Fredholm properties of radially symmetric, second order differential operators

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Abstract: We analyze Fredholm properties of radially symmetric second order systems in unbounded domains. The main theorem relates the Fredholm index to the Morse index at infinity. As a consequence, linear operators are Fredholm in exponentially weighted spaces for almost all weights. The result provides the basic tool for the analysis of perturbation and bifurcation problems in the presence of essential spectrum. We give a simple illustrative example, where we use the implicit function theorem to calculate the effect of a localized source term on a trimolecular chemical reaction-diffusion systems on the plane.

Keywords: differential operators; Fredholm properties; radial symmetry; bifurcation from essential spectrum; far–field matching.

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1 Introduction

When studying perturbation and bifurcation problems in unbounded domains, one is often confronted with the difficulty that the relevant linearized operator is not invertible, not even Fredholm, in convenient function spaces such as L^p-based spaces or spaces of continuous functions. This difficulty is caused by the non-compactness of the underlying physical space. The goal of this paper is to present and illustrate results on the calculation of Fredholm indices in exponentially weighted spaces that can be used to circumvent this difficulty.

We therefore first present a simple toy problem that captures some of the main difficulties. We then discuss in some more detail our main general results, Theorems 1.1–1.3, which characterize Fredholm properties of radially symmetric elliptic operators, and then apply those results to our toy problem. We also briefly comment on a more elaborate application towards bifurcation of eigenvalues from the essential spectrum.

1.1 Perturbation theory and the essential spectrum — a toy problem

Consider the apparently simple problem

$$\Delta u - u^3 + \varepsilon V_0(|x|) = 0, \quad x \in \mathbb{R}^2,$$

with V exponentially localized, and ε small. One would like to continue the trivial solution $u(x) \equiv 0$ at $\varepsilon = 0$ to $\varepsilon \neq 0$ in, say, $H^2(\mathbb{R}^2)$. When trying to invoke the implicit function theorem to that purpose, a difficulty arises from the fact that the Laplacian is not Fredholm from H^2 into L^2, so that a naive application of the implicit function theorem is not possible. This difficulty also manifests itself when different methods are employed, such as comparison principles or variational methods; see for instance the recent work [3] and references therein, or [2] for a situation where the linearization is invertible.

In this specific context, a simple remedy is to rewrite the elliptic problem as a first-order ordinary differential equation,

$$u_r = v, \quad v_r = -\frac{v}{r} + u^3 - \varepsilon V_0(r),$$

use dynamical systems methods to construct manifolds $\mathcal{W}_\varepsilon^\pm$ of solutions that decay at $r = +\infty$ and are bounded at $r = 0$, respectively, and then use a variant of Melnikov analysis to study the intersection of these manifolds.

This dynamical systems approach has been used quite successfully in much more elaborate problems, such as elliptic equations posed on infinite cylinders, when the dynamical systems setup is actually ill-posed due to Hadamard-type instabilities. Key technique then is often the construction of center manifolds as pioneered in [12], or global dichotomies and Melnikov analysis as in [15]. While quite successful in many circumstances, this method is somewhat indirect since PDE concepts need to be translated into dynamical systems language. For instance, dimensions of the generalized kernel are often encoded in geometric transverse crossing of stable and unstable manifolds; see for instance [22].

A somewhat different approach, in some sense more traditional, was outlined in [23] and successfully extended and applied in [16, 17]. The key idea in these papers was to decompose the solution into an exponentially localized part and a far-field component which can be computed to leading order from a simpler far-field problem. In our simple toy problem, we would decompose $u = u_{\rm loc} + u_{\rm ff}$, where $u_{\rm loc}$ belongs to a space of exponentially localized functions, and $u_{\rm ff}$ is a bounded function that solves the equation for $r \geq r_* \gg 1$ exactly; see (5.16) for the precise form of the decomposition. The upshot is that in this decomposition, one can rely on the fact that the Laplacian is actually Fredholm in spaces of exponentially localized functions and use a bordering lemma and the implicit function theorem to establish existence and asymptotic properties of solutions for $\varepsilon \neq 0$ in a fairly straightforward fashion.

One ingredient to such an analysis are Fredholm properties of differential operators in spaces with suitable exponential weights. Our main abstract results, Theorems 1.1–1.3, aim at precisely such Fredholm properties. As a simple corollary, they show that the Laplacian is Fredholm of index -1 on spaces of exponentially localized functions; see (1.5) for a precise definition.

Compared to other methods, the approach presented here is quite flexible and direct, generalizing to systems and problems in cylindrical domains $\Omega \times \mathbb{R}^k$. It can for instance be used to track eigenvalues as they merge into the essential spectrum, as we showed in [16, 17]; see Section 5.1 for more details.

1.2 Fredholm properties of radially symmetric elliptic operators

Before addressing our particular problem of radially symmetric elliptic problems, let us recall some results on abstract linear differential equations and Fredholm properties of operators on the real line. We refer to [1, 6, 7, 8, 9, 10, 11, 13, 14, 15, 18, 19, 20, 21] and references therein for details and applications.

Determining Fredholm properties of differential operators,

$$\mathcal{T}u = u' - A(t)u, \tag{1.1}$$

in unbounded domains, $t \in \mathbb{R}$, can sometimes be reduced to the study of relative Morse indices of asymptotic operators. Roughly speaking, denote by $\nu_\pm^j$ possible asymptotic rates of solutions $u \sim \mathrm{e}^{\nu_\pm^j t}$ at $\pm\infty$. The operator $\mathcal{T}$ then is Fredholm whenever $\mathrm{Re}\,\nu_\pm^j \neq 0$. In the presence of essential spectrum, we have $\mathrm{Re}\,\nu = 0$ for at least one of those growth rates. Introducing exponential weights η, $\|u(\cdot)\|_{L^2_\eta} = \|u(\cdot)\mathrm{e}^{\eta\cdot}\|_{L^2}$, shifts the asymptotic decay rates $\nu \mapsto \nu + \eta$, so that $\mathcal{T}$ may be Fredholm for non-zero choices of η. One can then determine Fredholm indices by counting the number $i_\pm$ of asymptotic growth rates $\nu_\pm$ with $\mathrm{Re}\,\nu_\pm > -\eta$: the Fredholm index $\mathrm{ind}(\mathcal{T})$ is given by the simple formula

$$\mathrm{ind}(\mathcal{T}) = i_- - i_+. \tag{1.2}$$

This strategy has been used successfully in a number of contexts, including cases where both i_- and i_+ are infinite as illustrated in the references cited above.

Here, we are concerned with perturbation problems that arise in the study of radially symmetric solutions to systems of second order equations. The linearized operators that we consider are of the form

$$\mathcal{L}_{\mathrm{rad}} = D(r)\left(\frac{d^2}{dr^2} + \frac{k-1}{r}\frac{d}{dr}\right) + Q(r)\frac{d}{dr} + R(r). \tag{1.3}$$

The operator $\mathcal{L}_{\mathrm{rad}}$ can be viewed as the restriction of

$$\mathcal{L} = D(|x|)\Delta + Q(|x|)\Big(\frac{x}{|x|}\cdot\nabla\Big) + R(|x|), \tag{1.4}$$

to the space of radially symmetric functions.

More precisely, we consider $\mathcal{L}_{\mathrm{rad}}$ as a closed operator on $L^2_{\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^m)$, the space of vector-valued functions in $L^2_{\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^m)$ which depend on $|x|$, only, that is, they are invariant under the rotations in $\mathbb{R}^k$. The domain of definition is $H^2_{\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^m) \subset H^2(\mathbb{R}^k, \mathbb{C}^m)$, again rotation-invariant functions.

We will assume throughout that $D, Q, R : [0, \infty) \to \mathcal{M}_m(\mathbb{C})$ are continuous functions with the following properties.

(N) *Nondegeneracy:* The matrix $D(r)$ is invertible and $|\det D(r)| \geq d_0 > 0$ for all $r \in [0, \infty)$[1];

(C) *Convergence:* We have that $D(r) \to D_\infty$, $Q(r) \to Q_\infty$ and $R(r) \to R_\infty$, as $r \to \infty$.

Our first main result will also assume asymptotic invertibility:

(A) *Asymptotic Invertibility:* The asymptotic operator, $D_\infty \partial_{tt} + Q_\infty \partial_t + R_\infty$ is invertible in $L^2(\mathbb{R})$. Equivalently, we require that $\det(D_\infty \nu^2 + Q_\infty \nu + R_\infty) \neq 0$ for all $\nu \in \mathrm{i}\mathbb{R}$, or that the matrix

$$T_\infty = \begin{bmatrix} 0 & I_m \\ -D_\infty^{-1} R_\infty & -D_\infty^{-1} Q_\infty \end{bmatrix},$$

is hyperbolic, that is, it does not possess purely imaginary eigenvalues.

We define the Morse index $i(T_\infty)$ of the hyperbolic matrix T_∞ as the number of eigenvalues of T_∞ with positive real part.

Theorem 1.1. *Assume Nondegeneracy (N) and Convergence (C). Then the operator $\mathcal{L}_{\mathrm{rad}}$ is Fredholm if and only if we have Asymptotic Invertibility (A). In this case, the Fredholm index is given by*

$$\mathrm{ind}(\mathcal{L}_{\mathrm{rad}}) = m - i(T_\infty),$$

where $i(T_\infty)$ is the Morse index of T_∞.

This conclusion here is in fact very similar to the formula for Fredholm indices for problems on the real line, (1.1) and (1.2), if one defines $i_- := m$. In fact, we do study Fredholm properties of $\mathcal{L}_{\mathrm{rad}}$ by writing the operator as a first-order differential operator on the real line. This is accomplished by using various weight functions and transformations of the independent variable, which eliminate the obvious difficulty caused by the $1/r$-singularity in the coefficients of $\mathcal{L}_{\mathrm{rad}}$; see Section 2. We emphasize, however, that the resulting problem is of a slightly different type than (1.1) and requires some additional arguments.

As pointed out, we are interested in Fredholm properties in exponentially weighted spaces. Therefore, consider the space $L^2_{\eta,\mathrm{rad}}$ of measurable functions such that

$$\|u\|^2_{L^2_{\eta,\mathrm{rad}}} := \int_0^\infty |u(r)\mathrm{e}^{\eta r}|^2 r^{k-1} dr, \tag{1.5}$$

is finite. Similarly, we define $H^2_{\eta,\mathrm{rad}}$ with norm

$$\|u\|^2_{H^2_{\eta,\mathrm{rad}}} := \|u\|^2_{L^2_{\eta,\mathrm{rad}}} + \|u_{rr}\|^2_{L^2_{\eta,\mathrm{rad}}}.$$

Asymptotic Hyperbolicity for such spaces can be restated as follows.

[1] With this assumption, one can actually reduce the problem to the case $D \equiv I_m$; since this does not simplify exposition, we prefer to retain the general form of D as it appears in applications [17]

$(\mathrm{A})_\eta$ *Asymptotic Invertibility:* The asymptotic operator, $D_\infty\partial_{tt}+Q_\infty\partial_t+R_\infty$ is invertible in L^2_η. Equivalently, we require that $\det(D_\infty\nu^2+Q_\infty\nu+R_\infty)\neq 0$ for all $\nu\in-\eta+\mathrm{i}\mathbb{R}$, or that the matrix

$$T_{\eta,\infty}=T_\infty+\eta I_{2m}=\begin{bmatrix}\eta I_m & I_m\\ -D_\infty^{-1}R_\infty & -D_\infty^{-1}Q_\infty+\eta I_m\end{bmatrix},$$

is hyperbolic.

Again, we denote the Morse index of the asymptotic problem by $\mathrm{ind}(T_{\eta,\infty})=\mathrm{ind}(T_\eta+\eta I_{2m})$.

Theorem 1.1 translates into a statement on Fredholm properties in exponentially weighted spaces as follows.

Theorem 1.2. *Assume Nondegeneracy (N) and Convergence (C). Then the operator $\mathcal{L}_{\mathrm{rad}}$ is Fredholm on $L^2_{\eta,\mathrm{rad}}$ if and only if we have Asymptotic Invertibility $(\mathrm{A})_\eta$. In this case, the Fredholm index is given by*

$$\mathrm{ind}(\mathcal{L}_{\mathrm{rad}})=m-i(T_{\eta,\infty})=m-i(T_\infty+\eta I_{2m}),$$

where $i(T_{\eta,\infty})$ is the Morse index of $T_{\eta,\infty}$.

We note that it follows from Theorem 1.2 that the operator $\mathcal{L}_{\mathrm{rad}}$ is Fredholm on $L^2_{\eta,\mathrm{rad}}(\mathbb{R}^k,\mathbb{C}^m)$ for all but finitely many values of $\eta>0$.

Another interesting case are isotropic systems of the form $D(r)\Delta u+M(r)u=f$ on $L^2(\mathbb{R}^k,\mathbb{C}^m)$, which can be simplified using the spectral decomposition of the Laplace-Beltrami operator Δ_B on S^{k-1}. In fact, the left-hand side decomposes into a direct sum of operators of the form

$$\mathcal{L}^\ell_{\mathrm{rad}}=D(r)\left(\frac{d^2}{dr^2}+\frac{k-1}{r}\frac{d}{dr}-\frac{\ell^2}{r^2}\right)+R(r), \tag{1.6}$$

where ℓ^2 is an eigenvalue of $-\Delta_B$. The topology of $L^2(\mathbb{R}^k,\mathbb{C}^m)$ and $H^2(\mathbb{R}^k,\mathbb{C}^m)$ naturally induce topologies which make $\mathcal{L}^\ell_{\mathrm{rad}}$ a closed operator with domain $H^{2,\ell}_{\mathrm{rad}}(\mathbb{R}^k,\mathbb{C}^m)$. Similarly, we define $H^{2,\ell}_{\eta,\mathrm{rad}}(\mathbb{R}^k,\mathbb{C}^m)$ in analogy to (1.5).

We then have the following theorem, similar to Theorem 1.1.

Theorem 1.3. *Assume Nondegeneracy (N) and Convergence (C). Then the operator $\mathcal{L}^\ell_{\mathrm{rad}}$ is*

(i) Fredholm on $L^2_{\mathrm{rad}}(\mathbb{R}^k,\mathbb{C}^m)$ if and only if we have Asymptotic Invertibility (A). In this case, the Fredholm index is given by

$$\mathrm{ind}(\mathcal{L}^\ell_{\mathrm{rad}})=0,$$

(ii) Fredholm on $L^2_{\eta,\mathrm{rad}}(\mathbb{R}^k,\mathbb{C}^m)$ if and only if we have Asymptotic Invertibility $(\mathrm{A})_\eta$. In this case, the Fredholm index is given by

$$\mathrm{ind}(\mathcal{L}^\ell_{\mathrm{rad}})=m-i(T_{\eta,\infty})=m-i(T_\infty+\eta I_{2m}),$$

where $i(T_{\eta,\infty})$ is the Morse index of $T_{\eta,\infty}$.

Theorems 1.1–1.3 can be generalized to various infinite-dimensional settings, using relative Morse indices as in [7, 9, 10, 21, 23]. These extensions can then cover systems of elliptic equations with radially symmetric domains $\mathbb{R}^k \times \Omega$, with $\Omega \subset \mathbb{R}^p$, bounded, and suitable boundary conditions on $\mathbb{R}^k \times \partial\Omega$. Another infinite-dimensional generalization concerns time-periodic solutions of parabolic equations in such domains; see for instance [21] for the case $k = 1$ and [24] for some applications in radially symmetric settings.

1.3 A perturbation result in the presence of essential spectrum

As pointed out in Section 1.1, we illustrate these results by applying them to a semilinear elliptic perturbation problem,

$$\Delta u - u^3 + \varepsilon V(|x|, u) = 0, \quad x \in \mathbb{R}^2, \tag{1.7}$$

for $\varepsilon \approx 0$. We think of this equation as a simple model for a chemical reaction of the form $A+2B \to C$, with reaction rate $k \cdot ab^2$ and non-dimensionalized concentrations $a = [A]$, $b = [B]$. Setting up this reaction in a large almost planar container and feeding A and B close to the center of the container, leads to a model of the form

$$\begin{cases} a_t &= d_a \Delta a - ab^2 + \varepsilon V_a(|x|, a, b) \\ b_t &= d_b \Delta b - 2ab^2 + \varepsilon V_b(|x|, a, b), \end{cases} \quad t \geq 0, \; x \in \mathbb{R}^k.$$

Assuming balance of concentrations in the feed mechanism,

$$2V_a(r, \kappa a, b) = V_b(r, \kappa b, a), \qquad \kappa = d_b/(2d_a),$$

one can find time-independent solutions in the system with $\kappa a = b$ from

$$d_a \Delta b - b^3 + \varepsilon \frac{d_a}{d_b} V_b(|x|, b/\kappa, b) = 0.$$

Scaling x now gives a system of the form (1.7).

We assume that $V : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a C^1-function that is exponentially localized. More precisely, assume that there exists $\delta_0 > 0$ such that

(V) *Exponential Decay:* $|V(r, u)| + |V_u(r, u)| \leq c\mathrm{e}^{-\delta_0 r}$, for all $r \in \mathbb{R}_+$, $u \in \mathbb{R}$.

(P) *Positivity:* $\int_0^\infty V(r, 0) r dr > 0$.

One would like to find solutions to this equation for ε small using the implicit function theorem near $u = 0, \varepsilon = 0$ in order to solve for u as a function of ε. The linearization with respect to u at $\varepsilon = 0$ is given by the Laplacian on $\mathbb{R}^2$, which is not Fredholm on L^p. The Laplacian is, however, Fredholm in spaces of exponentially localized functions, by Theorem 1.2, as we shall see later. We therefore use such exponentially weighted spaces together with a far-field matching ansatz in order to obtain a perturbation result based on an implicit function theorem.

Theorem 1.4. *Consider (1.7) with Exponential Decay (V) and Positivity (P). Then there exists $\delta > 0$ and $\eta \in (0, \delta_0/2)$ such that for any $\varepsilon \in [0, \delta]$ equation (1.7) has a smooth, radially symmetric solution with asymptotics*

$$u(r; \varepsilon) = \varepsilon v_*(r\varepsilon) + \mathcal{O}(\varepsilon^2) + \mathcal{O}(e^{-\eta r}), \quad \text{as} \quad r \to \infty.$$

The function $v_* : (0, \infty) \to \mathbb{R}$ satisfies the conditions

$$\Delta_r v_* = v_*^3, \quad \lim_{r \to 0} \frac{v_*(r)}{\ln r} \in (-\infty, 0) \quad \text{and} \quad \lim_{r \to \infty} v_*(r) = 0.$$

We note that the case $\varepsilon < 0$ can be reduced to the case $\varepsilon > 0$ by the simple change of variable $u \mapsto -u$ in equation (1.7).

Theorem 1.4 can be extended in many ways. The exponential decay assumption can be substantially weakened. One can also change the power of the nonlinear term and the dimension of the space slightly, with only minor changes to the proof. Some aspects of our analysis do however change for both small and large powers and/or space dimensions.

