Colliding dissipative pulses—the shooting manifold

Arnd Scheel
School of Mathematics
University of Minnesota
206 Church St. SE
Minneapolis, MN 55455
scheel@math.umn.edu

J. Douglas Wright
School of Mathematics
University of Minnesota
206 Church St. SE
Minneapolis, MN 55455
jdoug@math.umn.edu

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Abstract

We study multi-pulse solutions in excitable media. Under the assumption that a single pulse is asymptotically stable, we show that there is a well-defined "shooting manifold", consisting of two pulses traveling towards each other. In the phase space, the two-dimensional manifold is a graph over the manifold of linear superpositions of two pulses located at x_1 and x_2, with $x_1 - x_2 \gg 1$. It is locally invariant under the dynamics of the reaction-diffusion system and uniformly asymptotically attracting with asymptotic phase. The main difficulty in the proof is the fact that the linearization at the leading order approximation is strongly non-autonomous since pulses approach each other with speed of order one.

1 Introduction

Traveling pulses are a ubiquitous phenomenon in excitable media. The most prominent example is the FitzHugh-Nagumo equation, derived as a simplified model of signal propagation in nerve axons [5]. Similar models of excitable media have been used in a variety of physical systems, for instance the CO-oxidation on Pt(110)-surfaces[3] or the dynamics of calcium waves in cell tissue [8]. Existence and stability of a single traveling pulse are now well established; see [6] and the references therein. There are also numerous results on existence and stability of multi-pulses, where widely spaced copies of the single excitation pulse travel in the same direction; see for example [9, 4, 12]. Typically, excitation pulses appear to annihilate in a head-on collision when traveling in opposite directions. More systematic recent studies have however exhibited a plethora of possible different scattering phenomena in the head-on collision; see for example [10].

The purpose of this paper is to provide a rigorous foundation for the analysis of such nonlinear scattering processes. The main result shows that the scattering problem around the head-on collision of pulses is completely described by the ω-limit set of a single trajectory. We call this trajectory the *shooting manifold*, as it consists of two copies of the traveling pulse for $t < -T_* \ll -1$, located at positions $x_\pm$, so that $x_\pm \to \pm\infty$ as $t \to -\infty$.

To be specific, consider the FitzHugh-Nagumo equation,

$$\begin{aligned} u_t &= u_{xx} + f(u) - v \\ v_t &= d^2 v_{xx} + \epsilon\,(u - \gamma v) \end{aligned} \tag{1}$$

where the dependent variables u and v are real valued, $x, t \in \mathbf{R}$, $f(u) = u(\alpha - u)(u-1)$ and d, γ, ϵ, α are positive constants. Note that, compared to the original model, we included a small diffusion coefficient $d > 0$ for the second variable, as this reduces the complexity of several of our arguments. We believe that our results should be easily adaptable to the special case $d = 0$. Letting $U := (u, v)^t$, we will occasionally rewrite (1) as

$$U_t = DU_{xx} + F(U)$$

with D, F defined implicitly. Here and throughout this document, when we say $U \in X$, where X is some Banach space (like L^2 or BU), we mean that each component of U is in X. The existence and stability of pulses for this version of FitzHugh-Nagumo can be inferred from the existence and stability for $d = 0$, provided d is sufficiently small; see Alexander, Gardner & Jones (in [1]). Specifically, there exists a unique positive number c and function $Q(\xi)$ such that $U(x,t) = Q(x + ct)$ solves (1). The function Q is infinitely differentiable, unique modulo spatial translations and there exists $\beta_0 > 0$ such that $\mathrm{e}^{\beta_0|\xi|}\dfrac{d^nQ}{d\xi^n}(\xi)$ is in L^2 for all n. Note that (1) is invariant under reflection in x, so that $Q(-x+ct)$ solves the equation as well. We denote $Q_\pm(x \pm ct) := (q_\pm(x \pm ct), p_\pm(x \pm ct))^t := Q(\pm(x + ct))$. For brevity, we also write $\xi_\pm := x \pm ct$.[1]

The fact that the solutions $Q_\pm(\xi_\pm)$ are stable and exponentially localized indicates that there ought to be solutions which are roughly the linear superposition of well-separated pulses. For example,

$$U(x,t) \sim Q_+(\xi_+) + Q_-(\xi_-)$$

would be a solution in which a pair of pulses head in from spatial infinity towards each other. Numerical simulation bears out this intuition—the pulses do not appear to interact at all until they "hit" each other.[2]

In this paper, we analytically confirm the existence of this sort of solution for times up to the point where the pulses begin to strongly interact. Our first main result is:

Theorem 1. *There exist $\delta < 0$, $T^\star > 0$ and $b \in (0, \beta_0)$ such that the following is true. There exists a unique solution of (1) of the form*

$$U^\star(x,t) := Q_+(\xi_+) + Q_-(\xi_-) + \mathrm{e}^{-\delta t}\rho(x,t)$$

where $\rho(x,t) \in L^2((-\infty, -T^\star), H^2) \cap H^1((-\infty, -T^\star), L^2)$. Moreover $\rho(x,t) := \rho_+(\xi_+, t) + \rho_-(\xi_-, t)$ with $\mathrm{e}^{b\sqrt{\xi^2+1}}\rho_\pm(\xi, t) \in L^2((-\infty, -T^\star), H^2) \cap H^1((-\infty, -T^\star), L^2)$.

We will refer to $U^\star$ as the *two-pulse*. Notice that the error ρ goes to zero exponentially fast as $t \to -\infty$. The number $-T^\star$ should be thought of as the time at which linear superposition fails to provide an accurate view of the solution. Moreover, the error term is itself the sum of a right and left moving piece which decay exponentially quickly (in space) away from their centers of mass.

Equation (1) is invariant under translation in space and time, and thus we conclude that (inside H^1) there is an invariant two-dimensional manifold which consists of temporal and spatial shifts of $U^\star$. As solutions on this manifold consist of two pulses "shooting" at each other, we call this set *the shooting manifold.* It is

$$\mathcal{M}_{\text{shoot}} := \left\{ U^\star(\cdot - x_0, -T) \;\middle|\; x_0 \in \mathbf{R},\ T \geq T^\star \right\}.$$

One can think of this manifold as being parameterized by the time at which the pulses begin to interact and the spatial coordinate of the point halfway between the two fronts when this

[1] We note that we will always view $\xi_\pm$ as functions of x and t, not as independent variables in their own right. We use ξ as the independent variable which corresponds to ξ_+ or ξ_-.

[2] It is at this point when pulses typically annihilate in the FitzHugh-Nagumo equation.

event takes place. Notice that $\mathcal{M}_{\text{shoot}}$ as defined is a manifold with a boundary, $\partial\mathcal{M}_{\text{shoot}}$. The boundary, of course, is where the pulses begin to interact strongly.

Our second main result is that the shooting manifold is stable in the sense that functions in H^1 which are sufficiently close to $\mathcal{M}_{\text{shoot}}$ (but far from $\partial\mathcal{M}_{\text{shoot}}$), when evolved according to (1), decay exponentially quickly to a trajectory on $\mathcal{M}_{\text{shoot}}$ until such time as that solution reaches the boundary. Specifically we have:

Theorem 2. *There exist positive constants a, μ_1, μ_2 and C such that the following is true. Suppose that*

$$\operatorname{dist}_{H^1}(U_0, \mathcal{M}_{\text{shoot}}) \le \mu_1$$

and

$$\operatorname{dist}_{H^1}(U_0, \partial\mathcal{M}_{\text{shoot}}) \ge \mu_2,$$

where $\operatorname{dist}_{H^1}(U_0, A) := \inf\{|U_0 - U|_{H^1} \bigm| U \in A\}$*. Then there exist $x_0 \in \mathbf{R}$ and $T > T^\star$ such that, if $U(x,t)$ is the solution of* (1) *with initial data $U(x,0) = U_0(x)$ and*

$$\varphi(x,t) := \mathrm{e}^{at}\left(U(x,t) - U^\star(x - x_0, t - T)\right),$$

we have

$$\|\varphi\|_{L^2((0,-T^\star+T),H^2)\cap H^2((0,-T^\star+T),L^2)} \le C\operatorname{dist}_{H^1}(U_0, \mathcal{M}_{\text{shoot}}).$$

Remark 3. *Our proofs for Theorems 1 and 2 do not strongly rely on the specific form of the nonlinearity F, the matrix D or even the fact that we have a two-component system. We have chosen to work with a specific example to make our main ideas and strategies more concrete. All that really matters for our method is that the pulses are localized and stable. Moreover, the techniques we use here should be adaptable to reaction diffusion equations with $x \in \mathbf{R}^n$.*

Remark 4. *Taken together, Theorems 1 and 2 imply that the head-on collision of two pulses is a well-defined scattering problem. By this we mean that investigations into the nature of the strong interactions which take place during the collision will not be affected by specific choices for the initial data which lead to this collision. In particular, the result of the scattering process can be tested in a reliable way by direct numerical simulations, initiated by the sum of two copies of the traveling pulse, spaced sufficiently far apart; our result guarantees exponential convergence of this simulation as the initial spacing goes to infinity. When the collision results in a stable pattern, such as the quiescent state found after annihilation, our result together with an asymptotic stability result for the asymptotic state allows for semi-rigorous numerical studies, since errors of numerical simulations need only be controlled on finite time-intervals.*

A single pulse solution, $Q_\pm$, when viewed in the moving reference frame $\xi_\pm$ appears stationary. This simple observation is crucial to proving the existence and stability of the pulse. However, the two-pulse solution has non-trivial time dependence and this is the principal complication which we must address in order to prove Theorems 1 and 2. In the following, we give a rough outline of how we address these issues.

