x), u'_+(x))$. It is not hard to see that, in the reduced phase space, we still detect all possible eigenvalues λ with Floquet exponents $\mathrm{i}\gamma$. In any case, we shall see that there is a circle of unstable eigenvalues.

The bounded solution of the adjoint equation is given by

$$\psi(x) = \nabla H(u_+(x), u'_+(x)) = (u'_+(x), u''_+(x))^{\perp} = (-u''_+(x), u'_+(x)).$$

Also, the matrix B that indicates the type of the PDE is given by

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

so that

$$Bh'(x) = \begin{pmatrix} 0 \\ u'_+(x) \end{pmatrix}.$$

Therefore,

$$M = \int_{-\infty}^{\infty} \langle \psi(x), Bh'(x) \rangle \, \mathrm{d}x = \int_{-\infty}^{\infty} (u'_+(x))^2 \, \mathrm{d}x > 0.$$

On the other hand, with this choice of $\psi(x)$, we have that $v_0 = -w_0$ in Corollary 5.2. Indeed, for $x \to \infty$, the adjoint solution $\psi(x)$ converges to the part of the unstable manifold of $(1, 0)$ that is not included in the heteroclinic cycle. We conclude that $M\langle v_0, w_0 \rangle < 0$ and, by Corollary 5.2, the periodic waves are unstable.

The most unstable eigenvalue corresponds to $\gamma = \pi$. On account of the expansion (21) in Theorem 2.1, the associated eigenfunction $v_\pi(x)$ is a concatenation of $u'_+(x)$ and $-u'_-(x)$, and both of these functions are positive. Thus, the eigenfunction $v_\pi(x)$ that represents the most unstable mode is positive, at least to leading order. Physically, the resulting instability manifests itself in the following fashion: Upon adding a small constant positive multiple of $v_\pi(x)$ to the periodic pattern, the layers $u_+(x)$ move to the left, while the layers $u_-(x)$ move to the right. Therefore, as the instability develops, the distance between adjacent layers u_+ and u_- increases, while the distance between adjacent layers u_- and u_+ decreases. Adding a negative multiple of $v_\pi(x)$ has the opposite effect. In any case, the resulting instability is created by attracting and repelling forces between adjacent layers. That is in accordance with results on the interaction between finitely many layers; see [4, 10] and, for a more general theory, [35].

6.2. *The FitzHugh–Nagumo Equation*

The FitzHugh–Nagumo equation is given by

$$\begin{aligned} u_t &= u_{xx} + f(u) - w \\ w_t &= \varepsilon(u - \gamma w), \end{aligned} \tag{103}$$

where $x \in \mathbb{R}$, $f(u) = u(1-u)(u-a)$ and $a \in (0, \frac{1}{2})$. This equation is a simplification of the Hodgkin-Huxley equation that models the propagation of impulses in nerve axons. We are interested in travelling waves $(u, w)(x, t) = (u, w)(x + ct)$. In the new variables $(\xi, t) = (x + ct, t)$, the FitzHugh–Nagumo equation (103) takes the form

$$\begin{aligned} u_t &= u_{\xi\xi} - cu_\xi + f(u) - w \\ w_t &= -cw_\xi + \varepsilon(u - \gamma w). \end{aligned} \tag{104}$$

Waves that travel with speed c are then solutions of the ODE

$$U' = F(U, c), \qquad U = (u, v, w), \tag{105}$$

where $' = \mathrm{d}/\mathrm{d}\xi$ and $F(u, v, w, c) = (v, cv - f(u) + w, \frac{\varepsilon}{c}(u - \gamma w))$. It has been shown in [14] that (105) exhibits a homoclinic solution with positive speed to the equilibrium $U = (u, v, w) = 0$. We refer to this homoclinic orbit as the fast pulse.