On the other hand, the heart of the proof, is well suited to analyze more complicated problems, such as systems of elliptic equations. A straightforward generalization would consider nonlinearly coupled systems of the form

$$\begin{cases} \Delta u - u^3 + V_0(|x|)u + v g_1(u, v) + \varepsilon V_1(|x|, u, v) = 0 \\ D_v \Delta v - g_2(u, v) v = 0, \end{cases}$$

where $D_v \Delta + g_2(0, 0)$ is invertible on $L^2(\mathbb{R}^k)$, and $V_0(|x|)$ is exponentially localized. Theorem 1.4 then applies to this system, as well.

Outline: In Section 2, we show that Fredholm properties of $\mathcal{L}_{\text{rad}}$ are equivalent to Fredholm properties of suitably defined first-order differential operators on $L^2(0, \infty)$ and $L^2(\mathbb{R})$, equipped with appropriate weight functions. In Section 3, we study the Fredholm properties of the associated first-order differential operators on the real line and calculate their Fredholm index. Section 4 combines these results into the proofs of Theorems 1.1–1.3. Section 5 contains applications of our main theorems. We first briefly summarize the application towards instability of spikes in reaction-diffusion equations coupled to conservation laws and then prove Theorem 1.4.

Notations: We collect some notation that we will use throughout this paper. We write $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_- = (-\infty, 0]$, $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and $\mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Re}, z < 0\}$. For an operator T on a Banach or Hilbert space X we use T^*, $\operatorname{dom}(T)$, $\ker T$, $\operatorname{im} T$, $\sigma(T)$, $\rho(T)$ and $T_{|Y}$ to denote the adjoint, domain, kernel, range, spectrum, resolvent set and the restriction of T on a subspace Y of X. $\mathcal{B}(X, Y)$ is the space of all bounded linear operators from X to Y and $\mathcal{K}(X, Y)$ is the space of all compact linear operators from X to Y. We denote the space of all $m \times m$ matrices with complex entries by $\mathcal{M}_m(\mathbb{C})$. We recall that a matrix is called hyperbolic if it has no eigenvalues on the imaginary axis. For a matrix B we denote by $i(B)$ the Morse index of the matrix B, the dimension of the generalized eigenspace of all eigenvalues μ with $\operatorname{Re} \mu > 0$. Similarly, we denote by $j(B)$ the dimension of the generalized eigenspace of all eigenvalues μ with $\operatorname{Re}\mu \geq 0$. We denote by L^p the usual Lebesgue spaces, by H^q the usual Sobolev spaces and by AC the space of absolutely continuous functions. In addition to this notations, we add the subscript *rad* to denote the restriction to the set of radially-symmetric functions. For any $p \in [1, \infty]$ and any measurable function $\omega : E \to \mathbb{R}_+$, $\omega > 0$ almost everywhere, we define the space $L^p(E, \mathbb{C}^m; \omega(x)dx) = \{u : \omega(\cdot)u(\cdot) \in L^p(E, \mathbb{C}^m)\}$ with the weighted norm $\|u\|_{L^p(E,\omega(x)dx)} = \|\omega u\|_p$. For any $F \in L^\infty(E, \mathcal{M}_m(\mathbb{C}))$ we denote by M_F the operator of multiplication on $L^2(E, \mathbb{C}^m)$ with the matrix-valued function F. We denote by c a generic positive constant.

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2 Second order radially-symmetric differential operators

In this section we study the Fredholm properties of the second order radially-symmetric differential operators $\mathcal{L}_{\mathrm{rad}}$, defined in (1.3). Our approach to the problem at hand is as follows. First, we reduce the order of the differential operator in the problem, that is, we construct a first order operator $\mathcal{T}_{\mathrm{rad}}$, which is Fredholm if and only if $\mathcal{L}_{\mathrm{rad}}$ is Fredholm. In the second step, we change the independent variable $r > 0$ to $\tau = \log r \in \mathbb{R}$ and construct a weighted first order differential operator on the real line that is Fredholm if and only if $\mathcal{T}_{\mathrm{rad}}$ is Fredholm with equal indices. Throughout this section we assume Nondegeneracy (N) and Convergence (C) for the coefficients as defined in the introduction.

First, recall that $L^2_{\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^m)$ is isometrically isomorphic to a weighted L^2-space of functions defined on $(0, \infty)$. The theorem therefore is equivalent to a statement on differential operators on weighted L^2-spaces of a single variable $r = |x|$. The following simple lemma makes this notion precise.

Lemma 2.1. *The operator $\mathcal{L}_{\mathrm{rad}}$ is equivalent to a one-dimensional differential operator in the following sense.*

(i) The isometry $U_{\mathrm{rad}} : L^2((0, \infty), \mathbb{C}^m) \to L^2_{\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^m)$ defined by $(U_{\mathrm{rad}}u)(r) = r^{\frac{1-k}{2}}u(r)$ is surjective.

(ii) If we define $\tilde{\mathcal{L}} = U^{-1}_{\mathrm{rad}}\mathcal{L}_{\mathrm{rad}}U_{\mathrm{rad}} : \mathrm{dom}(\tilde{\mathcal{L}}) \to L^2((0, \infty), \mathbb{C}^m)$ then $\mathrm{dom}(\tilde{\mathcal{L}})$ consists of all functions $v \in L^2((0, \infty), \mathbb{C}^m)$ such that $v, v' \in AC_{\mathrm{loc}}((0, \infty), \mathbb{C}^m)$ and the vector-valued functions $r \mapsto v''(r) - \frac{(k-1)(k-3)}{4r^2}v(r)$, $r \mapsto v'(r) - \frac{k-1}{2r}v(r)$ belong to $L^2((0, \infty), \mathbb{C}^m)$. Moreover,

$$\tilde{\mathcal{L}} = D(r)\Big(\frac{d^2}{dr^2} - \frac{(k-1)(k-3)}{4r^2}\Big) + Q(r)\Big(\frac{d}{dr} - \frac{k-1}{2r}\Big) + R(r). \tag{2.1}$$

(iii) The operator $\mathcal{L}_{\mathrm{rad}}$ is Fredholm if and only if the operator $\tilde{\mathcal{L}}$ is Fredholm and their indices coincide.

Proof. The assertion (i) follows directly from the definition of radially-symmetric functions in $L^2(\mathbb{R}, \mathbb{C}^m)$. The proof of (ii) is a simple computation and (iii) follows immediately from (i). ∎

Next, we define the linear operators $S_j : \mathrm{dom}(S_j) \to L^2((0,\infty), \mathbb{C}^m)$, $j = 1, 2$ by

$$\begin{aligned}\mathrm{dom}(S_j) = \{u \in & L^2((0,\infty), \mathbb{C}^m) : u \in AC_{\mathrm{loc}}, r \mapsto u'(r) + (-1)^j \frac{k-1}{2r} u(r) \in L^2((0,\infty), \mathbb{C}^m)\} \\ & (S_j u)(r) = u'(r) + (-1)^j \frac{k-1}{2r} u(r). \end{aligned} \tag{2.2}$$

Remark 2.2. A direct computation shows that the operators S_1 and S_2 are closed, densely-defined linear operators and

(i) $(0, \infty) \subset \rho(S_j)$, $j = 1, 2$ and

$$\begin{aligned} [(S_1 - a)^{-1} g](r) &= r^{\frac{k-1}{2}} e^{ar} \int_r^\infty s^{-\frac{k-1}{2}} e^{-as} g(s) ds, \quad a > 0, \quad g \in L^2((0,\infty), \mathbb{C}^m). \\ [(S_2 - a)^{-1} g](r) &= r^{-\frac{k-1}{2}} e^{-ar} \int_0^r s^{\frac{k-1}{2}} e^{as} g(s) ds, \quad a > 0, \quad g \in L^2((0,\infty), \mathbb{C}^m). \end{aligned} \tag{2.3}$$

(ii) $S_2 S_1 = \frac{d^2}{dr^2} - \frac{(k-1)(k-3)}{4r^2}$.

(iii) $\mathrm{dom}\, \tilde{\mathcal{L}} = \mathrm{dom}(S_2 S_1)$.

(iv) $\tilde{\mathcal{L}} = M_D S_2 S_1 + M_Q S_1 + M_R$.

Lemma 2.3. *Define the linear operator* $\mathcal{T}_{\mathrm{rad}} : \mathrm{dom}(S_1) \times \mathrm{dom}(S_2) \to L^2((0,\infty), \mathbb{C}^{2m})$ *by*

$$\mathcal{T}_{\mathrm{rad}} = \begin{bmatrix} S_1 & -\mathrm{Id} \\ M_{D^{-1}R} & S_2 + M_{D^{-1}Q} \end{bmatrix}. \tag{2.4}$$

The operator $\tilde{\mathcal{L}}$ *is Fredholm if and only if* $\mathcal{T}_{\mathrm{rad}}$ *is Fredholm and their indices coincide.*

Proof. The proof of this lemma is similar to the the proof of [23, Thm. A.1]. There are however a few key differences and we give a complete proof here.

From the definition of the operator $\mathcal{T}_{\mathrm{rad}}$ and Remark 2.2(iv) one can easily see that

$$(u, v)^{\mathrm{T}} \in \ker \mathcal{T}_{\mathrm{rad}} \quad \text{if and only if} \quad u \in \ker \tilde{\mathcal{L}} \quad \text{and} \quad v = S_1 u.$$

It follows that the map $u \mapsto (u, S_1 u)^{\mathrm{T}}$ from $\ker \tilde{\mathcal{L}}$ to $\ker \mathcal{T}_{\mathrm{rad}}$ is surjective. Since it is clearly also injective, we have

$$\ker \tilde{\mathcal{L}} \cong \ker \mathcal{T}_{\mathrm{rad}}. \tag{2.5}$$

Define the operators $\mathcal{T}_0 : \mathrm{dom}(S_1) \times \mathrm{dom}(S_2) \to L^2(\mathbb{R}, \mathbb{C}^{2m})$, $\mathcal{B} : L^2(\mathbb{R}, \mathbb{C}^{2m}) \to L^2(\mathbb{R}, \mathbb{C}^{2m})$ by

$$\mathcal{T}_0 = \begin{bmatrix} S_1 & -\mathrm{Id} \\ -\mathrm{Id} & S_2 \end{bmatrix} \quad \mathcal{B} = \begin{bmatrix} 0 & 0 \\ B_1 + \mathrm{Id} & B_2 \end{bmatrix}, \tag{2.6}$$

where $B_1 = M_{D^{-1}R}$ and $B_2 = M_{D^{-1}Q}$ are the multiplication operators by the matrix-valued functions $D^{-1}(\cdot)R(\cdot)$ and $D^{-1}(\cdot)Q(\cdot)$, respectively. Thus, $\mathcal{T}_{\mathrm{rad}} = \mathcal{T}_0 + \mathcal{B}$. It follows from Remark 2.2(ii) that the operators $S_1S_2 - \mathrm{Id}$ and $S_2S_1 - \mathrm{Id}$ are invertible which implies that the operator $\mathcal{T}_0$ is invertible and

$$\mathcal{T}_0^{-1} = \begin{bmatrix} S_2(S_1S_2 - \mathrm{Id})^{-1} & (S_2S_1 - \mathrm{Id})^{-1} \\ (S_1S_2 - \mathrm{Id})^{-1} & S_1(S_2S_1 - \mathrm{Id})^{-1} \end{bmatrix}. \tag{2.7}$$

Next, we will prove that

$$\operatorname{im}\mathcal{T}_{\mathrm{rad}} \quad \text{is closed if and only if} \quad \operatorname{im}\tilde{\mathcal{L}} \quad \text{is closed} \tag{2.8}$$

Assume first that $\operatorname{im}\mathcal{T}_{\mathrm{rad}}$ is closed. To prove that $\operatorname{im}\tilde{\mathcal{L}}$ is closed assume that $f \in L^2(\mathbb{R}, \mathbb{C}^m)$ and that there exists a sequence $(u_n)_{n\geq 1}$ of elements of $\mathrm{dom}(\tilde{\mathcal{L}}) = \mathrm{dom}(S_2S_1)$, according to Remark 2.2(iii), such that $f_n := \tilde{\mathcal{L}}u_n \to f \in L^2(\mathbb{R}, \mathbb{C}^m)$. Then, $(u_n, S_1u_n)^{\mathrm{T}} \in \mathrm{dom}(S_1) \times \mathrm{dom}(S_2) = \mathrm{dom}(\mathcal{T}_{\mathrm{rad}})$ for all $n \geq 1$. Since the operator $M_{D^{-1}}$ is bounded on $L^2(\mathbb{R}, \mathbb{C}^m)$ by Nondegeneracy (N), we infer that $\mathcal{T}_{\mathrm{rad}}(u_n, S_1u_n)^{\mathrm{T}} = (0, M_{D^{-1}}f_n) \to (0, M_{D^{-1}}f)^{\mathrm{T}}$ as $n \to \infty$. Since $\operatorname{im}\mathcal{T}_{\mathrm{rad}}$ is closed, we obtain that $(0, M_{D^{-1}}f)^{\mathrm{T}} \in \operatorname{im}\mathcal{T}_{\mathrm{rad}}$. Using again the definition of $\mathcal{T}_{\mathrm{rad}}$, we conclude that $f \in \operatorname{im}\tilde{\mathcal{L}}$, proving that $\operatorname{im}\tilde{\mathcal{L}}$ is closed.

Assume next that $\operatorname{im}\tilde{\mathcal{L}}$ is closed and let $\{(u_n, v_n)^{\mathrm{T}}\}_{n\geq 1}$ be a sequence of elements of $\mathrm{dom}(\mathcal{T}_{\mathrm{rad}}) = \mathrm{dom}(S_1) \times \mathrm{dom}(S_2)$ such that $(f_n, g_n)^{\mathrm{T}} := \mathcal{T}_{\mathrm{rad}}(u_n, v_n)^{\mathrm{T}} \to (f, g)^{\mathrm{T}} \in L^2(\mathbb{R}, \mathbb{C}^{2m})$. Since the operator $\mathcal{T}_0$ is invertible, we have that $\mathcal{T}_0^{-1}(f_n, g_n)^{\mathrm{T}} \in \mathrm{dom}(\mathcal{T}_0) = \mathrm{dom}(S_1) \times \mathrm{dom}(S_2)$ for all $n \geq 1$ and $\mathcal{T}_0^{-1}(f_n, g_n)^{\mathrm{T}} \to \mathcal{T}_0^{-1}(f, g)^{\mathrm{T}}$ as $n \to \infty$. Thus,

$$(\tilde{u}_n, \tilde{v}_n)^{\mathrm{T}} := (u_n, v_n)^{\mathrm{T}} - \mathcal{T}_0^{-1}(f_n, g_n)^{\mathrm{T}} \in \mathrm{dom}(S_1) \times \mathrm{dom}(S_2) \quad \text{for} \quad n \geq 1. \tag{2.9}$$

In addition, from the definition of the sequence $\{(f_n, g_n)^{\mathrm{T}}\}_{n\geq 1}$, we have that

$$\begin{aligned} \mathcal{T}_{\mathrm{rad}}(\tilde{u}_n, \tilde{v}_n)^{\mathrm{T}} &= \mathcal{T}_{\mathrm{rad}}(u_n, v_n)^{\mathrm{T}} - \mathcal{T}_0\mathcal{T}_0^{-1}(f_n, g_n)^{\mathrm{T}} - \mathcal{B}(f_n, g_n)^{\mathrm{T}} \\ &= -\mathcal{B}(f_n, g_n)^{\mathrm{T}} = (0, B_1f_n + f_n + B_2g_n)^{\mathrm{T}} \quad \text{for all} \quad n \geq 1. \end{aligned} \tag{2.10}$$

It follows that

$$S_1\tilde{u}_n = \tilde{v}_n \quad \text{and} \quad S_2\tilde{v}_n + B_2\tilde{v}_n + B_1\tilde{u}_n = B_1f_n + f_n + B_2g_n \quad \text{for all} \quad n \geq 1, \tag{2.11}$$

which implies that $\tilde{u}_n \in \mathrm{dom}(S_2S_1) = \mathrm{dom}(\tilde{\mathcal{L}})$. Using (2.11) we calculate

$$\begin{aligned} \tilde{\mathcal{L}}\tilde{u}_n &= M_D(S_2S_1\tilde{u}_n + B_2S_1\tilde{u}_n + B_1\tilde{u}_n) = M_D(S_2\tilde{v}_n + B_2\tilde{v}_n + B_1\tilde{u}_n) \\ &= M_D(B_1f_n + f_n + B_2g_n) \quad \text{for all} \quad n \geq 1. \end{aligned} \tag{2.12}$$

Since M_D, B_1, and B_2 are bounded operators on $L^2(\mathbb{R}, \mathbb{C}^m)$, we have that $\tilde{\mathcal{L}}\tilde{u}_n \to M_D(B_1f + f + B_2g)$ as $n \to \infty$. Since $\operatorname{im}\tilde{\mathcal{L}}$ is closed, we infer that $= M_D(B_1f + f + B_2g) \in \operatorname{im}\tilde{\mathcal{L}}$. Using again the definitions of $\tilde{\mathcal{L}}$ and $\mathcal{T}_{\mathrm{rad}}$ we conclude that $(f, g)^{\mathrm{T}} \in \operatorname{im}\mathcal{T}_{\mathrm{rad}}$, proving that $\operatorname{im}\mathcal{T}_{\mathrm{rad}}$ is closed.

To finish the proof of the lemma, we need to show that $\ker \mathcal{T}_{\text{rad}}^*$ and $\ker \tilde{\mathcal{L}}^*$ are isomorphic. Similarly to the proof of (2.5), one can show that the map $w \mapsto ((S_2^* M_D^* + M_Q^*)w, M_D^* w)^{\mathrm{T}}$ from $\ker \tilde{\mathcal{L}}^*$ to $\ker \mathcal{T}_{\text{rad}}^*$ is bijective, and thus,

$$\ker \tilde{\mathcal{L}}^* \cong \ker \mathcal{T}_{\text{rad}}^*. \tag{2.13}$$

∎

In the next lemma, we construct a weighted first order differential operator $\tilde{\mathcal{T}}$ on $L^2(\mathbb{R}, \mathbb{C}^{2m})$ that is Fredholm if and only if $\mathcal{T}_{\text{rad}}$ is Fredholm with equal indices. First we construct an *increasing* C^∞-function Ψ such that

$$\Psi(\tau) = \begin{cases} e^\tau, & \tau \leq -1 \\ \tau, & \tau \geq 1 \end{cases}, \qquad \Phi = \Psi^{-1} \tag{2.14}$$

Lemma 2.4. *The following assertions hold:*

(i) The operator $U_\Psi : L^2(\mathbb{R}, \mathbb{C}^m) \to L^2((0,\infty), \mathbb{C}^m)$ defined by $(U_\Psi f)(r) = (\Phi'(r))^{\frac{1}{2}} f(\Phi(r))$ is an isometric isomorphism and its inverse is defined by $(U_\Psi^{-1} g)(\tau) = (\Phi'(\Psi(\tau)))^{-\frac{1}{2}} g(\Psi(\tau))$.