To prove existence of the two-pulse, we make the *Ansatz*

$$U^\star(x,t) := Q_+(\xi_+) + Q_-(\xi_-) + R(x,t) \tag{2}$$

and plug this into (1). (Note that $R(x,t) = \mathrm{e}^{-\delta t}\rho(x,t)$.) We arrive at the following equation for $R = (r_1, r_2)^t$:

$$R_t = A(t)R + H_1(t) + N_1(R) \tag{3}$$

where

$$A(t) := D\partial_x^2 + F'(Q_+(\xi_+) + Q_-(\xi_-)),$$
$$H_1(t) := F\left(Q_+(\xi_+) - Q_-(\xi_-)\right) - F\left(Q_+(\xi_+)\right) - F\left(Q_-(\xi_-)\right) q_+^2(\xi_+)$$

and

$$N_1(R) := F\left(Q_+(\xi_+) + Q_-(\xi_-) + R\right) - F\left(Q_+(\xi_+) + Q_-(\xi_-)\right) - F'\left(Q_+(\xi_+) + Q_-(\xi_-)\right) R.$$

Proving Theorem 1 is equivalent to showing that (3) possess a unique solution which decays exponentially quickly as $t \to -\infty$. Note that $H_1(t)$ is an inhomogeneous term and $N_1(R)$ is nonlinear. We will show that H_1 is small and that $N_1(R)$ is $O(R^2)$ later, and neither computation is particularly complicated. The difficulty is that the linear operator $A(t)$ is non-autonomous.

This operator also shows up when we examine the stability of $U^\star$. To study its stability we let $U(x,t) = U^\star(x,t) + W(x,t)$ and substitute this into (1). We find

$$W_t = A(t)W + G_1(t)W + N_2(W) \tag{4}$$

where

$$G_1(t) := F'(U^\star(x,t)) - F'(Q_+(\xi_+) + Q_-(\xi_-))$$

and

$$N_2(W) := F(U^\star + W) - F(U^\star) - F'(U^\star)W.$$

Notice that if R decays exponentially in time, then $G_1(t)$ will be small. Stability follows from showing that solutions of (4) with small initial data decay in time.

Since both (3) and (4) are abstractly of the form

$$W_t = A(t)W + H \tag{5}$$

where H is some combination of nonlinearities, small linear terms and inhomogeneities, proving the main theorems will be a consequence of solving this equation with good estimates for the solution.

Though non-autonomous, $A(t)$ is closely related to the autonomous operators

$$A_\pm := D\partial_\xi^2 \mp c\partial_\xi + F'(Q_\pm(\xi))$$

which arise when linearizing (1) about the $Q_\pm$ in the moving reference frames $\xi_\pm$. Much is known about these operators, and we discuss these in Section 2.

Our strategy for solving equation (5) is as follows. Let $\chi_+(x)$ be a C^∞ function which is equal to one on $(2,\infty)$ and zero on $(-\infty,-2)$ and satisfies $0 \le \chi'(x) \le 1$ for all x. Let $\chi_- := 1 - \chi_+$. (These are smooth cut off functions for the positive and negative half lines.) Let $H_\pm(\xi_\pm, t) := \chi_\pm(x)H(x,t)$. Suppose that $W_\pm$ are solutions of the equations

$$\partial_t W_\pm := A_\pm W + H_\pm. \tag{6}$$

If we set $\tilde{W}(x,t) := W_+(\xi_+,t) + W_-(\xi_-,t)$, then a direct computation yields

$$\tilde{W}_t = A(t)\tilde{W} + H - \mathrm{Res}$$

where

$$\mathrm{Res} := B_+(\xi_+,t)W_-(\xi_-,t) + B_-(\xi_-,t)W_+(\xi_+,t)$$

and

$$B_\pm(t) := F'(Q_+(\xi_+) + Q_-(\xi_-)) - F'(Q_\mp).$$

As we shall demonstrate, the error Res has small norm. We then can use a Neumann series to construct an actual solution of the problem.

Remark 5. *This strategy for solving* (5) *is an adaptation and extension of methods used by Zelik & Mielke in [13]. In that paper the authors prove (amongst other things) the existence of multi-pulse solutions in dissipative equations where the pulses do not move (or move only slowly) with respect to one another. Our use of this method to prove stability appears to be novel.*

The main issues in using this strategy are **(a)** to properly formulate the function spaces over which it is appropriate to work, **(b)** to solve equations of the form (6) so that the solutions have sufficient decay and regularity that the formal calculation above makes sense rigorously and **(c)** estimate the error terms and construct true solutions. The remainder of this paper is organized as follows. In Section 2 we discuss the operators $A_\pm$ and solutions of (6). In Section 3 we prove the existence of the two-pulse $U^\star$, and thus of $\mathcal{M}_{\text{shoot}}$. Section 4 contains the proof of Theorem 2, *i.e.* the proof that $\mathcal{M}_{\text{shoot}}$ is stable.

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2 Properties of the single-pulse linearization $A_\pm$

We begin this section with the following theorem, which is a special case of Theorem 4.10.7 in Amann [2]:

Theorem 6. *Let $X = L^2$ and $X^1 = H^2$ and $\mathcal{L}$ be an unbounded linear operator on X with domain X^1. Suppose that $\mathcal{L}$ generates an analytic semigroup, $\mathrm{e}^{\mathcal{L}t}$, on X and moreover that the spectrum of $\mathcal{L}$, $\sigma(\mathcal{L})$, lies in $\Sigma_1 := \{\lambda \mid |\arg(\lambda)| > \pi - \phi_0\}$ for some $\phi_0 \in (0, \pi/2)$ and does not contain zero. Moreover suppose that there is a positive number M such that resolvent estimate $(1 + |\lambda|)\|(\lambda - \mathcal{L})^{-1}\|_{X\to X} \le M$ is true for all $\lambda \notin \Sigma_1$. Let $I := (T_1, T_2)$ be an interval of $\mathbf{R}$, $\mathfrak{X} := L^2(I, L^2)$ and $\mathfrak{X}^1 := H^1(I, X) \cap L^2(I, X^1)$. Then*

$$V(t) := \mathrm{e}^{\mathcal{L}(t-T_1)}V_0 + \int_{T_1}^{t} \mathrm{e}^{\mathcal{L}(t-\tau)}H(\tau)d\tau$$

satisfies $\|V\|_{\mathfrak{X}^1} \le C(|V_0|_{H^1} + \|H\|_{\mathfrak{X}})$ and $V_t = \mathcal{L}V + H$ for $t \in I$ a.e.. The constant C is independent of T_1 and T_2. Finally, the conclusion holds true if $V_0 = 0$ and $T_1 = -\infty$.

Theorem 6 implies that we can solve equations like (6). We expand on this here. First we describe the spectral properties of $A_\pm$. In [1], the authors prove the following proposition:

Proposition 7. *The spectrum of $A_\pm$ (viewed as an operator on the Banach space of bounded uniformly continuous functions, BU) consists of a single simple eigenvalue at zero (due to the translation invariance of the problem) and the rest which lies in the set*

$$\{\lambda \mid \Re\lambda < -\alpha_0,\ |\arg(\lambda)| > \pi - \varphi_0\}$$

where $\alpha_0 > 0$ and $\varphi_0 \in (0, \pi/2)$.[3]

In this paper, we will work, not in BU, but rather with L^2 based spaces. The spectrum of $A_\pm$ when viewed as an (unbounded) operator on L^2 functions is identical the spectrum over BU. This follows from the fact that the eigenfunctions of $A_\pm$ satisfy ordinary differential equations, and as such decay exponentially at spatial infinity.