THEOREM 21. *Fix a in the interval* $(0, \frac{1}{2})$. *There exists a number* $\varepsilon_* = \varepsilon_*(a)$ *with the following property. For every* ε *with* $0 < \varepsilon < \varepsilon_*$, *there is an* $L_* = L_*(\varepsilon)$ *so that the fast pulse to the FitzHugh–Nagumo system is accompanied by periodic wave trains with period* $2L$ *for any* $L > L_*$, *and all these wave trains are spectrally stable.*

Note that the minimal period $L_*(\varepsilon)$ permitted in the above theorem will tend to infinity as ε tends to zero. In other words, our approach is valid only in the following limit: first, fix $\varepsilon > 0$ and consider the fast pulse for that value of ε; then consider the periodic wave trains that accompany the fast pulse for sufficiently large periods $L_* = L_*(\varepsilon)$ where $L_*(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Eszter [8] investigated the stability of periodic wave trains to the FitzHugh–Nagumo system in a different limit: he first fixed the period L of a singular spatially-periodic wave train and then varied $\varepsilon > 0$ near zero with $\varepsilon < \varepsilon_*(L)$; the maximal allowed value $\varepsilon_*(L)$ will tend to zero as the period L

tends to infinity. In this sense, his and our results are complementary: there is a region in parameter space in which neither his nor our results apply.

Proof. It has been proved in [17, 42] that the fast pulse is stable. It therefore suffices to calculate the spectrum of the periodic waves near $\lambda=0$.

Linearized stability of a wave (u, w) of (104) is determined by the spectrum of the linear operator

$$\mathscr{L}(\tilde{u}, \tilde{w}) = \begin{pmatrix} \tilde{u}_{\xi\xi} - c\tilde{u}_\xi + f_u(u)\,\tilde{u} - \tilde{w} \\ -c\tilde{w}_\xi + \varepsilon(\tilde{u} - \gamma\tilde{w}) \end{pmatrix}. \tag{106}$$

In particular, eigenvalues λ with corresponding eigenfunction $(\tilde{u}, \tilde{w})$ of $\mathscr{L}$ are given by bounded solutions of

$$V' = (F_U(U, c) + \lambda B)\,V, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{c} \end{pmatrix}, \tag{107}$$

where F_U is evaluated at the fast pulse $h(\xi) = (u, v, w)(\xi)$. In the following, we collect various properties of (105) and (107) that can be found in the literature [17, 42].

The spectrum of the linearization of (105) about the equilibrium $U=0$ in the relevant parameter regime is given by three simple eigenvalues $\nu^{ss} < \nu^{\mathrm{s}} < 0 < \nu^{\mathrm{u}}$ with $\nu^{\mathrm{u}} > |\nu^{\mathrm{s}}|$. Thus, (G4) is met for the time-reversed system. Note also that (G1) is automatically satisfied since the unstable manifold of (105) is one-dimensional. Furthermore, it follows from [6, Chapter 3.8] that there are eigenvectors V^{u} and V^{s} of $F_U(0, c)$ to the eigenvalues ν^{u} and ν^{s}, respectively, as well as eigenvectors W^{u} and W^{s} to $F_U(0, c)^*$ such that

$$h(\xi) = \begin{cases} V^{\mathrm{u}} \mathrm{e}^{\nu^{\mathrm{s}}\xi} + \mathrm{O}(\mathrm{e}^{(\nu^{\mathrm{s}} - \delta)\,\xi}), & \xi \to \infty \\ V^{\mathrm{s}} \mathrm{e}^{\nu^{\mathrm{u}}\xi} + \mathrm{O}(\mathrm{e}^{(\nu^{\mathrm{u}} + \delta)\,\xi}), & \xi \to -\infty \end{cases}$$

$$\psi(\xi) = \begin{cases} W^{\mathrm{u}} \mathrm{e}^{-\nu^{\mathrm{u}}\xi} + \mathrm{O}(\mathrm{e}^{-(\nu^{\mathrm{u}} + \delta)\,\xi}), & \xi \to \infty \\ W^{\mathrm{s}} \mathrm{e}^{-\nu^{\mathrm{s}}\xi} + \mathrm{O}(\mathrm{e}^{-(\nu^{\mathrm{s}} - \delta)\,\xi}), & \xi \to -\infty, \end{cases}$$