(ii) The operator $\mathcal{T}_{\text{rad}}$ is Fredholm if and only if the operator $\tilde{\mathcal{T}} := U_\Psi^{-1} \mathcal{T}_{\text{rad}} U_\Psi$ is Fredholm and their indices coincide.

(iii) The linear operator $\tilde{\mathcal{T}}$ is asymptotically equal to a weighted first order differential operator, of the general form described in (3.2)*, below, with $\alpha_- = 1$ and $\alpha_+ = 0$.*

Proof. We first prove (i). Using the change of variables $r = \Phi(\tau)$, $\tau \in \mathbb{R}$ one can readily check that U_Ψ is an isometry. Similarly, using the change of variables $\tau = \Psi(r)$, $r > 0$, one can see immediately that $(\Phi' \circ \Psi))^{-\frac{1}{2}} (g \circ \Psi) \in L^2(\mathbb{R}, \mathbb{C}^m)$ for any $g \in L^2((0,\infty), \mathbb{C}^m)$, proving the surjectivity of U_ψ and thus (i).

Assertion (ii) follows immediately from (i).

Assertion (iii) follows from the definition of Ψ in (2.14) after a long but straightforward computation. ∎

3 Weighted First order Differential Operators on $\mathbb{R}$

In this section we give necessary and sufficient conditions for Fredholm properties of weighted first order differential operators on $\mathbb{R}$. Given $\alpha_\pm \geq 0$, $A_\pm \in \mathcal{M}_m(\mathbb{C})$, we define $\alpha = (\alpha_-, \alpha_+)$ and the functions $\varphi_\alpha : \mathbb{R} \to [1, \infty)$ and $A : \mathbb{R} \to \mathcal{M}_m(\mathbb{C})$ by

$$\varphi_\alpha(\tau) = \begin{cases} e^{-\alpha_- \tau}, & \tau < 0 \\ e^{\alpha_+ \tau}, & \tau \geq 0 \end{cases}, \qquad A(\tau) = \begin{cases} A_-, & \tau < 0 \\ A_+, & \tau \geq 0 \end{cases}. \tag{3.1}$$

Next we define the operator $\mathcal{T}_\alpha^A : \mathrm{dom}(\mathcal{T}_\alpha^A) \subseteq L^2(\mathbb{R}, \mathbb{C}^m) \to L^2(\mathbb{R}, \mathbb{C}^m)$ as follows,

$$\mathrm{dom}(\mathcal{T}_\alpha^A) = \{u \in H^1(\mathbb{R}, \mathbb{C}^m) : \varphi_\alpha(u' - M_A u) \in L^2(\mathbb{R}, \mathbb{C}^m)\}, \quad \mathcal{T}_\alpha^A u = \varphi_\alpha(u' - M_A u). \tag{3.2}$$

We recall that, in general, if $B \in L^\infty(\mathbb{R}, \mathcal{M}_m(\mathbb{C}))$, M_B denotes the operator of multiplication by the matrix-valued function B.

We note that the linear operator $\mathcal{T}_\alpha^A$ is closed for every choice of $\alpha \in \mathbb{R}_+^2$ and $A_\pm \in \mathcal{M}_m(\mathbb{C})$.

The following compactness lemma is needed in the sequel.

Lemma 3.1. *If K is matrix-valued L^∞ function with bounded support, then M_K, the operator of multiplication by K, is a compact operator from* $\mathrm{dom}(\mathcal{T}_\alpha^A)$ *to* $L^2(\mathbb{R}, \mathbb{C}^m)$. *Here we consider* $\mathrm{dom}(\mathcal{T}_\alpha^A)$ *as a Hilbert space with the usual graph norm.*

Proof. Since the support of K is bounded, we infer that $M_K \in \mathcal{K}(H^1(\mathbb{R}, \mathbb{C}^m), L^2(\mathbb{R}, \mathbb{C}^m))$. In fact, using [25, Thm. 4.1], one can show that the operator M_K is Hilbert-Schmidt. To finish the proof of the lemma it is enough to show that the canonical inclusion from $\mathrm{dom}(\mathcal{T}_\alpha^A)$ into $H^1(\mathbb{R}, \mathbb{C}^m)$ is bounded, with the corresponding norms. Assume $u \in \mathrm{dom}(\mathcal{T}_\alpha^A)$ and let $f = \mathcal{T}_\alpha^A u = \varphi_\alpha(u - M_A u)$. Then $u' = M_A u + \frac{1}{\varphi_\alpha} f$, which implies that

$$\begin{aligned}\|u'\|_2 &\le \|M_A u\|_2 + \|\frac{1}{\varphi_\alpha} f\|_2 \le c\|u\| + \|\frac{1}{\varphi_\alpha}\|_\infty \|f\|_2 \le c\|u\| + \|f\|_2 \\ &\le c\|u\|_2 + \|u\|_{\mathrm{dom}(\mathcal{T}_\alpha^A)} \le c\|u\|_{\mathrm{dom}(\mathcal{T}_\alpha^A)}.\end{aligned}$$

∎

In the next example we show that the L^∞ condition in the previous lemma is necessary. Moreover, there is an example when the operator M_B is not even relatively bounded to $\mathcal{T}_\alpha^A$ in the absence of the L^∞-condition on B.

Example 3.2. Set $m = 1$, $B(\tau) = e^{-\tau}$, $\alpha_- = 2$, $\alpha_+ = 0$, $A_- = \frac{1}{2}$, $A_+ = 0$. In this setup, the operator of multiplication by B is not relatively bounded to $\mathcal{T}_\alpha^A$.

Indeed, define the function $u_0 : \mathbb{R} \to \mathbb{C}$ by $u_0(\tau) = e^{\frac{\tau}{2}}$ for $\tau < 0$ and $u_0(\tau) = e^{-\tau}$ for $\tau \ge 0$. One can check that $u_0 \in H^1(\mathbb{R}, \mathbb{C})$ and

$$u_0'(\tau) = \begin{cases} \frac{1}{2}e^{\frac{\tau}{2}}, & \tau < 0 \\ -e^{-\tau}, & \tau \ge 0 \end{cases} \qquad [\varphi_\alpha(u_0' - M_A u_0)](\tau) = \begin{cases} 0, & \tau < 0 \\ -e^{-\tau}, & \tau \ge 0 \end{cases}. \tag{3.3}$$

This shows that $\varphi_\alpha(u_0' - M_A u_0) \in L^2(\mathbb{R}, \mathbb{C})$, which implies that $u_0 \in \mathrm{dom}(\mathcal{T}_\alpha^A)$. However,

$$\int_{-\infty}^0 |B(\tau) u_0(\tau)|^2 d\tau = \int_{-\infty}^0 e^{-\tau} d\tau = \infty,$$

proving that $B(\cdot)u_0(\cdot) \notin L^2(\mathbb{R}, \mathbb{C})$.

In the next lemma we establish a connection between weighted exponential spaces and the domain of $\mathcal{T}_\alpha^A$.

Lemma 3.3. *If $\alpha_+ > 0$ then there exists $\eta_+ \in (0, \alpha_+)$ such that*

$$\int_0^\infty |e^{\eta_+ \tau} u(\tau)|^2 d\tau \le c\|u\|^2_{\mathrm{dom}(\mathcal{T}_\alpha^A)} \quad \textit{for all} \quad u \in \mathrm{dom}(\mathcal{T}_\alpha^A). \tag{3.4}$$

Proof. Since A_+ is a matrix, one can choose $\eta_+ \in (0,\alpha)$, small enough, such that $A_+ + \eta_+$ is hyperbolic and

$$\sigma(A_+ + \eta_+) \cap \mathbb{C}_+ = \{\mu + \eta_+ : \mu \in \sigma(A_+), \operatorname{Re}\mu \geq 0\}. \tag{3.5}$$

Next, we define the stable and the unstable subspaces $W_+^{\mathrm{s/u}}$of the hyperbolic matrix $A_+ + \eta_+$: let W_+^{s} and W_+^{u} be the subspaces of all $h \in \mathbb{C}^m$ such that $e^{(A_+ + \eta_+)\tau}h \to 0$ as $\tau \to \infty$ and $\tau \to -\infty$, respectively. Since $A_+ + \eta_+$ is hyperbolic, we have that

$$\mathbb{C}^m = W_+^{\mathrm{s}} \oplus W_+^{\mathrm{u}}. \tag{3.6}$$

We denote by P^{s} and P^{u} the projections onto W_+^{s} and W_+^{u} respectively associated to the decomposition (3.6). Define the operator $\mathcal{D}_+ : \operatorname{dom}(\mathcal{D}_+) \to L^2(\mathbb{R}, \mathbb{C}^m)$ by

$$\operatorname{dom}(\mathcal{D}_+) = \{u \in H^1(\mathbb{R}, \mathbb{C}^m) : u(0) \in W_+^{\mathrm{u}}\}, \quad \mathcal{D}_+ u = u' - (A_+ + \eta_+)u. \tag{3.7}$$

Let $u \in \operatorname{dom}(\mathcal{T}_\alpha^A)$ and define $F : \mathbb{R}_+ \to \mathbb{C}$ by $F(\tau) = e^{-A_+\tau}P^{\mathrm{u}}u(\tau)$. We will show in the sequel that $F \in H^1(\mathbb{R}_+, \mathbb{C}^m)$. First we need to show that $\lim_{\tau\to\infty} F(\tau) = 0$. The latter is not trivial since the matrix A_+ might have eigenvalues on the imaginary axis and hence, $e^{-A_+\tau}$ might grow at $+\infty$. Since $u \in H^1(\mathbb{R}, \mathbb{C}^m)$, one immediately concludes that $F \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^m)$ and, since A_+ commutes with P^{u},

$$\begin{aligned} F'(\tau) &= -A_+ e^{-A_+\tau}P^{\mathrm{u}}u(\tau) + e^{-A_+\tau}P^{\mathrm{u}}u'(\tau) = e^{-A_+\tau}P^{\mathrm{u}}(u'(\tau) - A_+u(\tau)) \\ &= e^{\alpha_+\tau}e^{-A_+\tau}P^{\mathrm{u}}f(\tau) = e^{-\alpha_+\tau}e^{-(A_+^{\mathrm{u}} - \eta_+)\tau}P^{\mathrm{u}}f(\tau) = e^{-(\alpha_+ - \eta_+)\tau}e^{-A_+^{\mathrm{u}}\tau}P^{\mathrm{u}}f(\tau). \end{aligned}$$

Next, we estimate

$$|F'(\tau)| \leq e^{-(\alpha_+ - \eta_+)\tau}\|e^{-A_+^{\mathrm{u}}\tau}\|\,|f(\tau)| \leq e^{-\nu\tau}|f(\tau)| \quad \text{for all} \quad \tau \geq 0, \tag{3.8}$$

for some $\nu > 0$. In this last estimate we used the fact that $\|e^{-A_+^{\mathrm{u}}\tau}\|$ decays exponentially. Since $f \in L^2(\mathbb{R}, \mathbb{C}^m)$, estimate (3.8) implies that $F' \in L^1(\mathbb{R}, \mathbb{C}^m)$. Hence, $F_\infty = \lim_{\tau\to\infty} F(\tau)$ exists in $\mathbb{C}^m$.

In what follows we will show that $F_\infty = 0$. First we note that we obtain from estimate (3.8) that

$$|F_\infty - F(\tau)| \leq \int_\tau^\infty |F'(s)|ds \leq \int_\tau^\infty e^{-\nu y}|f(y)|dy \leq \Big(\int_\tau^\infty e^{-2\nu y}dy\Big)^{1/2}\|f\|_2 \leq ce^{-\nu\tau}, \tag{3.9}$$

for all $\tau \geq 0$. Next, we decompose $W_+^{\mathrm{u}} = W_+^{\mathrm{uu}} \oplus W_+^{\mathrm{uc}}$, where W_+^{uu} and W_+^{uc} are the spectral subspaces associated to the spectral sets $\sigma(A_+) \cap \mathbb{C}_+$ and $\sigma(A_+) \cap \mathrm{i}\mathbb{R}$, respectively. We denote by P^{uu} and P^{uc} the projections onto W_+^{uu} and W_+^{uc}, respectively, associated to this spectral splitting. One can immediately see that $P^{\mathrm{u}} = P^{\mathrm{uu}} + P^{\mathrm{uc}}$.

Since $P^{\mathrm{uu}}P^{\mathrm{u}} = P^{\mathrm{uu}}$, it follows that $P^{\mathrm{uu}}F(\tau) = e^{-A_+P^{\mathrm{uu}}\tau}P^{\mathrm{uu}}f(\tau)$ for all $\tau \geq 0$, which implies that

$$|P^{\mathrm{uu}}F(\tau)| \leq \|e^{-A_+P^{\mathrm{uu}}\tau}\||u(\tau)| \leq e^{-\nu\tau}\,|u(\tau)| \quad \text{for all} \quad \tau \geq 0.$$

Since $u \in H^1(\mathbb{R}, \mathbb{C}^m)$, we obtain that $P^{\mathrm{uu}}F_\infty = 0$ by passing to the limit as $\tau \to \infty$.

Since $P^{\rm uc}P^{\rm u} = P^{\rm uc}$, we obtain that $P^{\rm uc}F(\tau) = e^{-A_+P^{\rm uc}\tau}P^{\rm uc}u(\tau)$ for all $\tau \geq 0$ which implies that

$$\begin{aligned} e^{A_+P^{\rm uc}\tau}P^{\rm uc}F_\infty &= e^{A_+P^{\rm uc}\tau}P^{\rm uc}F(\tau) + e^{A_+P^{\rm uc}\tau}P^{\rm uc}(F_\infty - F(\tau)) \\ &= P^{\rm uc}u(\tau) + e^{A_+P^{\rm uc}\tau}P^{\rm uc}(F_\infty - F(\tau)) \end{aligned}$$

for all $\tau \geq 0$. Since $\sigma(A_+P^{\rm uc}) \subseteq {\rm i}\mathbb{R}$, we infer that $\|e^{A_+P^{\rm uc}\tau}\| \leq c(1+\tau)^j$ for all $\tau \geq 0$ and for some $c > 0$ and j a positive integer. Using the estimate (3.9) we obtain that

$$|e^{A_+P^{\rm uc}\tau}P^{\rm uc}F_\infty| \leq c|u(\tau)| + c(1+\tau)^je^{-\nu\tau} \quad \text{for all} \quad \tau \geq 0,$$

which implies that $e^{A_+P^{\rm uc}\cdot}P^{\rm uc}F_\infty \in L^2(\mathbb{R}, \mathbb{C}^m)$. Using again the fact that $\sigma(A_+P^{\rm uc}) \subseteq {\rm i}\mathbb{R}$, we conclude that $P^{\rm uc}F_\infty = 0$. Moreover, from the definition of F, we have that $F_\infty \in W_+^{\rm u}$ which implies that

$$F_\infty = P^{\rm u}F_\infty = P^{\rm uu}F_\infty + P^{\rm uc}F_\infty = 0. \tag{3.10}$$

Since $F' \in L^1(\mathbb{R}, \mathbb{C}^m)$, we obtain that

$$\int_0^\infty F'(\tau)d\tau = -F(0) = -P^{\rm u}u(0). \tag{3.11}$$

It is well-known that the operator $\mathcal{D}_+$ is invertible (see for example [4] or [5]) and

$$(\mathcal{D}_+^{-1}f)(\tau) = \int_0^\tau e^{A_+^{\rm s}(\tau-y)}P^{\rm s}f(y)dy - \int_\tau^\infty e^{-A_+^{\rm u}(y-\tau)}P^{\rm u}f(y)dy. \tag{3.12}$$

Here $A_+^{\rm s}$ and $A_+^{\rm u}$ are the restrictions of $(A_+ + \eta_+)$ to the invariant subspaces $W_+^{\rm s}$ and $W_+^{\rm u}$ respectively.

Next we define the functions $g : \mathbb{R}_+ \to \mathbb{C}^m$ by $g(\tau) = e^{-(\alpha_+-\eta_+)\tau}f(\tau)$ and $z := \mathcal{D}_+^{-1}g$. Using (3.12) and (3.11), we calculate

$$\begin{aligned} z(0) &= -\int_0^\infty e^{-A_+^{\rm u}\tau}P^{\rm u}g(\tau)d\tau = -\int_0^\infty e^{-A_+^{\rm u}\tau}P^{\rm u}e^{\eta_+\tau}(u'(\tau) - A_+u(\tau))d\tau \\ &= -\int_0^\infty e^{-A_+\tau}e^{-\eta_+\tau}P^{\rm u}e^{\eta_+\tau}(u'(\tau) - A_+u(\tau))d\tau = -\int_0^\infty F'(\tau)d\tau = P^{\rm u}u(0). \end{aligned} \tag{3.13}$$

Next, we will show that

$$e^{\eta_+\tau}u(\tau) = z(\tau) + e^{A_+^{\rm s}\tau}P^{\rm s}u(0) \quad \text{for all} \quad \tau \geq 0. \tag{3.14}$$

Define $H : \mathbb{R}_+ \to \mathbb{C}^m$ by $H(\tau) = e^{\eta_+\tau}u(\tau) - z(\tau) - e^{A_+^{\rm s}\tau}P^{\rm s}u(0)$. One readily checks that $H \in H^1_{\rm loc}(\mathbb{R}, \mathbb{C}^m)$ and

$$\begin{aligned} H'(\tau) &= \eta_+e^{\eta_+\tau}u(\tau) + e^{\eta_+\tau}u'(\tau) - z'(\tau) - A_+^{\rm s}e^{A_+^{\rm s}\tau}P^{\rm s}u(0) \\ &= \eta_+e^{\eta_+\tau}u(\tau) + e^{\eta_+\tau}(A_+u(\tau) + e^{-\alpha_+\tau}f(\tau)) - (A_+ + \eta_+)z(\tau) - g(\tau) - (A_+ + \eta_+)e^{A_+^{\rm s}\tau}P^{\rm s}u(0) \\ &= (A_+ + \eta_+)\Big(e^{\eta_+\tau}u(\tau) - z(\tau) - e^{A_+^{\rm s}\tau}P^{\rm s}u(0)\Big) = (A_+ + \eta_+)H(\tau) \end{aligned}$$

for all $\tau \geq 0$. It follows from (3.13) that $H(0) = 0$, and therefore $H(\tau) = e^{(A_+ + \eta_+)\tau} H(0) = 0$ for all $\tau \geq 0$, proving (3.14). Thus,

$$e^{\eta_+ \cdot} u \in L^2(\mathbb{R}, \mathbb{C}^m) \quad \text{for all} \quad u \in \operatorname{dom}(\mathcal{T}_\alpha^A). \tag{3.15}$$

To finish the proof of the lemma, we define the operator $V_+ : \operatorname{dom}(\mathcal{T}_\alpha^A) \to L^2(\mathbb{R}_+, \mathbb{C}^m)$ by $(V_+ u)(\tau) = e^{\eta_+ \tau} u(\tau)$, $\tau \geq 0$. To show that V_+ is bounded it is enough to show that it is closed. Let $(u_n)_{n \geq 1}$ be a sequence of vectors from $\operatorname{dom}(\mathcal{T}_\alpha^A)$, $u \in \operatorname{dom}(\mathcal{T}_\alpha^A)$ and $g \in L^2(\mathbb{R}, \mathbb{C}^m)$ such that $u_n \to u$ in $\operatorname{dom}(\mathcal{T}_\alpha^A)$ and $V_+ u_n \to g$ in $L^2(\mathbb{R}_+, \mathbb{C}^m)$, as $n \to \infty$. It follows that $u_n \to u$ in $L^2(\mathbb{R}, \mathbb{C}^m)$, as $n \to \infty$, which implies that there exists a subsequence $(u_{n_k})_{k \geq 1}$ such that $u_{n_k} \to u$ almost everywhere as $k \to \infty$. From the definition of V_+ we infer that $V_+ u_{n_k} \to V_+ u$ almost everywhere as $k \to \infty$, which proves that $V_+ u = g$. Hence, V_+ is bounded, which finishes the proof. ∎

In the next corollary we extend and summarize the result proved in Lemma 3.3.