We are also interested in the behavior of $A_\pm$ on data which is spatially localized. Let $\theta_b(\xi) := \exp(b\sqrt{\xi^2+1})$, a smooth "version" of $\mathrm{e}^{b|\xi|}$. Then set $H^s_b := \{V \mid \theta_b(\xi)V(\xi) \in H^s\}$. In addition, we use the notations $X_b := L^2_b := H^0_b$ and $X^1_b := H^2_b$. The domains of $A_\pm$ in X_b are X^1_b. We denote norms with respect to these spaces using absolute value bars, *e.g.* $|\cdot|_{X_b}$. The spectrum of $A_\pm$ viewed as an operator on L^2_b is equivalent to that of the operator $\theta_b A_\pm \theta_{-b}$ viewed as an operator on L^2. For b positive and sufficiently small, $\theta_b A_\pm \theta_{-b}$ is relatively bounded with respect to $A_\pm$. As such its spectrum is a small perturbation of that of $A_\pm$. We summarize and expand on these facts with the following:

[3]We take the argument, arg, of a complex number to be in $(-\pi, \pi]$.

Proposition 8. *There exists $b_0 \in (0, \beta_0]$ and constants $a_0 > 0$, $\phi_0 \in (0, \pi/2)$ such that the following is true for all $b \in [0, b_0]$. The spectrum of $A_\pm$ as an unbounded operator on L^2_b consists of a simple eigenvalue at zero and the rest which lies in the set*

$$\Sigma := \left\{ \lambda \;\middle|\; \Re\lambda < -a_0,\ |\arg(\lambda)| > \pi - \phi_0 \right\}.$$

In addition, for each $\delta < 0$ there is a number $M(\delta) > 0$ such that we have the resolvent estimate $(1 + |\lambda|)\|(\lambda - \delta - A_\pm)^{-1}\|_{L^2_b \to L^2_b} \leq M(\delta)$ for $\lambda \in \Sigma_1$ where

$$\Sigma_1 := \left\{ \lambda \;\middle|\; |\arg(\lambda)| > \pi - \phi_0 \right\}.$$

Finally, $A_\pm$ generate analytic semigroups $\mathrm{e}^{A_\pm t}$ on X_b.

Remark 9. *The existence of the semigroups and the resolvent estimate in the above proposition is implied by the fact that $A_\pm$ are strongly elliptic.*

For the interval $I_1 := (-\infty, -T^\star)$, we define

$$\mathfrak{X}_{\delta,b,I_1} := \left\{ V(x,t) \;\middle|\; \mathrm{e}^{\delta t} V(x,t) \in L^2(I_1, X_b) \right\}$$

and

$$\mathfrak{X}^1_{\delta,b,I_1} := \left\{ V(x,t) \;\middle|\; \mathrm{e}^{\delta t} V(x,t) \in L^2(I_1, X^1_b) \cap H^1(I_1, X_b) \right\}.$$

We denote norms with respect to these spaces using double bars, *e.g.* $\|\cdot\|_{\mathfrak{X}_{a,b,I_2}}$.

Given the definition of R above, we expect that this function exists for all negative times and decays exponentially in that direction. Thus, for existence we will be working with the spaces $\mathfrak{X}_{\delta,b,I_1}$ and $\mathfrak{X}^1_{\delta,b,I_1}$ where δ is a small negative number. We have the following lemma, which is implied by Proposition 8 and Theorem 6. It concerns solutions of $W_t = A_\pm W + H$, with $H \in \mathfrak{X}_{\delta,b,I_1}$.

Lemma 10. *Fix $\delta < 0$, $b \in [0, b_0]$ and $I_1 = (-\infty, -T^\star)$. Let $H \in \mathfrak{X}_{\delta,b,I_1}$ and*

$$\Psi_\pm H := \int_{-\infty}^{t} \mathrm{e}^{A_\pm (t-\tau)} H(\xi, \tau) d\tau.$$

Then

$$\|\Psi_\pm H\|_{\mathfrak{X}^1_{\delta,b,I_1}} \leq C \|H\|_{\mathfrak{X}_{\delta,b,I_1}}$$

and

$$(\Psi_\pm H)_t = A_\pm \Psi_\pm H + H$$

for $t \in I_1$ a.e.. The constant C depends on δ, b, but not on $T^\star$.

Remark 11. *Note that it is crucial here that δ be negative. Since $H \in \mathfrak{X}_{\delta,b,I_1}$, we could equivalently study the problem $\tilde{W}_t = (A_\pm + \delta)\tilde{W} + \tilde{H}$ with $\tilde{H} \in \mathfrak{X}_{0,b,I_1}$. The spectrum of $A_\pm + \delta$ is simply that of $A_\pm$ shifted by an amount δ, which means it is in the left half plane. Without this consideration we could not allow to I_1 to be semi-infinite. Moreover we point out that $\Psi_\pm H$ is maximally regular, a fact which we exploit later on.*

When we examine the stability of $U^\star$, we wish to show that solutions decay exponentially as time evolves. As it happens we will have to work on the finite time interval $I_2 := (-T, -T^\star)$ and we define

$$\mathfrak{X}_{a,b,I_2} := \left\{ V(x,t) \;\middle|\; \mathrm{e}^{a(T+t)} V(x,t) \in L^2(I_2, X_b) \right\}$$

and

$$\mathfrak{X}^1_{a,b,I_2} := \left\{ V(x,t) \;\middle|\; \mathrm{e}^{a(T+t)} V(x,t) \in L^2(I_2, X^1_b) \cap H^1(I_2, X_b) \right\}.$$

We will study $W_t = A_\pm W + H$, for $H \in \mathfrak{X}_{a,b,I_2}$ with $a > 0$.

As in the previous remark, working with $\mathfrak{X}_{a,b,I_2}$ and $a > 0$ is equivalent to working with $A_\pm + a$ on $\mathfrak{X}_{0,b,I_2}$. This has the effect of pushing the zero eigenvalue into the right half plane, which implies we cannot solve $W_t = A_\pm W + H$ as an initial value problem with solutions whose norm is independent of I_2. Instead we will solve a boundary problem related to the center and stable eigenspaces of $A_\pm$, which we now describe.

The center eigenspace of $A_\pm$ is $X^{\mathrm{c}}_\pm := \mathrm{span}\{Q'_\pm\}$. Let $\Pi^{\mathrm{c}}_\pm$ be the spectral projection onto $X^{\mathrm{c}}_\pm$. This projection is given by

$$\Pi^{\mathrm{c}}_\pm \cdot = \langle e_\pm, \cdot \rangle Q'_\pm,$$

where $\langle \cdot, \cdot \rangle$ is the standard L^2 inner product and $e_\pm$ is the eigenfunction associated with the simple zero eigenvalue of $A^\dagger_\pm$ (the adjoint of $A_\pm$ with respect to $\langle \cdot, \cdot \rangle$). We remark that $e_\pm$ enjoys the same properties as $Q_\pm$ does. Let $X^{\mathrm{s}}_\pm = \ker \Pi^{\mathrm{s}}_\pm$, the stable eigenspace of $A_\pm$, and $\Pi^{\mathrm{s}}_\pm := 1 - \Pi^{\mathrm{c}}_\pm$.

The boundary value problem and its solution are described by:

Lemma 12. *Fix $a \in (0, a_0)$, $b \in [0, b_0]$. Let $H \in \mathfrak{X}_{a,b,I_2}$, $W^{\mathrm{s}} \in X^{\mathrm{s}}_\pm \cap H^1_b$ and*

$$\Gamma_\pm(W^{\mathrm{s}}, H) := \mathrm{e}^{A_\pm(T+t)} \Pi^{\mathrm{s}}_\pm W^{\mathrm{s}} + \int_{-T}^{t} \mathrm{e}^{A_\pm(t-\tau)} \Pi^{\mathrm{s}}_\pm H(\tau) d\tau - \int_{t}^{-T^\star} \mathrm{e}^{A_\pm(t-\tau)} \Pi^{\mathrm{c}}_\pm H(\tau) d\tau.$$

Then

$$\|\Gamma_\pm(W^{\mathrm{s}}, H)\|_{\mathfrak{X}^1{a,b,I_2}} \leq C \left(|W^{\mathrm{s}}|_{H^1_b} + \|H\|_{\mathfrak{X}_{a,b,I_2}} \right),$$
$$\left(\Gamma_\pm(W^{\mathrm{s}}, H)\right)_t = A_\pm \Gamma_\pm(W^{\mathrm{s}}, H) + H$$

for $t \in I_2$ a.e. and

$$\Pi^{\mathrm{s}}_\pm \Gamma_\pm(W^{\mathrm{s}}, H) \Big|_{t=-T} = W^{\mathrm{s}}, \quad \Pi^{\mathrm{c}}_\pm \Gamma_\pm(W^{\mathrm{s}}, H) \Big|_{t=-T^\star} = 0.$$

The constant C depends on a, b, but not on I_2.