where $\psi(\xi)$ is the bounded solution to the adjoint variational equation. We choose $\psi(\xi)$ such that $\langle V^{\mathrm{u}}, W^{\mathrm{u}} \rangle = 1$. With this choice, it follows from [42, (2.18–2.19)] and [34, (5.7–5.8)] that

$$M = \int_{-\infty}^{\infty} \langle \psi(\xi), Bh'(\xi) \rangle \,\mathrm{d}\xi = \int_{-\infty}^{\infty} \langle \psi(\xi), F_c(h(\xi), c) \rangle \,\mathrm{d}\xi > 0.$$

In particular, taking $\mu = c$, we see that the Melnikov condition (G2) is met so that the pulse unfolds transversely with respect to the wave speed c. In addition, Hypothesis (G3) is met, that is, zero is a simple eigenvalue of the PDE linearization about the fast pulse. Finally, it has been demonstrated in [20, Proposition 6] that the fast pulse is orientable. Given that $\langle V^{\mathrm{u}}, W^{\mathrm{u}} \rangle = 1$, orientability is equivalent to

$$\langle V^{\mathrm{s}}, W^{\mathrm{s}} \rangle > 0.$$

Therefore, we get

$$h'(\xi) = \nu^{\mathrm{s}} V^{\mathrm{s}} \mathrm{e}^{\nu^{\mathrm{s}} \xi} + \mathrm{O}(\mathrm{e}^{(\nu^{\mathrm{s}} - \delta)\, \xi})$$

and

$$\langle \psi(-L), h'(L) \rangle = \nu^{\mathrm{s}} \langle V^{\mathrm{s}}, W^{\mathrm{s}} \rangle\, \mathrm{e}^{2\nu^{\mathrm{s}} L} + \mathrm{O}(\mathrm{e}^{(2\nu^{\mathrm{s}} - \delta)\, L}).$$

Since $\nu^{\mathrm{s}} < 0$, it follows from (89), using the proof of Theorem 5.2, that the periodic waves that accompany the fast pulse are stable. ∎

6.3. *The PSA Equation*

As another application, we consider a fourth-order equation that arises when studying propagating pulses in optical fibers. It has been proposed in [22] to utilize periodically spaced phase-sensitive amplifiers to compensate for the attenuation of pulses inherent to such fibers. Each such amplifier exhibits an associated reference phase. The part of the signal in phase with this reference phase is amplified, while the out-of-phase component is attenuated; see again [22] for the details. In the last reference, it has also been shown that the dynamics of the in-phase component U of the pulse amplitude under the influence of phase-sensitive amplifiers is governed by the fourth-order equation

$$\frac{\partial U}{\partial t} + \frac{1}{4} \frac{\partial^4 U}{\partial x^4} + \left(\left(3 - \frac{\tanh \Gamma\ell}{\Gamma\ell} \right) U^2 - \frac{\kappa}{2} \right) \frac{\partial^2 U}{\partial x^2}$$
$$+ 3 \left(2 - \frac{\tanh \Gamma\ell}{\Gamma\ell} \right) U \left(\frac{\partial U}{\partial x} \right)^2 + \left(\frac{\kappa}{4} - \Delta\alpha \right) U - \kappa U^3 + U^5 = 0. \tag{108}$$

Here, $t \in \mathbb{R}$ measures the distance along the fiber, and $x \in \mathbb{R}$ is the time variable in a frame moving with the group velocity of light in the optical fiber. Furthermore, κ is related to the reference phase associated with each amplifier. The parameter $\Delta\alpha$ measures the amount of over-amplification, that is, the amount of energy remaining after compensating for the loss in the fiber. Finally, $\Gamma\ell$ is the product of the linear loss rate Γ in the fiber and the distance ℓ of the amplifiers. For $\Gamma\ell=0$, (108) exhibits the explicit pulse

$$Q(x)=\sqrt{\kappa+2\sqrt{\Delta\alpha}}\,\operatorname{sech}(\sqrt{\kappa+2\sqrt{\Delta\alpha}}\,x)$$

that exists for $0<\Delta\alpha<\kappa^2/4$ for each fixed $\kappa\geqslant 0$.