Corollary 3.4. *The following assertions hold true:*

(i) If $\alpha_- > 0$ then there exists $\eta_- \in (0, \alpha_-)$ such that

$$\int_{-\infty}^{0} |e^{-\eta_- \tau} u(\tau)|^2 d\tau \leq c \|u\|^2_{\operatorname{dom}(\mathcal{T}_\alpha^A)} \quad \textit{for all} \quad u \in \operatorname{dom}(\mathcal{T}_\alpha^A). \tag{3.16}$$

(ii) For any pair $\alpha = (\alpha_-, \alpha_+) \in \mathbb{R}_+^2$ there exists a pair $\eta = (\eta_-, \eta_+) \in \mathbb{R}_+^2$ such that $\eta_\pm \in [0, \alpha + \pm]$, $\eta_\pm > 0$ if $\alpha_\pm > 0$ and

$$\|\varphi_\eta u\|_2 \leq c \|u\|^2_{\operatorname{dom}(\mathcal{T}_\alpha^A)} \quad \textit{for all} \quad u \in \operatorname{dom}(\mathcal{T}_\alpha^A). \tag{3.17}$$

Proof. The proof of (i) is similar to the proof of Lemma 3.3. Assertion (ii) follows directly from Lemma 3.3 and (i). ∎

In the next lemma we give a more general $\operatorname{dom}(\mathcal{T}_\alpha^A)$-relative compactness result needed in the proof of the main result of this section.

Lemma 3.5. *If B is matrix-valued L^∞ function and $\lim\limits_{\tau \to \pm\infty} \frac{1}{\varphi_\alpha(\tau)} B(\tau) = 0$, then M_B, the operator of multiplication by B, is a compact operator from $\operatorname{dom}(\mathcal{T}_\alpha^A)$ to $L^2(\mathbb{R}, \mathbb{C}^m)$. Here, like in Lemma 3.1, we consider $\operatorname{dom}(\mathcal{T}_\alpha^A)$ as a Hilbert space with the usual graph norm.*

Proof. To prove the lemma we are going to approximate the matrix-valued function B with a sequence of matrix-valued functions K_n defined such that the operator M_{K_n} approximates the operator M_B in the operator norm. Let $(\psi_n)_{n \geq 1}$ be a sequence of real-valued C^∞ functions such that $0 \leq \psi_n(\tau) \leq 1$, $\psi_n(\tau) = 1$ for all $\tau \in [-n, n]$ and $\psi_n(\tau) = 0$ for all $\tau \notin [-n-1, n+1]$. Define the matrix-valued functions $K_n := \psi_n B_n$, $n \geq 1$.

Since $B \in L^\infty(\mathbb{R}, \mathbb{C}^m)$, we can assume without loss of generality that $|B(\tau)| \leq c$ for all $\tau \in \mathbb{R}$. It follows from Corollary 3.4(ii) that there exists $\eta = (\eta_-, \eta_+) \in \mathbb{R}_+^2$ such that $\eta_\pm \in [0, \alpha_\pm]$, $\eta_\pm > 0$

if $\alpha_\pm > 0$ such that (3.17) is satisfied. Since, by the hypothesis, we have that $\lim_{\tau\to\pm\infty} \frac{1}{\varphi_\alpha(\tau)} B(\tau) = 0$, we conclude that

$$\lim_{\tau\to\pm\infty} \frac{1}{\varphi_\eta(\tau)} B(\tau) = 0. \tag{3.18}$$

Thus,

$$\left\| \frac{1}{\varphi_\eta} K_n - \frac{1}{\varphi_\eta} B \right\|_\infty \leq \sup_{|\tau|\geq n} \left| \frac{1}{\varphi_\eta(\tau)} B(\tau) \right| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.19}$$

Next, we will show that $M_{K_n} \to M_B$ as $n \to \infty$ in the operator norm. Using (3.17), for any $u \in \operatorname{dom}(\mathcal{T}_\alpha^A)$, we estimate

$$\begin{aligned} \|\Big(M_{K_n} - M_B\Big)u\|_2 &= \left\| \frac{1}{\varphi_\eta} (K_n - B)(\varphi_\eta u) \right\|_2 \leq \left\| \frac{1}{\varphi_\eta} K_n - \frac{1}{\varphi_\eta} B \right\|_\infty \|\varphi_\eta u\|_2 \\ &\leq c \left\| \frac{1}{\varphi_\eta} K_n - \frac{1}{\varphi_\eta} B \right\|_\infty \|u\|_{\operatorname{dom}(\mathcal{T}_\alpha^A)}, \end{aligned}$$

which implies that

$$\|M_{K_n} - M_B\| \leq c \left\| \frac{1}{\varphi_\eta} K_n - \frac{1}{\varphi_\eta} B \right\|_\infty \quad \text{for all} \quad n \geq 1. \tag{3.20}$$

Applying Lemma 3.1 to the sequence of matrix valued functions $(K_n)_{n\geq 1}$ we obtain that $M_{K_n} \in \mathcal{K}(\operatorname{dom}(\mathcal{T}_\alpha^A), L^2(\mathbb{R}, \mathbb{C}^m))$ for all $n \geq 1$. From (3.19) and (3.20) we have that $M_{K_n} \to M_B$ as $n \to \infty$ in the operator norm, which implies that $M_B \in \mathcal{K}(\operatorname{dom}(\mathcal{T}_\alpha^A), L^2(\mathbb{R}, \mathbb{C}^m))$, proving the lemma. ∎

Recall that for a matrix B we denote by $i(B)$ the dimension of the generalized eigenspace of all eigenvalues μ with $\operatorname{Re}\mu > 0$. Similarly, we denote by $j(B)$ the dimension of the generalized eigenspace of all eigenvalues μ with $\operatorname{Re}\mu \geq 0$.

Assume that $\alpha_\pm > 0$ or that $A_\pm$ is hyperbolic. Then there exists $\eta = (\eta_-, \eta_+) \in \mathbb{R}_+^2$ (not necessarily unique) that satisfies the condition from Corollary 3.4(ii), (3.17). Moreover, we can choose $\eta_\pm$ so that $A_\pm \pm \eta_\pm$ is hyperbolic and

$$i_- := i(A_- - \eta_-) = i(A_-), \quad i_+ := i(A_+ + \eta_+) = \begin{cases} i(A_+), & \text{if } A_+ \text{ is hyperbolic} \\ j(A_+), & \text{if } A_+ \text{ is } \textit{not} \text{ hyperbolic} \end{cases}. \tag{3.21}$$

Definition 3.6. Assume that either $\alpha_\pm > 0$ or that $A_\pm$ is hyperbolic. Take $\eta_\pm$ as defined above satisfying (3.17) and (3.21). We define $\beta = (\beta_-, \beta_+) \in \mathbb{R}_+^2$ and $H = (H_-, H_+)$ through $\beta_\pm = \alpha_\pm - \eta_\pm$, $H_\pm = A_\pm \pm \eta_\pm$.

Lemma 3.7. *Assume that either $\alpha_\pm > 0$ or that $A_\pm$ is hyperbolic and let β and H as defined in Definition 3.6. Then the following hold true:*

(i) The operator $U_\eta : \operatorname{dom}(\mathcal{T}_\alpha^A) \to \operatorname{dom}(\mathcal{T}_\beta^H)$ defined by $U_\eta u = \varphi_\eta u$ is bounded with bounded inverse. In addition,

$$\mathcal{T}_\alpha^A = \mathcal{T}_\beta^H U_\eta. \tag{3.22}$$

(ii) The operator $\mathcal{T}_\alpha^A$ is Fredholm if and only if the operator $\mathcal{T}_\beta^H$ is Fredholm. In this case

$$\operatorname{ind}(\mathcal{T}_\alpha^A) = \operatorname{ind}(\mathcal{T}_\beta^H). \tag{3.23}$$

Proof. First we note that U_η is an injective operator. Next, we will show that

$$U_\eta\left(\operatorname{dom}(\mathcal{T}_\alpha^A)\right) \subseteq \operatorname{dom}(\mathcal{T}_\beta^H) \quad \text{and} \quad \mathcal{T}_\beta^H U_\eta u = \mathcal{T}_\alpha^A u \quad \text{for all} \quad u \in \operatorname{dom}(\mathcal{T}_\alpha^A). \tag{3.24}$$

Let $u \in \operatorname{dom}(\mathcal{T}_\alpha^A)$ and denote by $v = U_\eta u = \varphi_\eta u$ and $f = \mathcal{T}_\alpha^A u \in L^2(\mathbb{R}, \mathbb{C}^m)$. From Corollary 3.4(ii) we have that also $v \in L^2(\mathbb{R}, \mathbb{C}^m)$. Since $u \in H^1(\mathbb{R}, \mathbb{C}^m)$ and $\varphi_\eta \in H^1_{\text{loc}}(\mathbb{R})$, we obtain that

$$\begin{aligned} v' = \varphi_\eta' u + \varphi_\eta &= (\eta_+ \chi_{\mathbb{R}_+} - \eta_- \chi_{\mathbb{R}_-})\varphi_\eta u + M_A(\varphi u) + \frac{1}{\varphi_{\alpha-\eta}} f \\ &= M_H v + \frac{1}{\varphi_{\alpha-\eta}} f \quad \text{almost everywhere.} \end{aligned} \tag{3.25}$$

Here χ_E denotes the characteristic function of a set $E \subset \mathbb{R}$. Since the operator M_H is bounded on $L^2(\mathbb{R}, \mathbb{C}^m)$ and since $\varphi_{\alpha-\eta} \geq 1$, we obtain from (3.25) that $v' \in L^2(\mathbb{R}, \mathbb{C}^m)$. Thus, $v \in H^1(\mathbb{R}, \mathbb{C}^m)$. Moreover, using the definition of β in Definition 3.6, we have that $\varphi_\beta(v' - M_H v) = f \in L^2(\mathbb{R}, \mathbb{C}^m)$, which shows $v \in \operatorname{dom}(\mathcal{T}_\beta^H)$ and $\mathcal{T}_\beta^H v = f$, proving (3.24).

Similarly, one can show that if $v \in \operatorname{dom}(\mathcal{T}_\beta^H)$ and $\mathcal{T}_\beta^H v = g$ then $u = \frac{1}{\varphi_\eta} v \in \operatorname{dom}(\mathcal{T}_\alpha^A)$ and $\mathcal{T}_\alpha^A u = g$. This proves that

$$\operatorname{dom}(\mathcal{T}_\beta^H) \subseteq U_\eta\left(\operatorname{dom}(\mathcal{T}_\alpha^A)\right) \quad \text{and} \quad \mathcal{T}_\alpha^A U_\eta^{-1} v = \mathcal{T}_\beta^H v \quad \text{for all} \quad v \in \operatorname{dom}(\mathcal{T}_\beta^H). \tag{3.26}$$

The conclusions of (i) follows shortly from (3.24), (3.26) and the definition of the domain of the operators $\mathcal{T}_\alpha^A$ and $\mathcal{T}_\beta^H$ and their respective graph norms.

Assertion (ii) follows immediately from (i). ■

In the next lemma we give sufficient conditions that guarantee the Fredholm property of the operator $\mathcal{T}_\alpha^A$ and in this case we compute its index.

Lemma 3.8. *Assume that $\alpha_\pm > 0$ or $A_\pm$ is hyperbolic and β and H are defined in Definition 3.6. Then, the operator $\mathcal{T}_\alpha^A$ is Fredholm and* $\operatorname{ind}(\mathcal{T}_\alpha^A) = i_- - i_+$*. Here $i_\pm$ were defined in* (3.21)*.*

Proof. It follows from Lemma 3.7(ii) that to prove the lemma, it is enough to show that the operator $\mathcal{T}_\beta^H$ is Fredholm and to compute its index.

From Palmer's Classic Dichotomy Theorem in [13, 14] we know that $\mathcal{T}_0^H$ is a Fredholm operator and $\operatorname{ind}(\mathcal{T}_0^H) = i(H_-) - i(H_+) = i_- - i_+$. A direct computation immediately shows that $\ker \mathcal{T}_\beta^H = \ker \mathcal{T}_0^H$. To conclude the proof the lemma, we only need to show that $\operatorname{im} \mathcal{T}_\beta^H$ is a closed subspace of finite codimension and $\operatorname{codim}(\operatorname{im} \mathcal{T}_\beta^H) = \operatorname{codim}(\operatorname{im} \mathcal{T}_0^H)$.

Therefore, we define the operator $V_\eta : L^2(\mathbb{R}, \mathbb{C}^m) \to L^2(\mathbb{R}, \mathbb{C}^m)$ by $V_\eta u = \frac{1}{\varphi_\eta} u$. Since $\varphi_\eta \geq 1$, we have that V_η is a linear, injective and bounded operator. Moreover, for any matrix-valued continuous function h with compact support, the function $\varphi_\eta h \in L^2(\mathbb{R}, \mathbb{C}^m)$, which implies that $h = V_\eta(\varphi_\eta h) \in \operatorname{im} V_\eta$. This shows that $\operatorname{im} V_\eta$ is a dense subspace, that is $\overline{\operatorname{im} V_\eta} = L^2(\mathbb{R}, \mathbb{C}^m)$. Thus, the operator V_η and the subspace $\operatorname{im} \mathcal{T}_0^H$ satisfy the conditions of Lemma 6.1. In addition a direct computation shows that $\operatorname{im} \mathcal{T}_\beta^H = V_\eta^{-1}(\operatorname{im} \mathcal{T}_0^H)$, which proves the lemma. ∎

The main result of this section is the following theorem.

Theorem 3.9. *Assume that $\alpha_\pm \geq 0$, $A_\pm \in \mathcal{M}_m(\mathbb{C})$, and let $\alpha = (\alpha_-, \alpha_+)$. Recall the definitions of $\varphi_\alpha : \mathbb{R} \to [1, \infty)$ and $A : \mathbb{R} \to \mathcal{M}_m(\mathbb{C})$,*

$$\varphi_\alpha(\tau) = \begin{cases} e^{-\alpha_- \tau}, & \tau < 0 \\ e^{\alpha_+ \tau}, & \tau \geq 0 \end{cases}, \qquad A(\tau) = \begin{cases} A_-, & \tau < 0 \\ A_+, & \tau \geq 0 \end{cases}.$$

Let $B \in L^\infty(\mathbb{R}, \mathcal{M}_m(\mathbb{C}))$ and define the operator $\mathcal{T} : \operatorname{dom}(\mathcal{T}_\alpha^A) \to L^2(\mathbb{R}, \mathbb{C}^m)$ by

$$\mathcal{T} u = \varphi_\alpha(u' - M_A u) - M_B u. \tag{3.27}$$

If in addition, there exist $B_\pm \in \mathcal{M}_m(\mathbb{C})$ such that

(i) $\lim_{\tau \to \pm\infty} e^{\mp \alpha_\pm \tau}(B(\tau) - B_\pm) = 0$,

(ii) $\alpha_- > 0$ *or* $A_- + B_-$ *is hyperbolic, and*

(iii) $\alpha_+ > 0$ *or* $A_+ + B_+$ *is hyperbolic,*

then the operator $\mathcal{T}$ is Fredholm and $\operatorname{ind}(\mathcal{T}) = \bar{\imath}_- - \bar{\imath}_+$. *Here*

$$\bar{\imath}_- = \begin{cases} i(A_- + B_-), & \text{if } \alpha_- = 0 \\ i(A_-), & \text{if } \alpha_- > 0 \end{cases}, \quad \bar{\imath}_+ := \begin{cases} i(A_+ + B_+), & \text{if } \alpha_+ = 0 \\ j(A_+), & \text{if } \alpha_+ > 0 \end{cases}. \tag{3.28}$$

Proof. First we define the matrices $\tilde{A}_\pm$ as follows

$$\tilde{A}_\pm = \begin{cases} A_\pm + B_\pm, & \text{if } \alpha_\pm = 0 \\ A_\pm, & \text{if } \alpha_\pm > 0 \end{cases}. \tag{3.29}$$

Also, we define the matrix-valued function $\tilde{B} : \mathbb{R} \to \mathcal{M}_m(\mathbb{C}^m)$ by

$$\tilde{B}(\tau) = \begin{cases} B(\tau) - B_- \chi_{\mathbb{R}_-}(\tau) - B_+ \chi_{\mathbb{R}_+}(\tau), & \text{if } \alpha_- = \alpha_+ = 0 \\ B(\tau) - B_- \chi_{\mathbb{R}_-}(\tau), & \text{if } \alpha_- = 0,\ \alpha_+ > 0 \\ B(\tau) - B_+ \chi_{\mathbb{R}_+}(\tau), & \text{if } \alpha_- > 0,\ \alpha_+ = 0 \\ B(\tau), & \text{if } \alpha_- > 0,\ \alpha_+ > 0 \end{cases}. \tag{3.30}$$