Proof. The definition of $\Gamma_\pm$ consists of two pieces, each of which is dealt with in a different way. Let

$$\Gamma_1 H := \mathrm{e}^{A_\pm(T+t)} W^{\mathrm{s}} + \int_{-T}^{t} \mathrm{e}^{A_\pm(t-\tau)} \Pi^{\mathrm{s}}_\pm H(\tau) d\tau.$$

Let $A^{\mathrm{s}}_\pm = A_\pm$ restricted to $X^{\mathrm{s}}_\pm$. The spectrum of this operator lies in Σ and generates the analytic semigroup $\mathrm{e}^{A_\pm(T+t)}\Pi^{\mathrm{s}}_\pm$ (and restricted to $X^{\mathrm{s}}_\pm$). As Σ lies in the left half plane and is separated from the imaginary axis by a distance a_0, Theorem 6 implies (since $a \in (0, a_0)$)

$$\|\Gamma_1 H\|_{\mathfrak{X}^1{a,b,I_2}} \leq C \left(\|W^{\mathrm{s}}\|_{H^1_b} + \|H\|_{\mathfrak{X}_{a,b,I_2}} \right),$$
$$\partial_t \Gamma_1 H = A^{\mathrm{s}} \Gamma_1 H + \Pi^{\mathrm{s}}_\pm H$$

for $t \in I_2$ a.e. and $\Pi^{\mathrm{s}}_\pm \Gamma_1 H|_{t=-T} = W^{\mathrm{s}}$. Moreover $\Pi^{\mathrm{c}}_\pm \Gamma_1 H = 0$.

Consider the term

$$\Gamma_2 H := - \int_{t}^{-T^\star} S_\pm(t-\tau) \Pi^{\mathrm{c}}_\pm H(\xi, \tau) d\tau.$$

Note that $\mathrm{e}^{A_\pm t}\Pi^{\mathrm{c}}_\pm = \Pi^{\mathrm{c}}_\pm$ and that $\Pi^{\mathrm{c}}_\pm$ is infinitely smoothing (since $Q'_\pm$ lies in H^{s}_b for any $s \in \mathbf{R}$). Thus, the spatial regularity of $\Gamma_2 H$ is assured. Lebesgue's differentiation lemma guarantees that the time derivative of $\Gamma_2 H$ exists and is equal to $\Pi^{\mathrm{c}} H(\xi, t)$ a.e.. Also $\Pi^{\mathrm{c}}_\pm \Gamma_2 H|_{t=-T^\star} = 0$ and $\Pi^{\mathrm{s}}_\pm \Gamma_2 H = 0$. The only thing left to verify is that

$$\|\Gamma_2 H\|_{\mathfrak{X}^1_{a,b,I_2}} \leq C \|H\|_{\mathfrak{X}_{a,b,I_2}}$$

with the constant C independent of I_2. The fact that Γ_2 is an integral which goes backwards in time and that $a > 0$ combine to give us this result, which in turn implies Lemma 12. $\square$

3 Existence

In this section we will solve equation (3) and thus prove the existence of the two-pulse solution. To do so we would like to solve $W_t = A(t)W + H$. What is the appropriate function space for H? The function H_1 in (3) is given by

$$H_1(t) := \begin{pmatrix} (2+2\alpha)q_+(\xi_+)q_-(\xi_-) - 3q_+^2(\xi_+)q_-(\xi_-) - 3q_-^2(\xi_-)q_+(\xi_+) \\ 0 \end{pmatrix}.$$

Notice that this function consists of sums and products of functions which are spatially localized in the moving coordinates $\xi_\pm := x \pm ct$.

Let

$$m_b(x,t) := \chi_+(x)\theta_b(\xi_+) + \chi_-(x)\theta_b(\xi_-).$$

(This is a smooth "version" of $\min\{\theta_b(\xi_+), \theta_b(\xi_-)\}$.) We introduce the following function spaces, for the interval $I_1 := (-\infty, -T^\star)$,

$$\mathfrak{Z}_{\delta,b,I_1} := \left\{ V(x,t) \mid \mathrm{e}^{\delta t} m_b(x,t) V(x,t) \in L^2(I_1, L^2) \right\}$$

and

$$\mathfrak{Z}^1_{\delta,b,I_1} := \left\{ V(x,t) \mid \mathrm{e}^{\delta t} m_b(x,t) V(x,t) \in L^2(I_1, H^2) \cap H^1(I_1, L^2) \right\}.$$

Roughly, these spaces consist of functions which are sums of moving and localized functions. Note that here and in the next section, we are thinking of $T^\star$ as being a fixed, large, but as of yet unspecified positive number.

The following lemmata relate the $\mathfrak{X}$ spaces of Section 2 to the $\mathfrak{Z}$ spaces. The first says that a function in an $\mathfrak{X}$ space, when put in a moving either reference frame ξ_+ or ξ_-, belongs to an appropriate $\mathfrak{Z}$ space.

Lemma 13. *Suppose that $V \in \mathfrak{X}_{\delta,b,I_1}$ (resp. $\mathfrak{X}^1_{\delta,b,I_1}$) with $\delta \le 0$, $b \ge 0$. Let $\tilde{V}(x,t) := V(\xi_\pm, t)$. Then*

$$\|\tilde{V}\|_{\mathfrak{Z}_{\delta,b,I_1}} \le C\|V\|_{\mathfrak{X}_{\delta,b,I_1}}.$$

(resp. $\|\tilde{V}\|_{\mathfrak{Z}^1_{\delta,b,I_1}} \le \|V\|_{\mathfrak{X}^1_{\delta,b,I_1}}$.) The constant C does not depend on $T^\star$.

Proof. This result immediately follows from the fact that, for $t \le 0$ and $x \in \mathbf{R}$,

$$\sum_{j=0}^{2} \left|\frac{d^j m_b}{dx^j}(x,t)\right| \le C \sum_{j=0}^{2} \left|\frac{d^j \theta_b}{d\xi_\pm^j}(\xi_\pm)\right|.$$

□

The next lemma tells us that a function in a $\mathfrak{Z}$ space, when cut off to either the positive or negative half lines and then viewed in a moving frame, belongs to an $\mathfrak{X}$ space.

Lemma 14. *Suppose that $V \in \mathfrak{Z}_{\delta,b,I_1}$ (resp. $\mathfrak{Z}^1_{\delta,b,I_1}$) with $\delta \le 0$, $b \ge 0$. Let $\tilde{V}(\xi_\pm, t) := \chi_\pm(x)V(x,t)$. Then*

$$\|\tilde{V}\|_{\mathfrak{X}_{\delta,b,I_1}} \le C\|V\|_{\mathfrak{Z}_{\delta,b,I_1}}.$$

(resp. $\|\tilde{V}\|_{\mathfrak{X}^1_{\delta,b,I_1}} \le \|V\|_{\mathfrak{Z}^1_{\delta,b,I_1}}$.) The constant C does not depend on $T^\star$.

Proof. The proof follows from the fact that, for $t \le 0$ and $x \in \mathbf{R}$,

$$\sum_{j=0}^{2} \left|\frac{d^j}{dx^j}(\chi_\pm(x)\theta_b(\xi_\pm))\right| \le C \sum_{j=0}^{2} \left|\frac{d^j m_b}{dx^j}(x,t)\right|.$$

□

Finally, if we have a spatially localized function which moves left and another which moves right, their product is small.

Lemma 15. *Suppose that* $V_\pm \in \mathfrak{X}^1_{\delta,b,I_1}$ *with* $\delta \leq 0$, $b \geq 0$. *Let* $\tilde{V}_\pm(x,t) := V(\xi_\pm, t)$. *Then*

$$\|\tilde{V}_+\tilde{V}_-\|_{\mathfrak{Z}^1_{\delta,b,I_1}} \leq C\mathrm{e}^{-bcT^\star}\|V_+\|_{\mathfrak{X}^1_{\delta,b,I_1}}\|V_-\|_{\mathfrak{X}^1_{\delta,b,I_1}}.$$

The constant C *does not depend on* $T^\star$.