It has been demonstrated in [23] that this pulse is stable with respect to (108) provided $\Delta\alpha$ is sufficiently close to zero. In [2], it has been proved that the pulse persists for $\Gamma\ell>0$ small, and that the resulting pulses are stable for every fixed $\Delta\alpha$ with $0<\Delta\alpha<\kappa^2/4$ and every $\Gamma\ell>0$ sufficiently small. Furthermore, in [36], the existence of stable multi-hump pulses has been demonstrated. Here, we prove that the periodic waves that accompany the pulse for small $\Gamma\ell>0$ are also stable.

THEOREM 6.2. *Fix $\Delta\alpha$ with $0<\Delta\alpha<\kappa^2/4$. The periodic waves $P_L(x)$ that accompany the pulse $Q(x)$ are spectrally stable for every $\Gamma\ell>0$ sufficiently small and $L>L_*$ sufficiently large where L_* depends upon $\Gamma\ell$.*

Note that the PSA equation exhibits multi-bump pulses for $\Gamma\ell>0$ that resemble equally spaced concatenated copies of the primary pulse Q. These multi-bump pulses are unstable; see [36]. One would expect that the periodic waves that accompany the pulse Q would then also be unstable. Note, however, that Theorem 6.2 is only valid for sufficiently large periods L. The multi-bump pulses that bifurcate at $\Gamma\ell=0$ have distances at $\Gamma\ell>0$ that are smaller than L_*. Hence, we expect that the periodic waves destabilize once their period gets too small.

Proof. We write (108) as

$$\frac{\partial U}{\partial t}+\Phi(U)=0.$$

The linearization about a wave U is then given by $\mathscr{L}(U)=\Phi_U(U)$. Note, however, that the wave U is stable if, and only if, the spectrum of $\mathscr{L}(U)$ is in the right half-plane.

The steady-state equation $\Phi(U)=0$ associated with (108) is a fourth-order ODE that can be written in the form

$$u'=f(u), \tag{109}$$

where $u=(U, U_x, U_{xx}, U_{xxx})$. Furthermore, the eigenvalue problem associated with linearized operator $\mathscr{L}(U)$ about a wave $u=(U, U_x, U_{xx}, U_{xxx})$ is given by

$$v'=(f_u(u)+\lambda B)\,v, \tag{110}$$

where

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Let $h = (Q, Q_x, Q_{xx}, Q_{xxx})$ be the homoclinic orbit of (109) that corresponds to the pulse Q of (108). Equation (109) admits a reversibility operator $\mathscr{R}$. It has been proved in [2, 36] that (109) and the pulse $h(x)$ satisfy the Hypotheses (R1) and (R2). For $\Gamma\ell > 0$, (108) satisfies (R4); see [36]. Finally, it follows from the arguments given in [36, p. 197] that

$$\langle v_0, w_0 \rangle > 0,$$

while the calculations in [36, p. 199] demonstrate that

$$M = \int_{-\infty}^{\infty} \langle \psi(x), Bh'(x) \rangle \, \mathrm{d}x < 0.$$

Therefore, (R3) is met, and we conclude from Theorem 5.5 that the circle of eigenvalues to the periodic waves that accompany the pulse h is contained in the right half-plane. In this context, that means that the periodic waves are stable; see above. ∎

7. DISCUSSION

We have presented a reduction method that allows us to explore the stability and instability properties of long-wavelength periodic patterns in a systematic fashion. The reduction results in a bifurcation equation for the eigenvalue λ and the spatial Floquet exponent $\mathrm{i}\gamma$; in addition, we have derived the leading-order terms of the bifurcation equation. Utilizing these results, the critical spectrum of long-wavelength periodic waves can be calculated under quite minimal assumptions on the underlying PDE.