One can readily check that $\operatorname{dom}(\mathcal{T}) = \operatorname{dom}(\mathcal{T}_\alpha^{\tilde{A}}) = \operatorname{dom}(\mathcal{T}_\alpha^A)$ and $\mathcal{T} = \mathcal{T}_\alpha^{\tilde{A}} - M_{\tilde{B}}$. Since $\alpha_\pm > 0$ or $\tilde{A}_\pm$ is hyperbolic, we conclude from Lemma 3.8 that the operator $\mathcal{T}_\alpha^{\tilde{A}}$ is Fredholm and $\operatorname{ind}(\mathcal{T}_\alpha^{\tilde{A}}) = \bar{\imath}_- - \bar{\imath}_+$. Since $\lim_{\tau \to \pm\infty} \frac{1}{\varphi_\alpha(\tau)} \tilde{B}(\tau) = 0$ and $\tilde{B} \in L^\infty(\mathbb{R}, \mathcal{M}_m(\mathbb{C}))$, we obtain from Lemma 3.5 that $M_{\tilde{B}} \in \mathcal{K}(\operatorname{dom}(\mathcal{T}_\alpha^{\tilde{A}}), L^2(\mathbb{R}, \mathbb{C}^m))$. Thus, the operator $\mathcal{T}$ is Fredholm and $\operatorname{ind}(\mathcal{T}) = \operatorname{ind}(\mathcal{T}_\alpha^{\tilde{A}}) = \bar{\imath}_- - \bar{\imath}_+$. ∎

4 Proofs of Theorems 1.1–1.3

Proof. [of Theorem 1.1] From Lemma 2.1(iii), Lemma 2.3, and Lemma 2.4(ii), we have that the operators $\mathcal{L}_{\mathrm{rad}}$, $\tilde{\mathcal{L}}$, $\mathcal{T}_{\mathrm{rad}}$, and $\tilde{\mathcal{T}}$ are Fredholm if one of them is Fredholm and their indices coincide. Moreover, from Lemma 2.4(iii), we have that $(\tilde{\mathcal{T}}u)(\tau) = (\mathcal{T}u)(\tau)$ for all $|\tau| \geq 1$, where $\mathcal{T} = \mathcal{T}_\alpha^A - M_B$. Hence, $\tilde{\mathcal{T}} - \mathcal{T}$ is relatively compact. Here $\alpha_- = -1$, $\alpha_+ = 0$, $A_- = \begin{bmatrix} \frac{k}{2}I_m & 0 \\ 0 & -\frac{k-2}{2}I_m \end{bmatrix}$, $A_+ = 0$. The function $B \in L^\infty(\mathbb{R}, \mathcal{M}_m(\mathbb{C}^m))$ satisfies the condition $\lim_{\tau\to\pm\infty} e^{\mp\alpha_\pm\tau}(B(\tau) - B_\pm) = 0$ for $B_- = 0$ and $B_+ = T_\infty = \begin{bmatrix} 0 & I_m \\ -D_\infty^{-1}R_\infty & -D_\infty^{-1}Q_\infty \end{bmatrix}$. From Theorem 3.9 we now conclude that $\mathcal{T} = \mathcal{T}_\alpha^A - M_B$ is Fredholm if T_∞ is hyperbolic. In this case $\mathrm{ind}(\mathcal{T}) = \bar{i}_- - \bar{i}_+ = i(A_-) - i(T_\infty) = m - i(T_\infty)$. To see that the hyperbolicity condition on T_∞ is necessary, assume that $\mathcal{L}_{\mathrm{rad}}$ is Fredholm. It follows that the operator $\mathcal{T}$ is Fredholm, and thus, from Theorem 3.9 we have that $\mathcal{T}_\alpha^{\tilde{A}}$ is Fredholm. Since $\alpha_+ = 0$, we infer that the equation $u' = \tilde{A}_+ u$ has an exponential dichotomy on $\mathbb{R}_+$ which implies that $T_\infty = B_+ = \tilde{A}_+$ is hyperbolic. ∎

Proof. [of Theorem 1.2] First, we define the smooth function $\phi \in C^\infty(\mathbb{R}_+)$, $\phi' \leq 0$, such that

$$\phi(r) = \begin{cases} e^{-\eta r} \text{ for } r \geq 2, \\ e^{-\eta} \text{ for } r \in [0,1] \end{cases}. \tag{4.1}$$

One then readily checks that $L^2_{\eta,\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^m) = L^2_{\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^m; \phi(|x|)^{-2}dx)$ and that the operator $U_\phi : L^2_{\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^m) \to L^2_{\eta,\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^m)$ defined by $U_\phi u = \phi(|\cdot|)u$ is an isomorphism. The linear operator $\mathcal{L}_{\phi,\mathrm{rad}} = U_\phi^{-1}\mathcal{L}_{\mathrm{rad}}U_\phi : H^2_{\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^m) \to L^2_{\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^m)$ is defined in the radial variable by

$$\begin{aligned} \mathcal{L}_{\phi,\mathrm{rad}} &= D(r)\left[\left(\frac{d}{dr} + \frac{\phi'}{\phi}\right)^2 + \frac{k-1}{r}\left(\frac{d}{dr} + \frac{\phi'}{\phi}\right)\right] + Q(r)\left(\frac{d}{dr} + \frac{\phi'}{\phi}\right) + R(r) \\ &= D(r)\left(\frac{d^2}{dr^2} + \frac{k-1}{r}\frac{d}{dr}\right) + Q_\phi(r)\frac{d}{dr} + R_\phi(r), \end{aligned} \tag{4.2}$$

where

$$Q_\phi(r) := Q(r) + \frac{2\phi'(r)}{\phi(r)}D(r), \quad R_\phi(r) := R(r) + \left(\frac{(k-1)\phi'(r)}{r\phi(r)} + \frac{\phi''(r)}{\phi(r)}\right)D(r) + \frac{\phi'(r)}{\phi(r)}Q(r). \tag{4.3}$$

Since the matrix-valued functions Q and R are continuous and $\phi'(r) = 0$ for all $r \in [0,1]$, we infer that Q_ϕ and R_ϕ are continuous and in addition

$$Q_{\phi,\infty} := \lim_{t\to\infty} Q_\phi(r) = Q_\infty - 2\eta D_\infty, \quad R_{\phi,\infty} := \lim_{t\to\infty} R_\phi(r) = R_\infty - \eta Q_\infty + \eta^2 D_\infty. \tag{4.4}$$

To finish the proof of the theorem, we need to show that the matrix

$$T_{\phi,\infty} := \begin{bmatrix} 0 & I_m \\ -D_\infty^{-1}R_{\phi,\infty} & -D_\infty^{-1}Q_{\phi,\infty} \end{bmatrix} = \begin{bmatrix} 0 & I_m \\ -D_\infty^{-1}R_\infty + \eta D_\infty^{-1}Q_\infty - \eta^2 I_m & -D_\infty^{-1}Q_\infty + 2\eta I_m \end{bmatrix} \tag{4.5}$$

is hyperbolic. Let $J_\eta = \begin{bmatrix} I_m & 0 \\ \eta I_m & I_m \end{bmatrix}$. Since $\det J_\eta = 1$, the matrix J_η is invertible. Since, moreover, $J_\eta^{-1} T_{\phi,\infty} J_\eta = T_\infty + \eta I_{2m}$, we have that the matrices $T_{\phi,\infty}$ and $T_\infty + \eta I_m$ are conjugate. Thus, $T_{\phi,\infty}$ is hyperbolic if and only if $T_\infty + \eta I_{2m}$ is hyperbolic. Since U_ϕ is an isometric isomorphism, it follows that $\mathcal{L}_{\mathrm{rad}}$ is Fredholm on $L^2_{\eta,\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^m)$ if and only if $\mathcal{L}_{\phi,\mathrm{rad}}$ is Fredholm on $L^2_{\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^m)$ and their indices coincide. Now the conclusion follows shortly from Theorem 1.1. Moreover, in the case when the operators are Fredholm, we have that

$$\mathrm{ind}(\mathcal{L}_{\mathrm{rad}}) = \mathrm{ind}(\mathcal{L}_{\phi,\mathrm{rad}}) = m - \mathrm{ind}(T_{\phi,\infty}) = m - i(T_\infty + \eta I_m)$$

∎

Proof. [of Theorem 1.3] First, we note that (i) can be obtained from (ii) for $\eta = 0$. Since, in this case, $T_\infty = \begin{bmatrix} 0 & I_m \\ -D_\infty^{-1} R_\infty & 0 \end{bmatrix}$, we have that $\det(T_\infty - \lambda)$ depends only on λ^2, thus the Morse index is simply $i(T_\infty) = m$. To prove (ii), let $\phi \in C^\infty(\mathbb{R}_+)$ be the function defined in (4.1). We define the operator $\tilde{\mathcal{L}}^\ell_\psi = U_{\mathrm{rad}}^{-1} U_\psi^{-1} \mathcal{L}^\ell_{\mathrm{rad}} U_\psi U_{\mathrm{rad}} : \mathrm{dom}(\tilde{\mathcal{L}}^\ell_\psi) \to L^2((0,\infty), \mathbb{C}^m)$, where U_{rad} is defined in Lemma 2.1 and U_ψ is defined in the proof of Theorem 1.2. It is a simple computation to see that

$$\begin{aligned}
\tilde{\mathcal{L}}^\ell_\psi &= D(r)\Big(\frac{d^2}{dr^2} - \frac{(k-1)(k-3)+4\ell^2}{4r^2}\Big) + \frac{2\phi'(r)}{\phi(r)} D(r)\Big(\frac{d}{dr} - \frac{k-1}{2r}\Big) \\
&\quad + R(r) + \left(\frac{(k-1)\phi'(r)}{r\phi(r)} + \frac{\phi''(r)}{\phi(r)}\right) D(r) \\
&= D(r)\Big(\frac{d^2}{dr^2} - \frac{(\tilde{k}-1)(\tilde{k}-3)}{4r^2}\Big) + \tilde{Q}_\psi(r)\Big(\frac{d}{dr} - \frac{\tilde{k}-1}{2r}\Big) + \tilde{R}_\psi(r).
\end{aligned}$$

Here $\tilde{k} := 2 + \sqrt{(k-2)^2 + 4m^2}$ and the matrix-valued functions $\tilde{Q}_\phi, \tilde{R}_\phi : \mathbb{R}_+ \to \mathcal{M}(\mathbb{C})$ are defined by

$$\tilde{Q}_\phi(r) := \frac{2\phi'(r)}{\phi(r)} D(r), \quad \tilde{R}_\phi(r) := R(r) + \left(\frac{(\tilde{k}-1)\phi'(r)}{r\phi(r)} + \frac{\phi''(r)}{\phi(r)}\right) D(r). \tag{4.6}$$

Since the matrix-valued functions D and R are continuous and $\phi'(r) = 0$ for all $r \in [0,1]$, we infer that $\tilde{Q}_\phi$ and $\tilde{R}_\phi$ are continuous and in addition

$$\tilde{Q}_{\phi,\infty} := \lim_{t\to\infty} \tilde{Q}_\phi(r) = -2\eta D_\infty, \quad \tilde{R}_{\phi,\infty} := \lim_{t\to\infty} \tilde{R}_\phi(r) = R_\infty + \eta^2 D_\infty. \tag{4.7}$$

Similarly to Remark 2.2, one can show that $\mathrm{dom}(\tilde{\mathcal{L}}^\ell_\psi) = \mathrm{dom}(\tilde{S}_2 \tilde{S}_1)$ and

$$\tilde{\mathcal{L}}^\ell_\psi = M_D \tilde{S}_2 \tilde{S}_1 + M_{\tilde{Q}_\psi} \tilde{S}_1 + M_{\tilde{R}_\psi},$$

where the linear operators $\tilde{S}_j : \mathrm{dom}(\tilde{S}_j) \to L^2((0,\infty), \mathbb{C}^m)$, $j = 1, 2$ are defined by

$$(\tilde{S}_j u)(r) = u'(r) + (-1)^j \frac{\tilde{k}-1}{2r} u(r).$$

Since we can also prove that $\tilde{S}_2\tilde{S}_1 = \frac{d^2}{dr^2} - \frac{(\tilde{k}-1)(\tilde{k}-3)}{4r^2}$, we have that the linear operators $\mathrm{Id} - \tilde{S}_2\tilde{S}_1$ and $\mathrm{Id} - \tilde{S}_1\tilde{S}_2$ are invertible. Hence, similarly to Lemma 2.3 we can show that the operator $\mathcal{L}^\ell_{\mathrm{rad}}$ is Fredholm on $L^2_{\eta,\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^m)$ if and only if the operator $\mathcal{T}^\ell_{\psi,\mathrm{rad}} : \mathrm{dom}(\tilde{S}_1) \times \mathrm{dom}(\tilde{S}_2) \to L^2((0,\infty), \mathbb{C}^{2m})$, defined by

$$\mathcal{T}^\ell_{\psi,\mathrm{rad}} = \begin{bmatrix} \tilde{S}_1 & -\mathrm{Id} \\ M_{D^{-1}\tilde{R}_\psi} & \tilde{S}_2 + M_{D^{-1}\tilde{Q}_\psi} \end{bmatrix}, \tag{4.8}$$

is Fredholm on $L^2((0,\infty), \mathbb{C}^{2m})$, and their indices coincide.

Next, we define $\tilde{\mathcal{T}}^\ell_\psi = U_\Psi^{-1}\mathcal{T}^\ell_{\psi,\mathrm{rad}}U_\psi : \mathrm{dom}(\tilde{\mathcal{T}}^\ell_\psi) \to L^2(\mathbb{R}, \mathbb{C}^m)$. Here, the isometric isomorphism U_Ψ is defined in Lemma 2.4. Similarly to the proof of Theorem 1.1, the operator $\tilde{\mathcal{T}}^\ell_\psi$ is Fredholm if and only if the operator $\mathcal{T} = \mathcal{T}^A_\alpha - M_B$ is Fredholm and their indices coincide. Here, $\alpha_- = -1$, $\alpha_+ = 0$, $A_- = \begin{bmatrix} \frac{\tilde{k}}{2}I_m & 0 \\ 0 & -\frac{\tilde{k}-2}{2}I_m \end{bmatrix}$, $A_+ = 0$. The function $B \in L^\infty(\mathbb{R}, \mathcal{M}_m(\mathbb{C}^m))$ satisfies the condition $\lim_{\tau\to\pm\infty} e^{\mp\alpha_\pm\tau}(B(\tau) - B_\pm) = 0$ for $B_- = 0$ and $B_+ = \begin{bmatrix} 0 & I_m \\ -D_\infty^{-1}\tilde{R}_{\psi,\infty} & -D_\infty^{-1}\tilde{Q}_{\psi,\infty} \end{bmatrix}$. Moreover, we have that $J_\eta^{-1}B_+J_\eta = T_\infty + \eta I_{2m}$, where J_η was defined in the proof of Theorem 1.2. We conclude from Theorem 3.9 that the operator $\mathcal{L}^\ell_{\mathrm{rad}}$ is Fredholm on $L^2_{\eta,\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^m)$ if $T_\infty + \eta I_m$ is hyperbolic and $\mathrm{ind}(\mathcal{L}^\ell_{\mathrm{rad}}) = i(A_-) - i(B_+) = m - \mathrm{ind}(T_\infty + \eta I_{2m})$. To show that the hyperbolicity condition on $T_{\eta,\infty} = T_\infty + \eta I_{2m}$ is necessary, one can use the same argument given in the proof of Theorem 1.1. ∎

5 Applications

5.1 Lyapunov-Schmidt reduction in linear problems

An interesting application of our results arises in the study of stability of spikes in a class of spatially extended systems that are governed by a scalar reaction-diffusion equation, coupled to a conservation law,

$$\begin{cases} u_t = \nabla\cdot[a(u,v)\nabla u + b(u,v)\nabla v], \\ v_t = \Delta v + f(u,v), \end{cases} \quad t \geq 0,\ x \in \mathbb{R}^k. \tag{5.1}$$

The functions a, b, and f are of class $C^3(\mathbb{R}^2, \mathbb{R})$ and, in addition, $a(u,v) \geq a_0 > 0$ for all $(u,v) \in \mathbb{R}^2$. This model includes models such as the Keller-Segel model for chemotaxis, the phase-field models for undercooled liquids, models for precipitation patterns, and reaction-diffusion systems in closed reactors. The spike solutions are time independent steady states of equation (5.1). In [17], we proved the instability of exponentially localized, radially-symmetric spikes with a stable background, that is spikes solutions satisfying

(rs1)

$$|(u^* - u^\infty, v^* - v^\infty)(x)| \leq c\mathrm{e}^{-\delta_0|x|}, \quad \text{for all} \quad x \in \mathbb{R}^k, \quad (u^*, v^*) \not\equiv (u^\infty, v^\infty),$$

for some constants $u^\infty, v^\infty \in \mathbb{R}$, $f(u^\infty, v^\infty) = 0$ and $c, \delta_0 > 0$.

(rs2) Spikes are asymptotic to constant states that are stable for the pure kinetics,

$$u' = 0 \qquad v' = f(u,v),$$

that is, we assume $f_v(u^\infty, v^\infty) < 0$.

A key argument in our proof in [17] is to track the point spectrum at the edge of the essential spectrum of the operator $\mathcal{L}_{\text{rad}}$ defined as the linearization of the equation (5.1) along the spike (u^*, v^*),

$$\mathcal{L}_{\text{rad}} = \begin{bmatrix} \frac{1}{r^{k-1}}\frac{d}{dr}\left[r^{k-1}\left(a^*\frac{d}{dr}+l_1\right)\right] & \frac{1}{r^{k-1}}\frac{d}{dr}\left[r^{k-1}\left(b^*\frac{d}{dr}+l_2\right)\right] \\ f_u^*(r) & \frac{1}{r^{k-1}}\frac{d}{dr}(r^{k-1}\frac{d}{dr}) + f_v^*(r) \end{bmatrix}, \tag{5.2}$$

where

$$a^*(r) = a(u^*(r), v^*(r)), \quad b^*(r) = b(u^*(r), v^*(r)), \quad \partial^\alpha f(r) = \partial^\alpha f(u^*(r), v^*(r)), \tag{5.3}$$

$$l_1 = a_u^* u_r^* + b_u^* v_r^* \quad \text{and} \quad l_2 = a_v^* u_r^* + b_v^* v_r^*. \tag{5.4}$$

This operator satisfies the conditions Nondegeneracy (N) and Convergence (C) of Theorem 1.1 and Theorem 1.2, with

$$D(r) = \begin{bmatrix} a^*(r) & b^*(r) \\ 0 & 1 \end{bmatrix}. \tag{5.5}$$

Using the fact that the spike (u^*, v^*) decays exponentially at ∞ we obtain that limiting matrices are

$$D_\infty = \begin{bmatrix} a^\infty & b^\infty \\ 0 & 1 \end{bmatrix}, \quad Q_\infty = 0_2 \quad \text{and} \quad R_\infty = \begin{bmatrix} 0 & 0 \\ f_u^\infty & f_v^\infty \end{bmatrix}, \tag{5.6}$$

where

$$a^\infty = a(u^\infty, v^\infty), \quad b^\infty = b(u^\infty, v^\infty), \quad \partial^\alpha f^\infty = \partial^\alpha f(u^\infty, v^\infty), \tag{5.7}$$

It is easy to check that in this case the eigenvalues of the matrix $T_\infty = \begin{bmatrix} 0_2 & I_2 \\ -D_\infty^{-1} R_\infty & 0_2 \end{bmatrix}$ are $\pm\sqrt{\frac{b^\infty}{a^\infty} f_u^\infty - f_v^\infty}$ with multiplicity 1 and 0 with multiplicity 2. Let $\eta^* = \frac{1}{2}\sqrt{\frac{b^\infty}{a^\infty} f_u^\infty - f_v^\infty} > 0$[2]. We obtain from Theorem 1.2 that $\mathcal{L}_{\text{rad}}$ is Fredholm on $L^2_{\eta,\text{rad}}(\mathbb{R}^k, \mathbb{C}^2)$ and

$$\operatorname{ind}(\mathcal{L}_{\text{rad}}) = 2 - i(T_\infty + \eta I_2) = 2 - 3 = -1 \quad \text{for all} \quad \eta \in (0, \eta^*).$$

In order to solve the eigenvalue problem

$$(\mathcal{L}_{\text{rad}} - \gamma^2)(u,v)^{\mathrm{T}} = 0 \tag{5.8}$$

for $\gamma \approx 0$ we use the ansatz

$$(u,v)^{\mathrm{T}} = \underline{w} + \beta\alpha(\gamma) h_k(\gamma), \tag{5.9}$$

where $\underline{w} \in H^2_{\eta,\text{rad}}(\mathbb{R}^k, \mathbb{C}^2)$ and the function h_k is asymptotically equal at ∞ to the plain wave solutions of the operator $\mathcal{L}_{\text{rad}}^\infty := D_\infty \Delta_r + R_\infty$. For details we refer to [17, Section 4]. As shown in the proof of Proposition 4.10 in [17] it is essential for the Lyapunov-Schmidt reduction that $\mathcal{L}_{\text{rad}}$ is Fredholm on $L^2_{\eta,\text{rad}}(\mathbb{R}^k, \mathbb{C}^2)$ and $\operatorname{ind}(\mathcal{L}_{\text{rad}}) = -1$ for all $\eta \in (0, \eta^*)$.