Proof. First we remark that the space $\mathfrak{Z}^1_{\delta,b,I_1}$ is an algebra if $\delta \leq 0$ and $b \geq 0$. Second, for all $t \in I_1$ and $x \in \mathbf{R}$,

$$|\partial_t \chi_\pm(x)\theta_{-b}(\xi_\mp)| + \sum_{j=0}^{j=2}|\partial_x^j \chi_\pm(x)\theta_{-b}(\xi_\mp)| \leq C\mathrm{e}^{-bcT^\star}$$

This estimate is less complicated than it appears. If we replace $\chi_+(x)$ with the heaviside function $\mathrm{Heav}_+(x)$, then it is clear that $\mathrm{Heav}_+(x)\theta_{-b}(x-ct)$ achieves its maximum at $x=0$, provided $t<0$, of course. This maximum is roughly $C\mathrm{e}^{-bct}$. The derivatives are bounded in a similar fashion. Given this estimate, this lemma follows immediately from writing down the definition of the norms. ☐

Now we can prove:

Proposition 16. *If* $T^\star$ *is sufficiently large, the following is true. Fix* $\delta < 0$ *and* $b \in [0, b_0]$. *Then there exists a map* $\Psi : \mathfrak{Z}_{\delta,b,I_1} \longmapsto \mathfrak{Z}^1_{\delta,b,I_1}$ *with the following properties. First,* $\|\Psi\|_{\mathfrak{Z}_{\delta,b,I_1}\to\mathfrak{Z}^1_{\delta,b,I_1}} \leq C$ *with the constant* C *independent of* $T^\star$. *Second,*

$$(\Psi H)_t = A(t)(\Psi H) + H$$

for $t \in I_1$ *a.e..*

Proof. (In this proof, for brevity, we set $\mathfrak{Z} = \mathfrak{Z}_{\delta,b,I_1}$, $\mathfrak{Z}^1 = \mathfrak{Z}^1_{\delta,b,I_1}$ $\mathfrak{X} = \mathfrak{X}_{\delta,b,I_1}$ and $\mathfrak{X}^1 = \mathfrak{X}^1_{\delta,b,I_1}$.) For a function $J \in \mathfrak{Z}$, define $J_\pm(\xi_\pm, t) := \chi_\pm(x)J(x,t)$. Lemma 14 implies $J_\pm$ lie in $\mathfrak{X}$. Let $W_\pm = \Psi_\pm J_\pm$, $\Psi_\pm$ as in Lemma 10. Let

$$\tilde{\Psi}J := W_+(\xi_+, t) + W_-(\xi_-, t).$$

Proposition 10 and Lemma 13 imply $\tilde{\Psi}J \in \mathfrak{Z}^1$ and so we can insert it directly into (5). Doing so yields

$$(\tilde{\Psi}J)_t = A(t)(\tilde{\Psi}J) + J - EJ$$

where

$$EJ := B_+(\xi_+, t)W_-(\xi_-, t) + B_-(\xi_-, t)W_+(\xi_+, t).$$

The maps $B_\pm$ defined in Section 1 are specifically

$$B_\pm(\xi_\pm, t) = q_\pm(\xi_\pm)\begin{pmatrix} -3q_\pm(\xi_\pm) + 2(\alpha+1) - 6q_\mp(\xi_\mp) & 0 \\ 0 & 0 \end{pmatrix}.$$

The matrix in the above is bounded, and the prefactor of $q_\pm(\xi_\pm)$ allows us to estimate EJ *via* Lemma 15:

$$\|EJ\|_{\mathfrak{Z}^1} \leq C\mathrm{e}^{-bcT^\star}\|J\|_{\mathfrak{X}}.$$

Choose $T^\star$ large enough so that $C\mathrm{e}^{-bcT^\star} < 1/2$ and consider the following map defined by a Neumann series:

$$\Psi H := \tilde{\Psi} \circ \sum_{j=0}^{\infty} E^j H.$$

The sum converges and a direct computation shows ΨH solves (5). The bounds on the norm of Ψ follow immediately from the norms on $\Psi_\pm$ and E.

The solution of (5) given by ΨH is the unique solution. Uniqueness is shown if we can prove that the zero solution is the only solution of (5) for $H = 0$. Suppose that $W_0 \in \mathfrak{Z}^1$ is a function such that $W_{0,t} := A(t)W_0$. Consider the equation

$$V_t = -A^\dagger(t)V + W_0, \quad V(t = -T^\star) = 0.$$

Here $A^\dagger(t)$ is the adjoint of $A(t)$ computed with respect to the L^2 (spatial) inner product (pointwise in time). An argument completely parallel to the one used to prove the existence of Ψ can be used to solve this equation. Call this solution V_0. Let $\langle \cdot, \cdot \rangle_{xt}$ be the L^2 spacetime inner product. Then

$$\begin{aligned} \langle W_0, W_0 \rangle_{xt} &= \langle W_0, V_{0,t} + A^\dagger(t)V_0 \rangle_{xt} \\ &= \langle -W_{0,t} + A(t)W_0, V_0 \rangle_{xt} \\ &= 0 \end{aligned} \tag{7}$$

(Note that the initial condition on V_0 is necessary in the integration by parts in the above calculation.) This concludes the proof of Proposition 16. ☐

Now we are able to prove the existence and uniqueness of the solution $U^\star$. The following theorem is equivalent to Theorem 2:

Theorem 17. *There exists $\delta \in ((b_0 - 2\beta_0)c, 0)$ such that following is true if $T^\star$ is sufficiently large. There exists a unique $R \in \mathfrak{Z}^1_{\delta,b,I_1}$ (for each $b \in [0, b_0]$) which satisfies* (3) *for all $t \in I_1$ a.e..*

Proof. (In this proof, for brevity, we set $\mathfrak{Z} = \mathfrak{Z}_{\delta,b,I_1}$, $\mathfrak{Z}^1 = \mathfrak{Z}^1_{\delta,b,I_1}$ $\mathfrak{X} = \mathfrak{X}_{\delta,b,I_1}$ and $\mathfrak{X}^1 = \mathfrak{X}^1_{\delta,b,I_1}$.)

Define the map

$$\Phi(R) := \Psi(H_1 + N_1(R)),$$

with Ψ as in Proposition 16. Since $\mathfrak{Z}^1$ is an algebra and F is a polynomial in each component, we have:

Lemma 18. *Fix $\delta < 0$ and $b \geq 0$. Then there exists $C > 0$ independent of $T^\star$ such that*

$$\|N_1(R)\|_{\mathfrak{Z}^1_{\delta,b,I_1}} \leq C \left(\|R\|^2_{\mathfrak{Z}^1_{\delta,b,I_1}} + \|R\|^3_{\mathfrak{Z}^1_{\delta,b,I_1}} \right)$$

and

$$\|N_1(R_1) - N_1(R_2)\|_{\mathfrak{Z}^1_{\delta,b,I_1}} \leq C \left(\|R_1 + R_2\|_{\mathfrak{Z}^1_{\delta,b,I_1}} + \|R_1 + R_2\|^2_{\mathfrak{Z}^1_{\delta,b,I_1}} \right) \|R_1 - R_2\|_{\mathfrak{Z}^1_{\delta,b,I_1}}.$$

With this, we can view Φ as a map from $\mathfrak{Z}^1$ to itself. A fixed point of Φ corresponds to a solution of (3). Note that

$$\|\Phi(R)\|_{\mathfrak{Z}^1} \leq C \left(\|H_1\|_{\mathfrak{Z}} + \|R\|^2_{\mathfrak{Z}^1} + \|R\|^3_{\mathfrak{Z}^1} \right)$$

and

$$\|\Phi(R_1 - R_2)\|_{\mathfrak{Z}^1} \leq C \left(\|R_1 + R_1\|_{\mathfrak{Z}^1} + \|R_1 + R_1\|^2_{\mathfrak{Z}^1} \right) \|R_1 - R_2\|_{\mathfrak{Z}^1}.$$

Provided $\|H_1\|_{\mathfrak{Z}}$ is sufficiently small, the above estimates imply that Φ is a contraction on a ball around the origin in $\mathfrak{Z}^1$, which in turn implies it has a fixed point. Thus all that needs to be checked is that H_1 is small, which is implied by calculations similar to those which prove Lemma 15. Here is the computation:

$$\mathrm{e}^{\delta t} m_b(x,t) q_-(\xi_-) q_+(\xi_+) = \frac{\mathrm{e}^{\delta t}(\theta_b(\xi_+)\chi_+(x) + \theta_b(\xi_-)\chi_-(x))}{\theta_{\beta_0}(\xi_-)\theta_{\beta_0}(\xi_+)} \theta_{\beta_0}(\xi_-) q(\xi_-) \theta_{\beta_0}(\xi_+) q(\xi_+).$$

A routine calculation shows that for $t \leq -T^\star$

$$\frac{\mathrm{e}^{\delta t}\chi_+(x)}{\theta_{\beta_0 - b}(\xi_+)\theta_{\beta_0}(\xi_-)} \leq C\mathrm{e}^{((2\beta_0 - b)c + \delta)T^\star}.$$

So, provided $0 > \delta > -(2\beta_0 - b)c$ and by taking $T^\star$ sufficiently large, we can make H_1 as small as we like. This completes the existence proof.