Our main conclusions regarding the spectral stability of spatially periodic patterns that accompany a stable pulse can be summarized as follows. In general, the precise location of the critical spectrum near zero depends crucially on the decay properties of the tails of the pulse. If the tails decay in an oscillatory fashion, then locking occurs: The periodic patterns that exist for sufficiently large periods $2L$ stabilize and destabilize alternately with increasing period $2L$. This process of critical spectrum crossing the imaginary axis forth and back occurs periodically in $2L$, and its frequency is given by the frequency of the oscillations in the tails of the pulse. If, on the other hand, the tails of the pulse decay monotonically, then the periodic patterns are either all stable or else all unstable regardless of their period $2L$. Which of these two cases occurs depends on the sign of a certain coefficient in the expansion of the bifurcation equation. This coefficient can be calculated using spectral information for the primary pulse (for instance, from the Evans function associated with the pulse) and a geometric quantity that describes whether a certain bundle about the homoclinic orbit of the steady-state ODE is orientable or not.

An interesting aspect of the results in Section 5 is that, in every case considered there, spectral stability of a periodic pattern with period $2L$ on the real line is equivalent to spectral stability of the same pattern considered on an interval of length $4L$, that is twice the period, with periodic boundary conditions. Indeed, in any of the cases studied in Section 5, the Floquet exponent with $\gamma = \pi$ decides upon stability. The resulting eigenfunction is periodic with period $4L$ and is therefore visible once the underlying PDE is considered on an interval of length $4L$ with periodic boundary conditions.

In the remaining part of this section, we shall explain a number of generalizations of our results and comment on some open problems.

7.1. *Elliptic PDEs on Cylinders with One Unbounded Direction*

Consider the parabolic equation

$$U_t = U_{xx} + \Delta U + f(U), \qquad (x, y) \in \mathbb{R} \times \Omega \tag{111}$$

on an unbounded cylinder $\mathbb{R} \times \Omega$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, and Δ is the Laplace operator in the y-variable. We impose appropriate boundary conditions such as Neumann or Dirichlet conditions on $\partial\Omega$, and assume that f is analytic. In a moving frame, travelling waves to (111) satisfy the elliptic problem

$$U_{\xi\xi} + \Delta U + cU_\xi + f(U) = 0, \qquad (\xi, y) \in \mathbb{R} \times \Omega. \tag{112}$$

Suppose that (112) exhibits a pulse solution $h(\xi, y)$. In [25, Corollary 2.6], we demonstrated that, under appropriate assumptions, the pulse h is again accompanied by periodic wave trains of (112). Their spectral stability properties with respect to (111) are then determined by Theorem 2.2. Indeed, as emphasized in Remark 3.1, we utilized only exponential dichotomies for the proof of our results. In [29], we had demonstrated that elliptic problems such as (112) admit exponential dichotomies. We refer to [38] for more details.

7.2. *Modulated Pulses*

We were motivated to investigate the stability of periodic wave trains when we studied Hopf bifurcations of pulses in reaction-diffusion systems

$$U_t = U_{\xi\xi} + cU_\xi + F(U, \mu), \qquad \xi \in \mathbb{R}. \tag{113}$$

Suppose that $h(\xi)$ is a pulse to (113) that experiences a super-critical Hopf bifurcation at $\mu = 0$. Thus, for $\mu > 0$, there exists a time-periodic modulated pulse $h(\xi, t)$ to (147), possibly for a slightly different wave speed c. In particular, there is a $T > 0$ such that $h(\xi, t+T) = h(\xi, t)$ for all t. Since we assumed that the Hopf bifurcation is super-critical, the solution $h(\xi, t)$ is stable with respect to (147). Note that the pulse $h(\xi)$ is accompanied by L-periodic wave trains $p_L(\xi)$ that satisfy $p_L(\xi + L) = p_L(\xi)$ for all L. It is then natural to expect that the modulated pulses $h(\xi, t)$ that bifurcate at the Hopf-bifurcation point are accompanied by modulated periodic waves $p_L(\xi, t)$ that satisfy