[2] One can show that in this case the quantity under the square root is positive, see condition ODE-Hyperbolicity on [17, page 5]

5.2 Lyapunov-Schmidt reduction in nonlinear problems

In this subsection, we prove Theorem 1.4. Recall that we are interested in the equation

$$\Delta u - u^3 + \varepsilon V(|x|, u) = 0, \quad x \in \mathbb{R}^2, \tag{5.10}$$

for $\varepsilon \approx 0$. Writing equation (5.10) in the radial variable $r = |x|$, we obtain the equation

$$u'' + \frac{1}{r}u' - u^3 + \varepsilon V(r, u) = 0, \quad r > 0. \tag{5.11}$$

We first construct a suitable far-field solution by ignoring the perturbation term εV. We find a one-parameter family of far-field expansions by exploiting the scaling symmetry. These far-field solutions are singular at the origin $r = 0$. We therefore truncate them to a support in $r \geq 2$ and allow for a general exponentially localized, ε-dependent contribution.

$$u'' + \frac{1}{r}u' - u^3 = 0, \quad r > 0, \tag{5.12}$$

we make the change of independent variable $\tau = \ln r$. We also set $\tilde{u}(\tau) := e^\tau u(\tau)$. Then $\tilde{u}$ satisfies the equation

$$\tilde{u}'' - 2\tilde{u}' + \tilde{u} - \tilde{u}^3 = 0, \quad \tau \in \mathbb{R}. \tag{5.13}$$

Using a phase-portrait analysis we can find a solution $\tilde{u}_* \in L^\infty(\mathbb{R})$, $\tilde{u}_*(\tau) \to 0$ as $\tau \to -\infty$, and $\tilde{u}_*(\tau) \to 1$ as $\tau \to \infty$, whose rate of decay at $-\infty$ is given by

$$\tilde{u}_*(\tau) = a_0 \tau e^\tau + \mathcal{O}(\tau^3 e^{3\tau}), \quad \text{as} \quad \tau \to -\infty, \tag{5.14}$$

for some $a_0 < 0$. It follows that the function $u_* : (0, \infty) \to \mathbb{R}$, defined by

$$u_*(r) := \frac{\tilde{u}_*(\ln r)}{r}. \tag{5.15}$$

is a solution of equation (5.12), for $r > 0$. Note however that u_* is not bounded.

Ansatz for the perturbation problem

To find solutions of equation (5.11) we use the following ansatz:

$$u = h(\cdot, \mu) + w, \quad w \in H^2_{\eta,\mathrm{rad}}(\mathbb{R}^2),\ \eta \in (0, \delta_0/2). \tag{5.16}$$

Here, δ_0 is given in the assumption on exponential decay (V), and the function $h : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ is defined by

$$h(r, \mu) = \begin{cases} \frac{\tilde{u}_*(\ln(\mu r))}{r}\chi(r), & \mu > 0 \\ 0, \quad \mu = 0 \end{cases}, \tag{5.17}$$

where $\chi \in C^\infty(\mathbb{R}_+)$, $0 \leq \chi \leq 1$, $\chi(r) = 0$ for $r \in [0, 1]$ and $\chi(r) = 1$ for $r \geq 2$. In the next remark we collect a few elementary properties of the functions u_* and h needed in the sequel.

Remark 5.1. The following statements hold true:

(i) There exists $\alpha, \beta \in C^1(\mathbb{R}_+)$ such that

$$u_*(r) = (\ln r)\alpha(r) + \beta(r) \quad \text{for all} \quad r > 0, \tag{5.18}$$

and $\alpha(0) = a_0$, $\alpha'(0) = 0$, $\alpha(r) = 0$ for all $r \geq 1$, $\beta(r) = 0$ for all $r \in [0, e^{-3}]$, $|\beta(r)| \leq \frac{c}{r}$ for all $r \geq 1$.

(ii) The function h can be represented as

$$h(r,\mu) = (\mu \ln \mu)\alpha(\mu r)\chi(r) + \mu\big[(\ln r)\chi(r)\big]\alpha(\mu r) + \mu\beta(\mu r)\chi(r) \quad \text{for all} \quad r, \mu > 0. \tag{5.19}$$

(iii) If we denote by $\tilde{\mu} : [0, e^{-1}] \to [0, e^{-1}]$ the inverse of the function $\mu \mapsto -\mu \ln \mu$ from $[0, e^{-1}]$ to itself, then the function $\tilde{h} : [0,\infty) \times [0, e^{-1}] \to \mathbb{R}$ defined by $\tilde{h}(r,\nu) = h(r, \tilde{\mu}(\nu))$, is C^1.

(iv) The function $\tilde{h}$ satisfies the following estimates

$$|\tilde{h}(r,\nu)| \leq c\frac{\chi(r)}{r} \quad \text{for all} \quad r \in \mathbb{R}_+,\ \nu \in [0, e^{-1}], \tag{5.20}$$

$$|\tilde{h}_\nu(r,\nu)| \leq c\chi(r)(\ln r + 1) \quad \text{for all} \quad r \in \mathbb{R}_+,\ \nu \in [0, e^{-1}]. \tag{5.21}$$

Proof. To prove (i), one writes u_* as a sum of two functions, one smoothly localized in a neighborhood of 0, and another one smoothly localized in a neighborhood of ∞. Then the conclusion follows immediately from (5.14). Moreover, one readily checks that (ii) follows from (i), and (iii) follows from (ii).

It remains to check (iv). Since $\tilde{u}_* \in L^\infty(\mathbb{R})$, it follows that $|h(r,\mu)| \leq c\frac{\chi(r)}{r}$ for all $r \in \mathbb{R}_+$, $\mu \in \mathbb{R}_+$, which proves (5.20). We obtain from (5.14) that

$$\Big|\frac{\tilde{u}_*'(\ln r)}{r}\Big| \leq c|\ln r| \quad \text{for all} \quad r \in (0, e^{-1}].$$

Since $\tilde{u}_*' \in L^\infty(\mathbb{R})$, we have that $\lim_{r\to\infty} \frac{\tilde{u}_*'(\ln r)}{r} = 0$, which implies that

$$\Big|\frac{\tilde{u}_*'(\ln r)}{r}\Big| \leq c\max\{|\ln r|, 1\} \leq c(|\ln r| + 1) \quad \text{for all} \quad r > 0.$$

It follows that

$$\begin{aligned}
|\tilde{h}_\nu(r,\nu)| = |h_\mu(r,\tilde{\mu}(\nu))\tilde{\mu}'(\nu)| &= \Big|\frac{\tilde{u}_*'(\ln(\tilde{\mu}(\nu)r))}{\tilde{\mu}(\nu)r}\tilde{\mu}'(\nu)\Big|\chi(r) \leq c\Big(|\ln r + \ln\tilde{\mu}(\nu)| + 1\Big)\chi(r)|\tilde{\mu}'(\nu)| \\
&\leq c\chi(r)(\ln r + 1)|\tilde{\mu}'(\nu)| + c\chi(r)|1 + \tilde{\mu}'(\nu)| \leq c\chi(r)(\ln r + 1)
\end{aligned}$$

for all $r > 0$, $\nu \in (0, e^{-1}]$. ∎

Next, we define the function $\mathcal{F} : H^2_{\eta,\mathrm{rad}}(\mathbb{R}^2) \times [0, e^{-1}] \times \mathbb{R} \to L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2)$ by

$$\mathcal{F}(w,\nu,\varepsilon) = \Delta_r(w + \tilde{h}(\cdot,\nu)) - (w + \tilde{h}(\cdot,\nu))^3 + \varepsilon V(\cdot, w + \tilde{h}(\cdot,\nu)). \tag{5.22}$$

In the next lemma we are going to prove that the map $\mathcal{F}$ is well-defined and C^1.

Lemma 5.2. *We have the following smoothness properties for $\mathcal{F}$:*

(i) The function $\mathcal{F}_1 : H^2_{\eta,\mathrm{rad}}(\mathbb{R}^2) \times [0, e^{-1}] \to L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2)$, defined by

$$\mathcal{F}_1(w, \nu) = \Delta_r(w + \tilde{h}(\cdot, \nu)) - (w + \tilde{h}(\cdot, \nu))^3, \tag{5.23}$$

is well defined and C^1.

(ii) The function $\mathcal{F}_2 : H^2_{\eta,\mathrm{rad}}(\mathbb{R}^2) \times [0, e^{-1}] \to L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2)$ defined by

$$\mathcal{F}_2(w, \nu) = V(\cdot, w + \tilde{h}(\cdot, \nu)) \tag{5.24}$$

is well defined and C^1.

Proof. We first show (i). We group the terms of $\mathcal{F}_1$ as follows:

$$\mathcal{F}_1(w, \nu) = \Delta_r w - w^3 - 3w^2\tilde{h}(\cdot, \nu) - 3w\tilde{h}^2(\cdot, \nu) + (\Delta_r \tilde{h}(\cdot, \nu) - \tilde{h}^3(\cdot, \nu)).$$

In what follows we will show that every term is well-defined and C^1. Since Δ_r is a bounded linear operator from $H^2_{\eta,\mathrm{rad}}(\mathbb{R}^2)$ to $L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2)$, we have that the first term is well-defined and C^1. Next, define $\mathcal{H}_1 : H^2_{\eta,\mathrm{rad}}(\mathbb{R}^2) \times L^\infty_{\mathrm{rad}}(\mathbb{R}^2) \times L^\infty_{\mathrm{rad}}(\mathbb{R}^2) \to L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2)$ by $\mathcal{H}_1(u, v, z) = uvz$. The map $\mathcal{H}_1$ is well-defined, multilinear and bounded, which implies that $\mathcal{H}_1$ is a C^1 function. In addition, from Sobolev's Embedding Theorem, we have that $H^2_{\eta,\mathrm{rad}}(\mathbb{R}^2) \hookrightarrow L^\infty_{\mathrm{rad}}(\mathbb{R}^2)$. Hence, the map

$$w \mapsto w^3 = \mathcal{H}_1(w, w, w) \text{ from } H^2_{\eta,\mathrm{rad}}(\mathbb{R}^2) \text{ to } L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2) \text{ is } C^1. \tag{5.25}$$

We next use Lemma 6.2 that is stated and proved in the appendix in order to show that the function

$$\nu \mapsto \tilde{h}(\cdot, \nu) \text{ from } [0, e^{-1}] \text{ to } L^2_{-\eta,\mathrm{rad}}(\mathbb{R}^2) \text{ is } C^1. \tag{5.26}$$

Therefore, we choose, in the notation of Lemma 6.2, $\omega(r) := re^{-2\eta r}$, $g_1(r) := \frac{c}{r}\chi(r)$ and $g_2(r) := c\chi(r)(\ln r + 1)$. One can readily check that $g_j \in L^2(\mathbb{R}_+; \omega(r)dr)$, $j = 1, 2$. Now, since $\tilde{h}$ is a C^1-function that satisfies (5.20), (5.21) we can apply Lemma 6.2 and find (5.26).

Define $\mathcal{H}_2 : L^\infty_{\eta,\mathrm{rad}}(\mathbb{R}^2) \times L^\infty_{\eta,\mathrm{rad}}(\mathbb{R}^2) \times L^2_{-\eta,\mathrm{rad}}(\mathbb{R}^2) \to L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2)$ by $\mathcal{H}_2(u, v, z) = uvz$. Again, the map $\mathcal{H}_2$ is well-defined, multilinear and bounded, hence C^1. Moreover, from Sobolev's Embedding Theorem, we also have that $H^2_{\eta,\mathrm{rad}}(\mathbb{R}^2) \hookrightarrow L^\infty_{\eta,\mathrm{rad}}(\mathbb{R}^2)$. Hence, the map

$$(w, \nu) \mapsto w^2\tilde{h}(\cdot, \nu) = \mathcal{H}_2(w, w, \tilde{h}(\cdot, \nu)) \text{ from } H^2_{\eta,\mathrm{rad}}(\mathbb{R}^2) \times [0, e^{-1}] \text{ to } L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2) \text{ is } C^1. \tag{5.27}$$

We now again use Lemma 6.2 from the appendix to show that the function

$$\nu \mapsto \tilde{h}^2(\cdot, \nu) \text{ from } [0, e^{-1}] \text{ to } L^2(\mathbb{R}_+) \text{ is } C^1. \tag{5.28}$$

This time, we choose, in the notation of Lemma 6.2, $\omega := 1$, $g_1(r) = c\frac{\chi^2(r)}{r^2}$, $g_2(r) = c\chi^2(r)\frac{\ln r + 1}{r}$ and $f(r, \nu) = \tilde{h}^2(r, \nu)$. Since $\tilde{h}$ is a C^1-function, we know that f is a C^1-function on $\mathbb{R}_+ \times [0, e^{-1}]$ and $g_j \in L^2(\mathbb{R}_+)$, $j = 1, 2$. From (5.20) and (5.21) we have that

$$|f(r, \nu)| \leq g_1(r) \quad \text{and} \quad |f_\nu(r, \nu)| \leq g_2(r) \quad \text{for all} \quad r \in \mathbb{R}_+,\ \nu \in [0, e^{-1}].$$

Applying again Lemma 6.2, we obtain (5.28).

The map $\mathcal{H}_3 : L^\infty(\mathbb{R}_+; r^{1/2}e^{\eta r}dr) \times L^2(\mathbb{R}_+) \to L^2_{\eta,\text{rad}}(\mathbb{R}^2)$ defined by $\mathcal{H}_2(u,v) = uv$ is well-defined, multilinear and bounded, which implies that $\mathcal{H}_2$ is a C^1 function.

Let $\psi \in C^\infty(\mathbb{R}_+)$ be a function such that $0 \le \psi \le 1$, $\psi(r) = 0$ for all $r \in [0, \frac{1}{2}]$ and $\psi(r) = 1$ for all $r \ge 1$. Since $\tilde{h}(r,\nu) = 0$ for all $r \in [0,1]$, $\nu \in [0, e^{-1}]$, we have that

$$\tilde{h}^2(\cdot,\nu) = \psi \tilde{h}^2(\cdot,\nu) \quad \text{for all} \quad \nu \in [0, e^{-1}]. \tag{5.29}$$

Next, we will prove that the linear operator $w \to \psi w : H^2_{\eta,\text{rad}}(\mathbb{R}^2) \to L^\infty(\mathbb{R}_+; r^{1/2}e^{\eta r}dr)$ is bounded. If $w \in H^2_{\eta,\text{rad}}(\mathbb{R}^2)$ then

$$re^{2\eta r}w^2(r) = 1/2e^{\eta}w^2(1/2) + \int_{\frac{1}{2}}^{r} \left[(2\eta s + 1)e^{2\eta s}w^2(s) + se^{2\eta s}w(s)w'(s)\right] ds \quad \text{for all} \quad r \ge \frac{1}{2},$$

which implies that

$$\begin{aligned} re^{2\eta r}w^2(r) &\le c\|w\|_\infty^2 + \int_{\frac{1}{2}}^{r} se^{2\eta s}|w(s)|^2 ds + \int_{\frac{1}{2}}^{r} se^{2\eta s}|w(s)w'(s)| ds \\ &\le c\|w\|_\infty^2 + c\|w\|^2_{H^2_{\eta,\text{rad}}(\mathbb{R}^2)} + \left(\int_{\frac{1}{2}}^{r} se^{2\eta s}|w(s)|^2 ds\right)^{1/2} \left(\int_{\frac{1}{2}}^{r} se^{2\eta s}|w'(s)|^2 ds\right)^{1/2} \\ &\le c\|w\|_\infty^2 + c\|w\|^2_{H^2_{\eta,\text{rad}}(\mathbb{R}^2)} \quad \text{for all} \quad r \ge \frac{1}{2}. \end{aligned}$$

Using again that $H^2_{\eta,\text{rad}}(\mathbb{R}^2) \hookrightarrow L^\infty_{\text{rad}}(\mathbb{R}^2)$, we find,

$$re^{2\eta r}\psi(r)w^2(r) \le c\|w\|_{H^2_{\eta,\text{rad}}(\mathbb{R}^2)} \quad \text{for all} \quad r \in \mathbb{R}_+,$$

which proves that the linear operator $w \to \psi w : H^2_{\eta,\text{rad}}(\mathbb{R}^2) \to L^\infty(\mathbb{R}_+; r^{1/2}e^{\eta r}dr)$ is bounded. Since $\mathcal{H}_3$ is C^1, we conclude from (5.28) and (5.29) that

$$(w,\nu) \mapsto w\tilde{h}^2(\cdot,\nu) = \mathcal{H}_3(\psi w, \tilde{h}^2(\cdot,\nu)) \text{ from } H^2_{\eta,\text{rad}}(\mathbb{R}^2) \times [0,e^{-1}] \text{ to } L^2_{\eta,\text{rad}}(\mathbb{R}^2) \text{ is } C^1. \tag{5.30}$$

It follows from (5.25), (5.27), and (5.30) that to finish the proof of (i) it is enough to show that the function $\nu \mapsto \Delta_r \tilde{h}(\cdot,\nu) - \tilde{h}^3(\cdot,\nu)$ from $[0,e^{-1}]$ to $L^2_{\eta,\text{rad}}(\mathbb{R}^2)$ is C^1. Since $\Delta_r u_* = u_*^3$ and $\tilde{h}(r,\nu) = \tilde{\mu}(\nu)u_*(r\tilde{\mu}(\nu))\chi(r)$ for all $r \in \mathbb{R}_+$, $\nu \in [0,e^{-1}]$, it follows that

$$\Delta_r \tilde{h}(r,\nu) - \tilde{h}^3(r,\nu) = 0 \quad \text{for all} \quad r \in \mathbb{R}_+ \setminus (1,2).$$

Thus, again from Lemma 6.2 in the appendix, it follows that $\nu \mapsto \Delta_r \tilde{h}(\cdot,\nu) - \tilde{h}^3(\cdot,\nu)$ from $[0,e^{-1}]$ to $L^2_{\eta,\text{rad}}(\mathbb{R}^2)$ is C^1, finishing the proof of (i).