$\square$

4 Stability

In this section, we prove Theorem 2, which states that the shooting manifold $\mathcal{M}_{\mathrm{shoot}}$ is stable. In what follows, we let $I_2 := (-T, -T^\star)$,

$$\mathfrak{Z}_{a,b,I_2} := \left\{ V(x,t) \;\middle|\; \mathrm{e}^{a(T+t)} m_b(x,t) V(x,t) \in L^2(I_2, L^2) \right\}$$

and

$$\mathfrak{Z}^1_{a,b,I_2} := \left\{ V(x,t) \;\middle|\; \mathrm{e}^{a(T+t)} m_b(x,t) V(x,t) \in L^2(I_2, H^2) \cap H^1(I_2, L^2) \right\}.$$

These spaces are the "two-sided" versions of $\mathfrak{X}_{a,b,I_2}$ and $\mathfrak{X}^1_{a,b,I_2}$.

Recall that if we set $U(x,t) = U^\star(x,t) + W(x,t)$, then W solves (4). If we were studying the stability of a single pulse, we would like to have detailed information about the center and stable eigenspaces of $A_\pm$. However, since $A(t)$ is non-autonomous these eigenspaces are not really well-defined objects. Nonetheless we can define subspaces of H^1 which play the roles that $X^{\mathrm{c}}_\pm$ and $X^{\mathrm{s}}_\pm$ do in proving the stability of $Q_\pm$. Let

$$X^{\mathrm{c}}_T := \operatorname{span}\{Q'_+(x - cT), Q'_-(x + cT)\},$$

which is roughly the center direction of the linearization about the two pulse at time $-T$. Also set $X^{\mathrm{s}}_{\pm,T} := \left\{ f(x \pm cT) \;\middle|\; f \in X^{\mathrm{s}}_\pm \right\}$—the "shifted" stable subspaces. The approximate stable direction is $X^{\mathrm{s}}_T := X^{\mathrm{s}}_{+,T} \cap X^{\mathrm{s}}_{-,T}$, which, by counting dimensions, we expect is a co-dimension two subspace of L^2. The following lemma justifies this claim:

Lemma 19. *For each $T \geq T^\star$, there exists a projection, Π^{c}_T, of L^2 onto X^{c}_T along X^{s}_T.*

Proof. We construct the projection explicitly. Set $c^\pm_\mp := \langle e_\pm(\cdot \mp cT), Q'_\mp(\cdot \pm cT)\rangle$ and $M := \begin{bmatrix} 1 & c^-_+ \\ c^+_- & 1 \end{bmatrix}$. The adjoint eigenfunctions decay exponentially at spatial infinity, so we have $|\langle e_\pm(\cdot \mp cT), Q'_\mp(\cdot \pm cT)\rangle| \leq C\mathrm{e}^{-2\beta_0 cT}$. Thus M is certainly invertible. Letting $(a_+, b_+)^t := M^{-1}(1,0)^t$, $(a_-, b_-)^t := M^{-1}(0,1)^t$, $v_\pm(x) := a_\pm e_+(x - cT) + b_\pm e_-(x + cT)$ guarantees $\langle v_\pm(\cdot), Q'_\pm(\cdot \mp cT)\rangle = 1$ and $\langle v_\pm(\cdot), Q'_\mp(\cdot \pm cT)\rangle = 0$. Clearly

$$\Pi^{\mathrm{c}}_T := \langle v_+, \cdot\rangle Q'_+(x - cT) + \langle v_-, \cdot\rangle Q'_-(x + cT)$$

has its range in X^{c}_T. The conditions on $v_\pm$ guarantee that Π^{c}_T restricted to X^{c}_T is the identity and $(\Pi^{\mathrm{c}}_T)^2 = \Pi^{\mathrm{c}}_T$. In a similar vein, the kernel of Π^{c}_T is X^{s}_T since being in this kernel implies being orthogonal to both $e_+(x-cT)$ and $e_-(x+cT)$ (and vice versa). It is precisely those conditions which characterize elements of $X^{\rm s}_{\pm,T}$. Thus

$$\Pi^{\rm s}_T := 1 - \Pi^{\rm c}_T$$

is a projection onto $X^{\rm s}_T$. This completes the proof. $\square$

We can now state the following proposition which is crucial to proving Theorem 2.

Proposition 20. *If $T^\star$ is sufficiently large, the following is true. Fix $a \in (0, a_0)$. Then there exists a map $\Gamma : X^{\rm s}_T \times \mathfrak{Z}_{a,0,I_2} \longmapsto \mathfrak{Z}^1_{a,0,I_2}$ with the following properties.*

First, $\|\Gamma\|_{X^{\rm s}_T \times \mathfrak{Z}_{\delta,b,I_1} \to \mathfrak{Z}^1_{\delta,0,I_1}} \le C$ with the constant C independent of T. Second, if $W := \Gamma(W^{\rm s}, H)$, then for $t \in I_2$ a.e.

$$\begin{aligned} &W_t = A(t)W + H, \quad \textit{and} \\ &\Pi^{\rm s}_T W = W^{\rm s}, \quad \Pi^{\rm c}_{T^\star} W = 0. \end{aligned} \tag{8}$$

Here, and in the following, when we write $\Pi^{\rm s}_T V$ or $\Pi^{\rm c}_T V$ of a time dependent function $V(t)$, we mean $\Pi^{\rm s}_T V(t=-T)$ or $\Pi^{\rm c}_T V(t=-T)$ unless otherwise explicitly stated. Also, X^s_T is considered as a subspace of H^1 with corresponding topology.

To prove this proposition we utilize the following lemmata, which are completely parallel to those in Section 3. We omit their proofs.

Lemma 21. *Suppose that $V \in \mathfrak{X}_{a,b,I_2}$ (resp. $\mathfrak{X}^1_{\delta,b,I_2}$) with $a \ge 0$, $b \ge 0$. Let $\tilde V(x,t) := V(\xi_\pm, t)$. Then*

$$\|\tilde V\|_{\mathfrak{Z}_{a,b,I_2}} \le C \|V\|_{\mathfrak{X}_{a,b,I_2}}.$$

(resp. $\|\tilde V\|_{\mathfrak{Z}^1_{a,b,I_2}} \le \|V\|_{\mathfrak{X}^1_{a,b,I_2}}$.) The constant C does not depend on T or $T^\star$.

Lemma 22. *Suppose that $V \in \mathfrak{Z}_{a,b,I_2}$ (resp. $\mathfrak{Z}^1_{a,b,I_2}$) with $a \ge 0$, $b \ge 0$. Let $\tilde V(\xi_\pm, t) := \chi_\pm(x) V(x,t)$. Then*

$$\|\tilde V\|_{\mathfrak{X}_{a,b,I_2}} \le C \|V\|_{\mathfrak{Z}_{a,b,I_2}}.$$

(resp. $\|\tilde V\|_{\mathfrak{X}^1_{a,b,I_2}} \le \|V\|_{\mathfrak{Z}^1_{a,b,I_2}}$.) The constant C does not depend on T or $T^\star$.

Lemma 23. *Suppose that $V_\pm \in \mathfrak{X}^1_{a,b,I_2}$ with $a \ge 0$, $b \ge 0$. Let $\tilde V_\pm(x,t) := V(\xi_\pm, t)$. Then*

$$\|\tilde V_+ \tilde V_-\|_{\mathfrak{Z}^1_{a,b,I_2}} \le C \mathrm{e}^{-bcT^\star} \|V_+\|_{\mathfrak{X}^1_{a,b,I_2}} \|V_-\|_{\mathfrak{X}^1_{a,b,I_2}}.$$

The constant C does not depend on T or $T^\star$.