$$p_L(\xi + L, t) = p_L(\xi, t) = p_L(\xi, t+T)$$

for all (ξ, t) with a suitably chosen $c = c(L)$. Here, T is close to the Hopf period of the pulse, while $L > L_*$ for some sufficiently large L_*. Given that these modulated periodic waves exist, one would then like to investigate their stability. The existence and stability analysis of these waves is carried out in [38] utilizing the results presented here. Note that such an analysis requires the investigation of the time-T map of (113) about a solution $p_L(\xi, t)$ that is periodic in space and time with periods L and T, respectively. It turns out, however, that, from a technical viewpoint, the spectral analysis of the time-T map associated with a modulated wave is quite similar to the spectral analysis for pulses in cylinders; we refer to the previous section and to [38] for more details.

We remark that a similar issue arises when a periodic wave train $h_L(\xi)$ is considered in the original coordinate $x = \xi + ct$. The resulting spatially–periodic wave train $h_L(x - ct)$ is periodic in time with period T, say. The spectrum of the linearization of the time-T map can again be computed from the spectrum of the linearization in the moving frame $\xi = x - ct$; we refer to [38, 37], and in the particular to [37, Prop. 1 and Section 5.1], for more details.

7.3. *Exponential Weights*

Suppose that $\mathscr{L}$ is the linearization about a wave train with spatial period $2L$. In addition to computing the spectrum of $\mathscr{L}$ in the space of bounded continuous functions, we may also compute its spectrum in the exponentially weighted space $\mathscr{C}_\eta$ with norm

$$\|u\|_{\mathscr{C}_\eta} = \sup_{\xi \in \mathbb{R}} \mathrm{e}^{\eta\xi} \, |u(\xi)|.$$

It follows from Floquet theory that λ is in the spectrum of $\mathscr{L}$ posed on the space $\mathscr{C}_\eta$ if, and only if, the boundary-value problem (9) has a non-trivial solution for some γ with $\operatorname{Im} \gamma = \eta/2L$. By analyticity of the bifurcation function E, see Theorem 2.1, the critical eigenvalues of $\mathscr{L}$ can be computed by substituting γ with $\operatorname{Im} \gamma = \eta/2L$ into the expression for E in Theorem 2.1. In particular, the radius of the circle of critical eigenvalues is changed to $e^{-\eta/2L}$ if the pulse decays faster at the right than at the left tail; see (93). Note that η positive is then necessary for spectral stability. In the reversible case, the wave trains are always unstable in exponentially weighted norms since $|\cos(\gamma)| > 1$ for $\operatorname{Im} \gamma \neq 0$.

7.4. *Homoclinic Bifurcations and N-periodic Waves*

As pointed out in the introduction, our results demonstrate that periodic waves with large period typically destabilize at homoclinic bifurcation points. This is illustrated in Theorem 5.3: The periodic waves $p_L(x)$ that accompany a pulse $h(x)$ to a saddle-focus destabilize and stabilize periodically in L. It is known that these periodic solutions undergo many saddle-node and period-doubling bifurcations as L increases. These are induced by the horseshoes that accompany the pulse.

At certain homoclinic bifurcation points, N-pulses are created that resemble many concatenated copies of the primary pulse $h(x)$. Denote the distances between consecutive copies of $h(x)$ that appear in the multi-bump orbit by $2L_j$. These multi-bump pulses are then also accompanied by periodic waves. Their spectrum is much harder to calculate since every N-pulse has N eigenvalues near zero instead of one simple eigenvalue at zero. In [33], a method had been presented that can be used to calculate these critical eigenvalues near an N-pulse. Augmented with the results in this article, it is possible to compute the spectrum of the N-periodic waves that bifurcate near a homoclinic bifurcation point by combining and adapting the results given in [33] and here appropriately.