We next prove (ii). Choosing $\omega(r) := e^{\eta r}$, $g_1(r) := \frac{c}{r}\chi(r)$ and $g_2(r) := c\chi(r)(\ln r + 1)$, one readily checks that $\lim_{r\to\infty} g_j(r)\omega(r) = 0$, $j = 1,2$. Since $\tilde{h}$ is a C^1-function, it follows from (5.20), (5.21), and Lemma 6.4 in the appendix that the function

$$\nu \mapsto \tilde{h}(\cdot,\nu) \text{ from } [0,e^{-1}] \text{ to } L^\infty_{-\eta,\text{rad}}(\mathbb{R}^2) \text{ is } C^1. \tag{5.31}$$

The embeddings $H^2_{\eta,\mathrm{rad}}(\mathbb{R}^2) \hookrightarrow L^\infty_{\mathrm{rad}}(\mathbb{R}^2) \hookrightarrow L^\infty_{-\eta,\mathrm{rad}}(\mathbb{R}^2)$ imply that

$$(w,\nu) \mapsto w + \tilde{h}(\cdot,\nu) \text{ from } H^2_{\eta,\mathrm{rad}}(\mathbb{R}^2) \times [0,e^{-1}] \text{ to } L^\infty_{-\eta,\mathrm{rad}}(\mathbb{R}^2) \text{ is } C^1. \tag{5.32}$$

From Lemma 6.5 we know that the map $u \to V(\cdot,u)$ from $L^\infty_{-\eta,\mathrm{rad}}(\mathbb{R}^2)$ to $L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2)$ is C^1, and we conclude that $\mathcal{F}_2$ is C^1. ∎

The next lemma, crucial in our Lyapunov-Schmidt reduction argument, is an immediate consequence of Theorem 1.2

Lemma 5.3. *We have the following Fredholm properties of the linearization.*

(i) Δ_r *is Fredholm on* $L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2)$ *with index* -1*;*

(ii) $\ker(\Delta_r, L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2)) = \{0\}$*;*

(iii) $\mathrm{im}(\Delta_r, L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2)) = \{f \in L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2) : \int_0^\infty rf(r)dr = 0\}$

Proof. To prove (i), note that $\Delta_r = \mathcal{L}_{\mathrm{rad}}$ with the special choice of $D=1$, $Q=R=0$ and $m=1$. It follows from Theorem 1.2 that Δ_r is Fredholm on $L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2)$ and $\mathrm{ind}(\Delta_r) = m - i(\eta I_2) = 1-2 = -1$.
Assertion (ii) follows from the fact that $1, \ln \notin H^2_{\eta,\mathrm{rad}}(\mathbb{R}^2)$.
It remains to show (iii). It follows from (i) and (ii) that $\mathrm{codim}\,\mathrm{im}(\Delta_r, L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2)) = 1$. Since, in addition, $1 \in \ker(\Delta_r, L^2_{-\eta,\mathrm{rad}}(\mathbb{R}^2))$, we infer that $\mathrm{im}(\Delta_r, L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2)) = \{1\}^\perp$, proving the lemma. ∎

Lemma 5.4. *The following assertions hold true:*

(i) There exists $\delta > 0$ *and* $E : [0,\delta] \times [-\delta,\delta] \to \mathbb{R}$ *a* C^1*-function such that*

$$\varepsilon \in [-\delta,\delta], \quad \nu \in [0,\delta] \quad \textit{and} \quad E(\nu,\varepsilon) = 0 \implies \textit{equation } (5.11) \textit{ has a solution.} \tag{5.33}$$

(ii) The function E *can be represented as follows*

$$E(\nu,\varepsilon) = E_0(\tilde{\mu}(\nu)) + \tilde{E}_1(\nu,\varepsilon) + \varepsilon E_2(\nu,\varepsilon), \tag{5.34}$$
$$\tilde{E}_1(\nu,\varepsilon) = \nu^2 z_{11}(\nu,\varepsilon) + \nu\varepsilon^2 z_{12}(\nu,\varepsilon) + \varepsilon^3 z_{13}(\varepsilon), \tag{5.35}$$
$$\tilde{E}_{1,\nu}(\nu,\varepsilon) = \nu^2 z_{21}(\nu,\varepsilon) + \nu\varepsilon z_{22}(\nu,\varepsilon) + \varepsilon^2 z_{23}(\nu,\varepsilon), \tag{5.36}$$

for all $\nu \in [0,\delta]$, $\varepsilon \in [-\delta,\delta]$*. Here* $E_0 \in C^1(\mathbb{R}_+)$, $E_0(0) = 0$, E_1, $\tilde{E}_1$, *and* E_2 *are* C^1*-functions on* $[0,\delta]\times[-\delta,\delta]$*, and* $E_0'(0) = a_0 \neq 0$ *and* z_{ij}, $i=1,2$, $j=1,2,3$ *are continuous functions.*

Remark 5.5. The expansions (5.35) and (5.36) are not sharp. For instance, $z_11(0,0) = 0$. Since those terms appear only as higher-order terms in the expansion, we do not attempt to isolate leading order terms in $\tilde{E}_1$.

Proof. To prove (i), let P_0 be the projection onto $\mathrm{im}(\Delta_r, L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2))$ and define the function $\tilde{\mathcal{F}} : H^2_{\eta,\mathrm{rad}}(\mathbb{R}^2) \times [0, e^{-1}] \times \mathbb{R} \to \mathrm{im}(\Delta_r, L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2))$ by $\tilde{\mathcal{F}}(w, \nu, \varepsilon) = P_0\mathcal{F}(w, \nu, \varepsilon)$. From Lemma 5.2 we have that $\mathcal{F}$ is C^1, which implies that $\tilde{\mathcal{F}}$ is C^1. Moreover, a simple computation shows that $\tilde{\mathcal{F}}(0, 0, 0) = 0$ and $\tilde{\mathcal{F}}_w(0, 0, 0) = P_0\Delta_r$. Since $P_0\Delta_r$ is a bounded, invertible linear operator from $H^2_{\eta,\mathrm{rad}}(\mathbb{R}^2)$ to $\mathrm{im}(\Delta_r, L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2))$, we conclude from the Implicit Function Theorem that there exists a $\delta > 0$, small enough, and a C^1-function $w_* : [0, \delta] \times [-\delta, \delta] \to H^2_{\eta,\mathrm{rad}}(\mathbb{R}^2)$ such that $w_*(0, 0) = 0$ and for any $(\nu, \varepsilon) \in [0, \delta] \times [-\delta, \delta]$

$$\tilde{\mathcal{F}}(w, \nu, \varepsilon) = 0 \Longleftrightarrow w = w_*(\nu, \varepsilon).$$

It follows from Lemma 5.3(iii) that for any $(\nu, \varepsilon) \in [0, \delta] \times [-\delta, \delta]$,

$$\mathcal{F}(w, \nu, \varepsilon) = 0 \Longleftrightarrow w = w_*(\nu, \varepsilon) \quad \text{and} \quad \langle \mathcal{F}(w, \nu, \varepsilon), 1\rangle = 0.$$

Here $\langle \cdot, \cdot \rangle$ represents the usual $L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2) - L^2_{-\eta,\mathrm{rad}}(\mathbb{R}^2)$ pairing. At this point, (i) follows immediately for

$$E(\nu, \varepsilon) := \langle \mathcal{F}(w_*(\nu, \varepsilon), \nu, \varepsilon), 1\rangle. \tag{5.37}$$

To prove (ii), we define the functions $E_1, E_2, \tilde{E}_1 : [0, \delta] \times [-\delta, \delta] \to \mathbb{R}$ and $E_0 : \mathbb{R}_+ \to \mathbb{R}$ as follows

$$E_j(\nu, \varepsilon) = \langle \mathcal{F}_j(w_*(\nu, \varepsilon), \nu, \varepsilon), 1\rangle, \ E_0(\mu) = -\int_0^\infty rh^3(r, \mu)dr, \ \tilde{E}_1(\nu, \varepsilon) = E_1(\nu, \varepsilon) - E_0(\tilde{\mu}(\nu)).$$

Let $F_\mu(r) = \mu u_*(\mu r)$. Since $\Delta_r u_* = u_*^3$, we calculate that $\Delta_r F_\mu = F_\mu^3$, which allows us to compute

$$\begin{aligned}
E_0(\mu) &= -\int_0^\infty rF_\mu^3(r)\chi^3(r)dr = -\int_0^\infty r\Delta_r F_\mu(r)\chi^3(r)dr = \int_0^\infty (rF'_\mu(r))'\chi^3(r)dr \\
&= -rF'_\mu(r)\chi(r)\Big|_{r=0}^{r=\infty} + 3\int_0^\infty rF'_\mu(r)\chi^2(r)\chi'(r)dr = 3\int_1^2 rF'_\mu(r)\chi^2(r)\chi'(r)dr
\end{aligned}$$

From Remark 5.1(i) we have that

$$F'_\mu(r) = (\mu^2 \ln \mu)\alpha'(\mu r)\mu\frac{\alpha(\mu r)}{r} + \mu(\ln r)\alpha'(\mu r) + \mu^2\beta'(\mu r) \quad \text{for all} \quad r \in [1, 2],$$

which implies that

$$E_0(\mu) = \mu a_1(\mu) + (\mu^2 \ln \mu)a_2(\mu) + \mu^2 a_3(\mu) \quad \text{for all} \quad \mu > 0,$$

where

$$a_1(\mu) = 3\int_1^2 \alpha(\mu r)\chi^2(r)\chi(r)dr, \quad a_2(\mu) = 3\int_1^2 r\alpha'(\mu r)\chi^2(r)\chi(r)dr,$$
$$a_3(\mu) = 3\int_1^2 \big[(\ln r)\alpha'(\mu r) + \beta'(\mu r)\big]\chi^2(r)\chi(r)dr.$$

Using again Remark 5.1(i), we conclude that $E_0 \in C^1(\mathbb{R}_+)$, $E_0(0) = 0$ and $E'_0(0) = a_0 \neq 0$.

To prove the expansions for $\tilde{E}_1$, we first note that, since $w_*(\nu,\varepsilon) \in H^2_{\eta,\mathrm{rad}}(\mathbb{R}^2)$ for all $(\nu,\varepsilon) \in [0,\delta]\times[-\delta,\delta]$, we have

$$\langle \Delta_r w_*(\nu,\varepsilon), 1\rangle = 0 \quad \text{for all} \quad (\nu,\varepsilon) \in [0,\delta]\times[-\delta,\delta]. \tag{5.38}$$

Also, since $\tilde{u}_*, \tilde{u}'_* \in L^\infty(\mathbb{R})$, we have

$$\langle \Delta_r \tilde{h}(\cdot,\nu), 1\rangle = \int_0^\infty r\Delta_r \tilde{h}(r,\nu) = r\tilde{h}_r(r,\nu)\Big|_{r=0}^{r=\infty} = 0 \quad \text{for all} \quad \nu \in [0,\delta]. \tag{5.39}$$

From (5.38) and (5.39) we obtain that

$$\tilde{E}_1(\nu,\varepsilon) = \langle \tilde{h}^3(\cdot,\nu) - (w_*(\nu,\varepsilon) + \tilde{h}(\cdot,\nu))^3, 1\rangle \quad \text{for all} \quad (\nu,\varepsilon) \in [0,\delta]\times[-\delta,\delta]. \tag{5.40}$$

To finish the proof of (ii), we note that we can write $\tilde{h}(\cdot,\nu) = \nu\tilde{\tilde{h}}(\cdot,\nu)$ and $w_*(\nu,\varepsilon) = \varepsilon\tilde{\tilde{w}}_*(\varepsilon) + \nu\tilde{w}_*(\nu,\varepsilon)$. Here, the functions $\tilde{w}_*$, $\tilde{\tilde{w}}_*$ and $\nu \mapsto \tilde{\tilde{h}}(\cdot,\nu)$ from $[0,\delta]$ to $L^2_{-\eta,\mathrm{rad}}(\mathbb{R}^2)$ are continuous functions. Plugging these expansions into (5.40), we obtain the representations (5.35) and (5.36). ∎

Proof. [of Theorem 1.4] After choosing $\delta > 0$ small enough, we can define the functions $F, \tilde{F}_1, F_2 : [0,\delta]\times[-\delta,\delta] \to \mathbb{R}$ by $F(\mu,\varepsilon) = E(-\mu\ln\mu,\varepsilon)$, $\tilde{F}_1(\mu,\varepsilon) = \tilde{E}_1(-\mu\ln\mu,\varepsilon)$ and $F_2(\mu,\varepsilon) = E_2(-\mu\ln\mu,\varepsilon)$. Next, since $E_0(0) = 0$ we can extend the function E_0 to $\mathbb{R}$ by $E_0(-\mu) = -E_0(\mu)$ so that $E_0 \in C^1(\mathbb{R})$. We also extend the functions $F, \tilde{F}_1, F_2$ to $[-\delta,\delta]^2$ by setting $F(-\mu,\varepsilon) = -F(\mu,\varepsilon) + 2F(0,\varepsilon)$, $\tilde{F}_1(-\mu,\varepsilon) = -\tilde{F}_1(\mu,\varepsilon) + 2\tilde{F}_1(0,\varepsilon)$ and $F_2(-\mu,\varepsilon) = -F_2(\mu,\varepsilon) + 2F_2(0,\varepsilon)$. One can readily verify that $F, \tilde{F}_1$, and F_2 are continuous on $[-\delta,\delta]^2$ and C^1 on $[-\delta,\delta]^2 \setminus (\{0\}\times[-\delta,\delta])$.

Define the functions $\gamma : [-\delta,\delta] \to \mathbb{R}$ and $\Lambda : [-\delta,\delta]^2 \to \mathbb{R}$ by $\gamma(\varepsilon) = \varepsilon F_2(0,\varepsilon)$ and

$$\Lambda(\xi,\varepsilon) = \begin{cases} \frac{1}{\gamma(\varepsilon)} F(-\frac{\gamma(\varepsilon)}{a_0}(1+\xi),\varepsilon), & \varepsilon \neq 0 \\ 0, \quad \varepsilon = 0 \end{cases}$$

Since E_2 is a C^1-function, we have the representation

$$E_2(\nu,\varepsilon) = E_2(0,\varepsilon) + \nu\tilde{E}_2(\nu,\varepsilon),$$

where $\tilde{E}_2$ is a continuous function. It follows that

$$\frac{\varepsilon}{\gamma(\varepsilon)} E_2(\nu,\varepsilon) = 1 + \frac{\varepsilon\nu}{\gamma(\varepsilon)}\tilde{E}_2(\nu,\varepsilon) \quad \text{for all} \quad \varepsilon \neq 0. \tag{5.41}$$

Using the fact that the functions E and E_0 are C^1, and using the representations (5.35), (5.36), and (5.41), we can show that Λ is continuous on $[-\delta,\delta]^2$, Λ_ξ is continuous on $[-\delta,\delta]^2$, $\Lambda(0,0) = 0$ and $\Lambda_\xi(0,0) = a_0 \neq 0$. From the Implicit Functions Theorem it follows that after taking $\delta > 0$ small enough for any $\varepsilon \in [-\delta,\delta]$ the equation $\Lambda(\xi,\varepsilon) = 0$ has a solution. It follows that for any $\varepsilon \in [-\delta,\delta]$ the equation $E(\nu,\varepsilon) = 0$ has a solution, $\nu_\varepsilon = -\mu_\varepsilon \ln\mu_\varepsilon$, where $\mu_\varepsilon = -\frac{\gamma(\varepsilon)}{a_0}(1+\xi_\varepsilon)$. Since $a_0 < 0$, $\xi_\varepsilon \in [-\delta,\delta]$ is small enough and from condition Positivity (P), $\gamma(\varepsilon) > 0$ for $\varepsilon \in [0,\delta]$ small enough, we conclude that $\mu_\varepsilon \geq 0$ and $\mu_\varepsilon = b_0\varepsilon + \mathcal{O}(\varepsilon^2)$, where

$$b_0 = -\frac{1}{a_0}\int_0^\infty V(r,0)rdr > 0.$$

Thus, by Lemma 5.4(i) we obtain that equation (5.11) has a solution. Moreover, using the ansatz (5.16) we infer that

$$u(r;\varepsilon) = h(r,\mu_\varepsilon) + w(r;\varepsilon),$$

for some function $w(\cdot;\varepsilon) \in H^2_{\eta,\mathrm{rad}}(\mathbb{R}^k,\mathbb{C}^m)$, $\eta \in (0,\delta_0/2)$. From the definition of h in (5.17) and Remark 5.1we have that for all $r \geq 2$

$$\begin{aligned}
u(r;\varepsilon) &= \mu_\varepsilon u_*(r\mu_\varepsilon) + \mathcal{O}(e^{-\eta r}) = b_0\varepsilon u_*(r\mu_\varepsilon) + \mathcal{O}(\varepsilon^2) + \mathcal{O}(e^{-\eta r}) \\
&= b_0\varepsilon \left[\ln(r\mu_\varepsilon)\alpha(r\mu_\varepsilon) + \beta(r\mu_\varepsilon)\right] + \mathcal{O}(\varepsilon^2) + \mathcal{O}(e^{-\nu r}) \\
&= b_0\varepsilon\Big[\big(\ln(b_0 r\varepsilon) + \mathcal{O}(\varepsilon)\big)\big(\alpha(b_0 r\varepsilon) + \mathcal{O}(\varepsilon^2)\big) + \beta(b_0 r\varepsilon) + \mathcal{O}(\varepsilon^2)\Big] + \mathcal{O}(\varepsilon^2) + \mathcal{O}(e^{-\nu r}) \\
&= b_0\varepsilon\left[\ln(b_0 r\varepsilon)\alpha(b_0 r\varepsilon) + \beta(b_0 r\varepsilon)\right] + \mathcal{O}(\varepsilon^2) + \mathcal{O}(e^{-\nu r}) = b_0\varepsilon u_*(b_0 r\varepsilon) + \mathcal{O}(\varepsilon^2) + \mathcal{O}(e^{-\nu r}) \\
&= \varepsilon v_*(r\varepsilon) + \mathcal{O}(\varepsilon^2) + \mathcal{O}(e^{-\nu r}),
\end{aligned}$$

where $v_* : (0,\infty) \to \mathbb{R}$ is defined by $v_*(r) = b_0 u_*(b_0 r)$. From (5.14) and the definition of h in (5.17) we obtain that

$$\Delta_r v_* = v_*^3, \quad \lim_{r\to 0}\frac{v_*(r)}{\ln r} \in (-\infty, 0) \quad \text{and} \quad \lim_{r\to\infty} v_*(r) = 0.$$

∎

6 Appendix

In this appendix, we state and prove some auxiliary lemmas needed in the proof of our main results.