Proof. (**of Proposition 20**) Suppose $H \in \mathfrak{Z}_{a,0,I_2}$. (Note that there is no spatial decay in H). Our approach here is similar to that used to prove Proposition 16. We proceed as follows. Let $H_\pm(\xi_\pm, t) = \chi_\pm H(x,t)$. We have $H_\pm \in \mathfrak{X}_{a,0,I_2}$ by Lemma 22. Let

$$W^{\rm bc}_+ := \Gamma_+(W^{\rm s}, H_+), \quad W^{\rm bc}_- := \Gamma_-(0, H_-),$$

where $\Gamma_\pm$ are as in Proposition 12.[4]

Setting $\Gamma_{\rm bc}(W^{\rm s}, H) := W^{\rm bc}_+(\xi_+, t) + W^{\rm bc}_-(\xi_-, t)$, we find that

$$(\Gamma_{\rm bc}(W^{\rm s}, H))_t = A(t)\Gamma_{\rm bc}(W^{\rm s}, H) + H - E_{\rm bc}(W^{\rm s}, H)$$

[4]Note that we have placed the entirety of the stable direction's initial data into the solution $W^{\rm bc}_+$. This is an arbitrary decision, which does not affect uniqueness.

with

$$E_{\rm bc}(W^{\rm s},H) := B_+(\xi_+,t)R_-(\xi_-,t) + B_-(\xi_-,t)R_+(\xi_+,t).$$

Moreover, $\Pi^{\rm s}_T\Gamma_{\rm bc}(W^{\rm s},H) = W^{\rm s}$ and $\Pi^{\rm c}_{T^\star}\Gamma_{\rm bc}(W^{\rm s},H) = 0$.

The function $E_{\rm bc}(W^{\rm s},H)$ is not small, since $W^{\rm bc}_\pm$ are not in spatially decaying spaces. However, the maps $B_\pm(t)$ contain only infinitely differentiable functions which decay at the rate $\mathrm{e}^{-\beta_0|\xi|}$ as $|\xi|\to\infty$. *Ergo,* if $V\in H^1$ we have $B_\pm(t)V\in H^1_{\beta_0}$ and this in turn implies $E_{\rm bc}(W^{\rm s},H)\in\mathfrak{Z}_{a,\beta_0,I_2}$! Thus if we can solve

$$\begin{aligned} &W^{\rm d}_t = A(t)W^{\rm d} + E_{\rm bc}(W^{\rm s},H)\\ &\Pi^{\rm s}_T W^{\rm d} = 0, \qquad \Pi^{\rm c}_{T^\star}W^{\rm d} = 0 \end{aligned} \tag{9}$$

then $W^{\rm d} + \Gamma_{\rm bc}(W^{\rm s},H)$ would solve (8).

Now suppose $J\in\mathfrak{Z}_{a,b,I_2}$. Let $J_\pm(\xi_\pm,t) := \chi_\pm(x)J(x,t)$. By Lemma 22, $J_\pm\in\mathfrak{X}_{a,b,I_2}$. Let

$$W_\pm := \Gamma_\pm(0,J_\pm)$$

and $\tilde\Gamma J := W_+(\xi_+,t) + W_-(\xi_-,t)$. Note that $\tilde\Gamma J\in\mathfrak{Z}^1_{a,b,I_2}$. Additionally,

$$(\tilde\Gamma J)_t := A(t)(\tilde\Gamma J) + J - E_{\rm d}J$$

with

$$E_{\rm d}J := B_+(\xi_+,t)W_-(\xi_-,t) + B_-(\xi_-,t)W_+(\xi_+,t).$$

$\tilde\Gamma J$ also meets meets the boundary conditions in (9). Lemma 23 implies that

$$\|E_{\rm d}J\|_{\mathfrak{Z}_{a,b,I_2}} \le C\mathrm{e}^{-bcT^\star}\|J\|_{\mathfrak{Z}_{a,b,I_2}}.$$

If $T^\star$ is large enough so that $C\mathrm{e}^{-bcT^\star} < 1/2$, then

$$\Gamma_d J := \tilde\Gamma\circ\sum_{n=0}^{\infty} E^n_{\rm d}J$$

converges and satisfies

$$(\Gamma_d J)_t := A(t)(\Gamma_d J) + J,$$

and $\Pi^{\rm s}_T\Gamma_d J = 0$, $\Pi^{\rm c}_{T^\star}\Gamma_d J = 0$. Therefore $\Gamma(W^{\rm s},H) := \Gamma_{\rm bc}(W^{\rm s},H) + \Gamma_d E_{\rm bc}(W^{\rm s},H)$ solves (8).

Finally, we note that the solution computed above is in fact unique. This argument is nearly identical to that used to prove the uniqueness of R in the previous section, and so we omit it. □

We now prove a nonlinear version of Proposition 20.

Proposition 24. *For $T^\star$ sufficiently large the following is true. Fix $a\in(0,a_0)$. There is a constant C independent of T such that the following is true. There is a map $\Gamma^\star_T(W^{\rm s})$ defined for $W^{\rm s}\in X^{\rm s}_T\cap H^1$ with $|W^{\rm s}|_{H^1}\le C$ and taking values in in $\mathfrak{Z}^1_{a,0,I_2}$ such that $\Gamma^\star_T(W^{\rm s})$ satisfies (4) for $t\in I_2$ a.e.,*

$$\|\Gamma^\star_T(W^{\rm s})\|_{\mathfrak{Z}^1_{a,0,I_2}} \le C\|W^{\rm s}\|_{H^1}$$

and

$$\Pi^{\rm s}_T\Gamma^\star_T(W^{\rm s}) = W^{\rm s}, \quad \Pi^{\rm c}_{T^\star}\Gamma^\star(W^{\rm s}) = 0.$$

Proof. Let

$$\Omega(W^{\mathrm{s}}, W) := \Gamma(W^{\mathrm{s}}, G_1(t)W + N_2(W)).$$

If $\Omega(W^{\mathrm{s}}, W^\star) = W^\star$, then $W^\star$ is a solution of (4) and meets the boundary conditions in (9).

Given that the solution R of (3) from Theorem 17 lies in $\mathfrak{Z}^1_{\delta,b,I_1}$, (that is to say, it decays as t goes to $-\infty$ like $\mathrm{e}^{\delta t}$) we have the following estimate on $G_1(t)$:

$$\|G_1 V\|_{\mathfrak{Z}^1_{a,b,I_2}} \leq C\mathrm{e}^{\delta T^\star} \|V\|_{\mathfrak{Z}^1_{a,b,I_2}}.$$

Moreover, we have the following lemma which is analogous in all respects to Lemma 18:

Lemma 25. *Fix $a \in (0, a_0)$ and $b \geq 0$. Then there exists $C > 0$ independent of $T^\star$ such that*

$$\|N_2(W)\|_{\mathfrak{Z}^1_{a,b,I_2}} \leq C \left(\|W\|^2_{\mathfrak{Z}^1_{a,b,I_2}} + \|W\|^3_{\mathfrak{Z}^1_{a,b,I_2}} \right)$$

and

$$\|N_2(W_1) - N_1(W_2)\|_{\mathfrak{Z}^1_{a,b,I_2}} \leq C \left(\|W_1 + W_2\|_{\mathfrak{Z}^1_{a,b,I_2}} + \|W_1 + W_2\|^2_{\mathfrak{Z}^1_{a,b,I_2}} \right) \|W_1 - W_2\|_{\mathfrak{Z}^1_{a,b,I_2}}.$$

These estimates imply

$$\|\Omega(W^{\mathrm{s}}, W)\|_{\mathfrak{Z}^1_{a,0,I_2}} \leq C \left(|W^{\mathrm{s}}|_{H^1} + \mathrm{e}^{\delta T^\star} \|W\|_{\mathfrak{Z}^1_{a,0,I_2}} + \|W\|^2_{\mathfrak{Z}^1_{a,0,I_2}} + \|W\|^3_{\mathfrak{Z}^1_{a,0,I_2}} \right)$$

and

$$\|\Omega(W^{\mathrm{s}}, W_1 - W_2)\|_{\mathfrak{Z}^1_{a,0,I_2}} \leq C \left(\|W_1 + W_1\|_{\mathfrak{Z}^1_{a,0,I_2}} + \|W_1 + W_1\|^2_{\mathfrak{Z}^1_{a,0,I_2}} \right) \|W_1 - W_2\|_{\mathfrak{Z}^1_{a,0,I_2}}.$$