Note that the spectrum about a periodic pattern that accompanies, for instance, a 2-pulse is expected to behave in a quite peculiar way. Typically, such periodic waves can be parametrized by two distances L_1 and L_2. As the periodic wave approaches the 2-pulse, one of these numbers, say L_2, approaches infinity. On the other hand, as L_2 approaches L_1, the periodic wave disappears in a period-doubling bifurcation. We observe that, in the limit $L_2 \to \infty$, the critical spectrum of the periodic wave consists of two circles that are attached to the two critical discrete eigenvalues of the 2-pulse. In the limit $L_2 \to L_1$, however, we recover the spectrum of the long-wavelength periodic wave that accompanies the primary pulse; see, for instance, [11]. Thus, while L_2 varies between L_1 and ∞, the aforementioned two critical circles somehow have to merge.

7.5. *Generalizations and Open Problems*

It is possible to calculate the algebraic multiplicity of an eigenvalue λ associated with a periodic wave train. In fact, the algebraic multiplicity is equal to the multiplicity of λ as a zero of the function $E(\lambda, \gamma)$. This is a consequence of the results in [11] and [33, Lemma 4.2]. In Section 2, we had assumed that the geometric and the algebraic multiplicity of the eigenvalue $\lambda = 0$ of the pulse are equal. We emphasize that it is possible, and in fact not difficult, to extend our results to the case where the algebraic multiplicity is larger than the geometric multiplicity.

Localized patterns of a parabolic equation

$$u_t = \Delta u + f(u), \qquad x \in \mathbb{R}^n$$

are still accompanied by a family of long-wavelength periodic patterns [25]. Periodicity, however, can be imposed successively in any of the n spatial directions. If the primary pattern is not rotationally symmetric, the stability of the accompanying periodic patterns might depend on the direction of periodicity. A detailed investigation of possible phenomena would be interesting and seems to be feasible. Note that Floquet theory is well developed for elliptic problems on $\mathbb{R}^n$ that are periodic on a lattice. The relevant results make use of the so-called Bloch decomposition of $L^2(\mathbb{R}^n)$; see [21, 26, 27].

Another aspect of the stability problem is the relation between linear and nonlinear stability. In dissipative equations, linear stability of a pulse implies nonlinear stability with exponential rate and asymptotic phase. Nonlinear stability of periodic patterns, however, is a much more delicate question. Stability with respect to localized perturbations that exclude phase shifts has recently been established in a fairly general setting using renormalization group theory [7, 40]. The convergence towards the periodic pattern is much slower: Perturbations typically decay like $t^{-1/2}$ instead of exponentially. It is an interesting issue that, in the limit of infinite wavelength, this diffusive aspect of nonlinear stability seems to become less important and we expect that the exponential decay associated with the pulse takes over as the wavelength approaches infinity even though, for any finite wavelength, the approach should be polynomial.

A related issue is the interaction of individual pulses in a spatially-periodic wave train with long wavelength. Consider

$$U_t = U_{\xi\xi} + cU_\xi + F(U), \qquad \xi \in \mathbb{R}$$

on the space $X = \mathscr{C}^0_{\text{unif}}(\mathbb{R}, \mathbb{R}^N)$, and focus on a stable pulse solution that is accompanied by spatially-periodic wave trains with long wavelength. As a consequence of [15, Theorems 6.1.2 and 6.2.1], any such wave train possesses an infinite-dimensional, Lipschitz-continuous local center manifold associated with the critical spectrum that we computed in this article. The approach towards this center manifold is exponential; the linear eigenmodes tangent to the center manifold at the periodic wave train describe the interaction of the individual pulses in the wave train on an infinitesimal level. The nonlinear dynamics on the center manifold is, however, quite limited as the supremum-norm on X is extremely restrictive as far as possible perturbations are concerned. It would be interesting to explore whether there is a global invariant manifold that contains all periodic wave trains of long wavelength and that could be used to describe the complete dynamics of the pulses in these wave trains.

Finally, we remark that our results as well as those of Gardner apply only near isolated eigenvalues with finite multiplicity in the spectrum of the homoclinic orbit. The spectrum of the linearization about a pulse contains also essential spectrum. The fate of the essential spectrum under truncation to periodic (and other) boundary conditions has recently been determined in [39].

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