Lemma 6.1. *Let X and Y be two Banach spaces and Z a closed subspace of Y of finite codimension. If $T \in \mathcal{B}(X,Y)$ is a bounded linear operator, injective with dense image. Then $T^{-1}Z$ is a closed subspace of X, of finite codimension and*

$$\operatorname{codim}(T^{-1}Z) = \operatorname{codim}(Z). \tag{6.1}$$

Proof. Since the operator T is bounded and the subspace Z is closed, $T^{-1}Z$ is closed. In what follows we denote by $[y] = y + Z$, $y \in Y$ the elements of the quotient space Y/Z with the usual norm. Next, we define the operator $S : X \to Y/Z$ by $Sx = [Tx]$. Since the operator T is linear and bounded, one readily checks that the operator S is linear and bounded.

Next, we will prove that the image of S is dense in Y/Z, that is $\overline{\operatorname{im} S} = Y/Z$. Let $[y]$, $y \in Y$, be an element of the closure of $\operatorname{im} S$. Since, by hypothesis, the image of the operator T is dense in Y, we can construct a sequence $(x_n)_{n\geq 1}$ of vectors of X such that $Tx_n \to y$ as $n \to \infty$ in Y. Since the canonical inclusion $y \to [y] : Y \to Y/Z$ is a bounded linear operator, we obtain that $Sx_n \to [y]$ as $n \to \infty$ in Y/Z, proving that $\overline{\operatorname{im} S} = Y/Z$. Since Z has finite codimension, it follows that the quotient space Y/Z has finite dimension, which implies that all of its subspaces are closed. Thus, $\operatorname{im} S = Y/Z$.

From the definition of S we can easily see that $\ker S = T^{-1}Z$. It follows that

$$X/T^{-1}Z = X/\ker S \cong \operatorname{im} S = Y/Z.$$

Thus,

$$\operatorname{codim}(T^{-1}Z) = \dim(X/T^{-1}Z) = \dim(Y/Z) = \operatorname{codim}(Z).$$

■

A key element of the argument given in Section 4 is to prove that several Banach space valued functions are C^1. Below we prove a couple of auxiliary lemmas that give necessary conditions for such functions to be C^1.

Lemma 6.2. *Let $f : \mathbb{R}_+ \times [0,a] \to \mathbb{R}$ be a C^1-function, $g_1, g_2 \in L^2(\mathbb{R}_+; \omega(r)dr)$ such that*

(i) $|f(r,\nu)| \le g_1(r)$ for all $r \in \mathbb{R}_+$, $\nu \in [0,a]$;

(ii) $|f_\nu(r,\nu)| \le g_2(r)$ for all $r \in \mathbb{R}_+$, $\nu \in [0,a]$.

Then the map $H_f : [0,a] \to L^2(\mathbb{R}_+; \omega(r)dr)$ defined by $H_f(\nu) = f(\cdot,\nu)$ is a C^1-function and $H_f'(\nu) = f_\nu(\cdot,\nu)$ for all $\nu \in [0,a]$.

Proof. From (i) and the fact that the function f is C^1 it follows that the map H_f is well-defined. Similarly, from (ii) and the fact that f is a C^1-function, we conclude that $f_\nu(\cdot,\nu) \in L^2(\mathbb{R}_+; \omega(r)dr)$.

Let $\nu_0 \in [0,a]$ and $(\nu_n)_{n\ge1}$ be a numerical sequence in $[0,a]$ such that $\nu \to \nu_0$ as $n \to \infty$ and $\nu_n \ne \nu_0$ for all $n \ge 1$. Then

$$\left\| \frac{1}{\nu_n - \nu_0}(H_f(\nu_n) - H_f(\nu_0)) - f_\nu(\cdot,\nu_0) \right\|^2_{L^2(\mathbb{R}_+,\omega(r)dr)} = \int_0^\infty F_n(r)dr,$$

where

$$F_n(r) = \left| \frac{1}{\nu_n - \nu_0}(f(r,\nu_n) - f(r,\nu_0)) - f_\nu(r,\nu_0) \right|^2 \omega(r),\ n \ge 1. \tag{6.2}$$

Since f is a C^1-function, it follows that $F_n(r) \to 0$ as $n \to \infty$ for all $r \in \mathbb{R}_+$. In addition we estimate

$$|F_n(r)| \le \left[2\left| \frac{(f(r,\nu_n) - f(r,\nu_0))}{\nu_n - \nu_0} \right|^2 + 2|f_\nu(r,\nu_0|^2 \right] \omega(r) \le 4g_2^2(r)\omega(r) := F(r)$$

for all $r \in \mathbb{R}_+$, $n \ge 1$. Since $g_2 \in L^2(\mathbb{R}_+, \omega(r)dr)$, we have that $F \in L^1(\mathbb{R}_+)$. From Lebesgue's Dominated Convergence Theorem we conclude that

$$\int_0^\infty F_n(r)dr \to 0 \quad \text{as} \quad n \to \infty,$$

which proves that the map H_f is differentiable on $[0,a]$ and $H_f'(\nu) = f_\nu(\cdot,\nu)$ for all $\nu \in [0,a]$.

To prove the continuity of H'_f consider again $\nu_0 \in [0,a]$ and $(\nu_n)_{n\geq 1}$ a numerical sequence in $[0,a]$ such that $\nu \to \nu_0$ as $n \to \infty$. Then

$$\|(H'_f(\nu_n) - H'_f(\nu_0))\|^2_{L^2(\mathbb{R}_+,\omega(r)dr)} = \int_0^\infty \tilde{F}_n(r)dr,$$

where

$$\tilde{F}_n(r) = |(f_\nu(r,\nu_n) - f_\nu(r,\nu_0))|^2\omega(r),\ n \geq 1. \tag{6.3}$$

Since f is a C^1-function, it follows that $\tilde{F}_n(r) \to 0$ as $n \to \infty$ for all $r \in \mathbb{R}_+$. Moreover, it follows from (ii) that

$$|F_n(r)| \leq 4g_2^2(r)\omega(r) := F(r) \quad \text{for all} \quad r \in \mathbb{R}_+,\ n \geq 1.$$

Using Lebesgue's Dominated Convergence Theorem again, we conclude that

$$\int_0^\infty \tilde{F}_n(r)dr \to 0 \quad \text{as} \quad n \to \infty,$$

which shows that the map H_f is C^1, proving the lemma. ∎

Next, we recall a well-known result from the theory of uniformly continuous functions.

Remark 6.3. If $f : \mathbb{R}_+ \times [0,a]$ is continuous function, $g : \mathbb{R}_+ \to \mathbb{R}_+$, $(\nu_n)_{n\geq 1}$ is a numerical sequence in $[0,a]$ with $\nu_n \to \nu_0$ as $n \to \infty$ such that

(i) $|f(r,\nu)| \leq h(r)$ for all $r \in \mathbb{R}_+$, $\nu \in [0,a]$;

(ii) $\lim\limits_{r\to\infty} h(r) = 0$,

then $f(\cdot,\nu_n) \to f(\cdot,\nu_0)$ as $n \to \infty$ in $L^\infty(\mathbb{R}_+)$.

Proof. One can readily see that it follows from (i) and (ii) that f is uniformly continuous on $\mathbb{R}_+ \times [0,a]$, which proves the remark. ∎

Lemma 6.4. *Let $f : \mathbb{R}_+ \times [0,a] \to \mathbb{R}$ be a C^1-function, ω a continuous weight function, $g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

(i) $|f(r,\nu)| \leq g_1(r)$ for all $r \in \mathbb{R}_+$, $\nu \in [0,a]$;

(ii) $|f_\nu(r,\nu)| \leq g_2(r)$ for all $r \in \mathbb{R}_+$, $\nu \in [0,a]$;

(iii) $\lim\limits_{r\to\infty} g_j(r)\omega(r) = 0$, $j = 1,2$.

Then the map $H_f : [0,a] \to L^\infty(\mathbb{R}_+,\omega(r)dr)$ defined by $H_f(\nu) = f(\cdot,\nu)$ is a C^1-function and $H'_f(\nu) = f_\nu(\cdot,\nu)$ for all $\nu \in [0,a]$.

Proof. Since f is a C^1-function and ω is continuous, we have that $\omega(\cdot)f(\cdot,\nu), \omega(\cdot)f_\nu(\cdot,\nu) \in L^\infty_{\mathrm{loc}}$. In addition, from the hypothesis, it follows that $\lim\limits_{r\to\infty} f(r,\nu)\omega(r) = \lim\limits_{r\to\infty} f_\nu(r,\nu)\omega(r) = 0$ for all $\nu \in [0,a]$, which proves that $f(\cdot,\nu), f_\nu(\cdot,\nu) \in L^\infty(\mathbb{R}_+, \omega(r)dr)$. Thus, H_f is well-defined.

Let $\nu_0 \in [0,a]$ and $(\nu_n)_{n\geq 1}$ be a numerical sequence in $[0,a]$ such that $\nu \to \nu_0$ as $n \to \infty$ and $\nu_n \neq \nu_0$ for all $n \geq 1$. First, we define the sequence of functions $(G_n)_{n\geq 1}$, $G_n : \mathbb{R}_+ \to \mathbb{R}_+$ as follows

$$G_n(r) = \left| \frac{1}{\nu_n - \nu_0}(f(r,\nu_n) - f(r,\nu_0)) - f_\nu(r,\nu_0) \right| \omega(r). \tag{6.4}$$

Next, we define the function $\tilde{f} : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\tilde{f}(r,\nu) = \left[\int_0^1 f_\nu(r, \nu_0 + s(\nu-\nu_0))ds - f_\nu(r,\nu_0) \right] \omega(r). \tag{6.5}$$

Since f is C^1 and ω is continuous, we infer that $\tilde{f}$ is continuous on $\mathbb{R}_+ \times [0,a]$. Moreover, it follows from (ii) that

$$|\tilde{f}(r,\nu)| \leq 2g_2(r)\omega(r) =: h(r).$$

In addition, it follows from (iii) that $\tilde{f}$ satisfies the conditions from Remark 6.3 and, since $G_n(r) = \tilde{f}(r,\nu_n)$, we conclude that

$$G_n \to 0 \quad \text{as} \quad n \to \infty \quad \text{in} \quad L^\infty(\mathbb{R}_+).$$

Thus, the map H_f is differentiable and $H_f'(\nu) = f_\nu(\cdot,\nu)$.

To prove the continuity of H_f' consider again $\nu_0 \in [0,a]$ and $(\nu_n)_{n\geq 1}$ a numerical sequence in $[0,a]$ such that $\nu \to \nu_0$ as $n \to \infty$. Define the function $\tilde{\tilde{f}} : \mathbb{R}_+ \times [0,a] \to \mathbb{R}$ by $\tilde{\tilde{f}}(r) = \omega(r)f_\nu(r,\nu)$. Since f is C^1 and ω is continuous, we obtain that $\tilde{\tilde{f}}$ is continuous on $\mathbb{R}_+ \times [0,a]$. In addition, from (ii) we have that

$$|\tilde{\tilde{f}}(r,\nu)| \leq g_2(r)\omega(r) \leq h(r).$$

Finally, from (iii) and Remark 6.3 it follows that

$$H_f(\nu_n) = \tilde{\tilde{f}}(\cdot,\nu_n) \to \tilde{\tilde{f}}(\cdot,\nu_0) = H_f(\nu_0) \quad \text{as} \quad n \to \infty \quad \text{in} \quad L^\infty(\mathbb{R}_+),$$

proving the continuity of H_f' and finishing the proof of the lemma. ∎

Lemma 6.5. *Assume that condition (V) holds. Then, for every $\eta \in (0, \delta_0/2)$ the map $\mathcal{V} : L^\infty_{-\eta,\mathrm{rad}}(\mathbb{R}^2) \to L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2)$ defined by $\mathcal{V}(u) = V(\cdot,u)$ is C^1.*

Proof. Let $u_0 \in L^\infty_{-\eta,\mathrm{rad}}(\mathbb{R}^2)$ and define the linear operator $L_0 : L^\infty_{-\eta,\mathrm{rad}}(\mathbb{R}^2) \to L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2)$ by $L_0u = V_u(\cdot,u_0)u$. To show that the operator L_0 is bounded, we estimate

$$\begin{aligned}\|L_0u\|^2_{L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2)} &= \int_0^\infty re^{2\eta r}|V_u(r,u_0(r))u(r)|^2dr \leq c\int_0^\infty re^{-2(\delta_0-2\eta)r}|e^{-\eta r}u(r)|^2dr \\ &\leq c\left(\int_0^\infty re^{-2(\delta_0-2\eta)r}dr\right)\|u\|^2_{L^\infty_{-\eta,\mathrm{rad}}(\mathbb{R}^2)}.\end{aligned}$$

Let $(u_n)_{n\geq 1}$ be a sequence of functions from $L^\infty_{-\eta,\mathrm{rad}}(\mathbb{R}^2)$, such that $u_n \to 0$ as $n \to \infty$ in $L^\infty_{-\eta,\mathrm{rad}}(\mathbb{R}^2)$ and $u_n \neq 0$ for all $n \geq 1$. Then

$$\frac{1}{\|u_n\|_{L^\infty_{-\eta,\mathrm{rad}}(\mathbb{R}^2)}}\Big\|\mathcal{V}(u_0+u_n)-\mathcal{V}(u_0)-L_0u_n\Big\|^2_{L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2)} = \int_0^\infty H_n(r)dr,$$

where

$$H_n(r) = \frac{1}{\|u_n\|_{L^\infty_{-\eta,\mathrm{rad}}(\mathbb{R}^2)}}\Big|V(r,u_n(r)+u_0(r))-V(r,u_0(r))-V_u(r,u_0(r))u_n(r)\Big|^2,\ n\geq 1.$$

Since $|u_n(r)| \leq e^{\eta r}\|u_n\|_{L^\infty_{-\eta,\mathrm{rad}}(\mathbb{R}^2)}$ for all $r \in \mathbb{R}_+$ and $n \geq 1$, we have that $u_n(r) \to 0$ as $n \to \infty$ for all $r \in \mathbb{R}_+$. From the fact that V is a C^1-function we obtain that

$$H_n(r) \to 0 \quad \text{as} \quad n \to \infty \quad \text{for all} \quad r \in \mathbb{R}_+. \tag{6.6}$$

From condition (V) and using again the fact that V is a C^1 function we find the estimate

$$\begin{aligned}|H_n(r)| &\leq 2re^{2\eta r}\frac{1}{\|u_n\|_{L^\infty_{-\eta,\mathrm{rad}}(\mathbb{R}^2)}}\left[\Big|V(r,u_n(r)+u_0(r))-V(r,u_0(r))\Big|^2+\Big|V_u(r,u_0(r))u_n(r)\Big|^2\right]\\ &\leq cre^{2(\eta-\delta_0)r}\frac{|u_n(r)|}{\|u_n\|_{L^\infty_{-\eta,\mathrm{rad}}(\mathbb{R}^2)}} \leq cre^{-(\delta_0-2\eta)r} \quad \text{for all} \quad r\in\mathbb{R}_+,\ n\geq 1.\end{aligned} \tag{6.7}$$

From (6.6), (6.7) and Lebesgue's Dominated Convergence Theorem it follows that

$$\int_0^\infty H_n(r)dr \to 0 \quad \text{as} \quad n\to\infty,$$

which proves that the map $\mathcal{V}$ is Frechet differentiable.

To finish the proof of the lemma, consider again $u_0 \in L^\infty_{-\eta,\mathrm{rad}}(\mathbb{R}^2)$ and $(u_n)_{n\geq 1}$ a sequence of functions from $L^\infty_{-\eta,\mathrm{rad}}(\mathbb{R}^2)$ such that $u_n \to 0$ as $n\to\infty$ in $L^\infty_{-\eta,\mathrm{rad}}(\mathbb{R}^2)$. Define the linear operators $L_n, L_0 : L^\infty_{-\eta,\mathrm{rad}}(\mathbb{R}^2) \to L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2)$ by $L_nu = V_u(\cdot,u_n+u_0)u$, $L_0u = V_u(\cdot,u_0)u$. To prove the lemma it is enough to show that the $L_n \to L$ as $n\to\infty$ in the operator norm. For any $u \in L^\infty_{-\eta,\mathrm{rad}}(\mathbb{R}^2)$ and any $n \geq 1$ we estimate

$$\begin{aligned}\|L_nu-L_0u\|^2_{L^2_{\eta,\mathrm{rad}}(\mathbb{R}^2)} &= \int_0^\infty re^{2\eta r}|V_u(r,u_n(r)+u_0(r))-V_u(r,u_0(r))|^2|u(r)|^2dr\\ &\leq \Big(\int_0^\infty re^{4\eta r}|V_u(r,u_n(r)+u_0(r))-V_u(r,u_0(r))|^2dr\Big)\|u\|^2_{L^\infty_{-\eta,\mathrm{rad}}(\mathbb{R}^2)}.\end{aligned}$$

This estimate implies that

$$\|L_n-L_0\| \leq \int_0^\infty \tilde{H}_n(r)dr, \tag{6.8}$$

where

$$\tilde{H}_n(r) := re^{4\eta r}|V_u(r,u_n(r)+u_0(r))-V_u(r,u_0(r))|^2,\ n\geq 1. \tag{6.9}$$

Since $|u_n(r)| \le e^{\eta r}\|u_n\|_{L^\infty_{-\eta,\mathrm{rad}}(\mathbb{R}^2)}$ for all $r \in \mathbb{R}_+$ and $n \ge 1$, we have that $u_n(r) \to 0$ as $n \to \infty$ for all $r \in \mathbb{R}_+$. Using again the fact that V is a C^1 function we obtain that $\tilde{H}_n(r) \to 0$ as $n \to \infty$ for all $r \in \mathbb{R}_+$. In addition, we have from condition (V) that

$$|\tilde{H}_n(r)| \le cre^{-2(\delta_0 - 2\eta)r} \quad \text{for all} \quad r \in \mathbb{R}_+,\ n \ge 1.$$

From Lebesgue's Dominated Convergence Theorem it follows that

$$\int_0^\infty \tilde{H}_n(r)dr \to 0 \quad \text{as} \quad n \to \infty.$$

Together (6.8), the lemma follows shortly. ∎

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