For $T^\star$ sufficiently large and $|W^{\mathrm{s}}|_{H^1}$ sufficiently small, $\Omega(W^{\mathrm{s}}, W)$ is a contraction (in the W argument). Letting $\Gamma^\star(W^{\mathrm{s}})$ be the map which sends W^{s} to this fixed point finishes the proposition. □

Now we can prove the following theorem, which is equivalent to Theorem 2:

Theorem 26. *The following is true for $T^\star$ sufficiently large. Fix $a \in (0, a_0)$. There exist $T^{\star\star} > T^\star$ and positive constants μ and C such that the following is true for all $T_0 > T^{\star\star}$. Suppose that $U_0(x)$ is such that*

$$|U_0 - U^\star(-T_0)|_{H^1} \leq \mu.$$

Then, there exist ΔT, $\Delta x \in \mathbf{R}$ which have the following properties. First, ΔT and ΔX depend smoothly on $U_0 \in H^1$, and $|\Delta T| + |\Delta x| \leq C|U_0 - U^\star(-T_0)|_{H^1}$. Second, if $U(x,t)$ is the solution of (1) *with initial data $U(x, -T) = U_0(x)$, where $T = T_0 + \Delta T$, then*

$$\|U^\star(\cdot - \Delta x, \cdot) - U(\cdot, \cdot)\|_{\mathfrak{Z}^1_{a,0,I_2}} \leq C|U_0 - U^\star(-T_0)|_{H^1}.$$

(Here $I_2 := (-T, -T^\star)$, as per normal.)

Proof. First let $W_0(x) := U_0(x) - U^\star(x, -T_0)$. We define

$$\eta(W^{\mathrm{s}}; T) := \Pi^{\mathrm{s}}_T \Gamma^\star(W^{\mathrm{s}}).$$

Notice that if

$$W_0(x) = W^{\mathrm{s}}(x) + \eta(W^{\mathrm{s}}; T_0)$$

then Proposition 24 implies that the solution of (1) with initial data $U^\star + W_0$ does indeed decay as in Theorem 26. In essence, η is a stable fibration around $U^\star$. We remark that η is smooth in both of its arguments. This is a consequence of the fact that the maps $\Gamma_\pm(W^{\mathrm{s}}, H)$ are smooth in W^{s} and T. It is also true that these derivatives are uniformly bounded in T.

If we can find Δx, ΔT and W^{s} such that

$$U^\star(x, -T_0) + W_0(x) = U^\star(x - \Delta x, -T_0 - \Delta T) + W^{\mathrm{s}}(x - \Delta x) + \eta(W^{\mathrm{s}}; T_0 + \Delta T).$$

then we would be done. Let $T := T_0 + \Delta T$, $\tilde{x} := x - \Delta x$ and $\tilde{W}_0(\tilde{x}) := W_0(x)$. Define

$$\theta(W^{\mathrm{s}}, \Delta x, \Delta T; \tilde{W}_0) := U^\star(\tilde{x} + \Delta x, -T + \Delta T) - U^\star(\tilde{x}, -T) + W^{\mathrm{s}}(\tilde{x}) + \eta(W^{\mathrm{s}}; T) - \tilde{W}_0(\tilde{x}).$$

We have $\theta(0, 0, 0; 0) = 0$. We view $\tilde{W}_0$ as a parameter in this function, and we will use the implicit function theorem to show there exist functions $W^{\mathrm{s}}(\tilde{W}_0)$, $\Delta x(\tilde{W}_0)$ and $\Delta T(\tilde{W}_0)$ defined for $\tilde{W}_0$ in a neighborhood of the origin (in H^1) such that the equation

$$\theta\left(W^{\mathrm{s}}(\tilde{W}_0), \Delta x(\tilde{W}_0), \Delta T(\tilde{W}_0); \tilde{W}_0\right) = 0.$$

To do so, we need only show that the derivative of θ with respect to $(W^{\mathrm{s}}, \Delta x, \Delta T)$ is invertible. To do so, first let $\theta^{\mathrm{s}} := \Pi^{\mathrm{s}}_T \theta$, $\theta^{\mathrm{c}} := \Pi^{\mathrm{c}}_T \theta$ and

$$\Theta := \begin{pmatrix} \theta^{\mathrm{s}} \\ \theta^{\mathrm{c}} \end{pmatrix}.$$

Then, the derivative of θ with respect to $(W^{\mathrm{s}}, \Delta x, \Delta T)$ is the derivative of Θ with respect to the same variables. This is

$$L := \begin{pmatrix} \partial_{W^{\mathrm{s}}} \theta^{\mathrm{s}} & \partial_{\Delta x} \theta^{\mathrm{s}} & \partial_{\Delta T} \theta^{\mathrm{s}} \\ \partial_{W^{\mathrm{s}}} \theta^{\mathrm{c}} & \partial_{\Delta x} \theta^{\mathrm{c}} & \partial_{\Delta T} \theta^{\mathrm{c}} \end{pmatrix}.$$

Note that since X^{c}_T is two-dimensional, this matrix is "square." (We remark that all derivatives are evaluated at $(W^{\mathrm{s}}, \Delta x, \Delta T) = (0, 0, 0)$.)

We have

$$\begin{aligned} \partial_{\Delta x} \theta =& \partial_x U^\star(\tilde{x}, T) \\ =& Q'_+(\tilde{x} - cT) + Q'_-(\tilde{x} + cT) + \partial_x R(\tilde{x}, -T) \\ =& Q'_+(\tilde{x} - cT) + Q'_-(\tilde{x} + cT) + O(\mathrm{e}^{\delta T}) \end{aligned}$$

and

$$\begin{aligned} \partial_{\Delta T} \theta =& \partial_t U^\star(\tilde{x}, T) \\ =& cQ'_+(\tilde{x} - cT) - cQ'_-(\tilde{x} + cT) + \partial_t R(\tilde{x}, -T) \\ =& cQ'_+(\tilde{x} - cT) - cQ'_-(\tilde{x} + cT) + O(\mathrm{e}^{\delta T}). \end{aligned}$$

Thus, if we make the identification of $\begin{pmatrix} k_+ \\ k_- \end{pmatrix} \in \mathbf{R}^2$ with $k_+ Q'_+(\tilde{x} - cT) + k_- Q'_-(\tilde{x} + cT) \in X^{\mathrm{c}}_T$, we have

$$(\partial_{\Delta x} \theta^{\mathrm{c}} \;\; \partial_{\Delta T} \theta^{\mathrm{c}}) = \begin{pmatrix} 1 & c \\ 1 & -c \end{pmatrix} + O(\mathrm{e}^{\delta T}) \tag{10}$$

and

$$(\partial_{\Delta x} \theta^{\mathrm{s}} \;\; \partial_{\Delta T} \theta^{\mathrm{s}}) = O(\mathrm{e}^{\delta T}). \tag{11}$$

Since $\eta \in X^{\mathrm{c}}_T$, we have $\Pi^{\mathrm{s}}_T \eta = 0$ and this in turn implies

$$\partial_{W^{\mathrm{s}}} \theta^{\mathrm{s}} = \mathrm{id}$$

where by id we mean the identity on X_T^{s}. Therefore

$$L = \begin{pmatrix} \mathrm{id} & 0 \\ \partial_{W^{\mathrm{s}}}\theta^{\mathrm{c}} & L_c \end{pmatrix} + O(\mathrm{e}^{\delta T})$$

with

$$L_c := \begin{pmatrix} 1 & c \\ 1 & -c \end{pmatrix}.$$

We remark that $\partial_{W^{\mathrm{s}}}\theta$ is uniformly bounded in T, and so we see that L is a small perturbation of an invertible map and is thus invertible. This concludes the proof. □

Remark 27. *The map from the perturbation to the phase shifts ΔT and Δx defines the smooth strong stable foliation of a neighborhood of the manifold. If the perturbation W^{s} is localized around the pulse positions, one can compute Δx and T_0 to leading order by projecting along the strong stable foliations of the single pulses, separately. In the proof, this is reflected by the fact that E_{bc} is small in this case and therefore $\partial_{W^{\mathrm{s}}}\theta^{\mathrm{c}}$ is small; our approximation for the stable subspace, composed as the direct sum of the stable subspaces for the individual pulses, Lemma 19, is exponentially close to the "true" stable subspace to the non-autonomous evolution.*

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