Subcritical bifurcation to infinitely many rotating waves

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Abstract

We consider the equation $u'' + \frac{1}{r}u' - \frac{k^2}{r^2}u = \lambda u + au|u|^2$ on $r \in \mathbb{R}_+$ with $k \in \mathbb{N}$, $a, \lambda \in \mathbb{C}$, $\mathrm{Re}\,\lambda > 0 > \mathrm{Re}\,a$, and $|\,\mathrm{Im}\,\lambda| + |\,\mathrm{Im}\,a| << 1$. Bounded solutions possess an interesting interpretation as rotating wave solutions to reaction-diffusion systems in the plane. Our main results claim that there are countably many solutions which are decaying to zero at infinity. The proofs rely on nodal properties of the equation and a Melnikov analysis.

1 Introduction

The problem of finding spiral wave solutions in reaction-diffusion systems has been studied intensively throughout the last fifteen years. In order to be able to address this problem, many authors assumed the reaction term to be in a specific form which allows for a decoupling of Fourier modes. These reaction-diffusion equations

$$u_t = D\triangle u + ug(|u|), \quad u \in \mathbb{C} \tag{1.1}$$

were called λ-ω systems and many interesting results on the existence of nonlinear waves under various assumptions on the particular structure of the reaction term have been derived [1, 2, 3, 4, 5, 6].

Recently, a systematic and mathematically rigorous procedure has been found, which allows to prove the approximation property of λ-ω systems for general reaction-diffusion equations [8]. The crucial assumption is that a homogeneous steady state is close to a Hopf bifurcation point in the pure reaction system. The ODE describing the shape of spiral wave solutions is the same as the one which can be derived from λ-ω systems. In a typical example, it is shown in [8] that the equations are of the form

$$u'' + \frac{1}{r}u' - \frac{k^2}{r^2}u = \lambda u + au|u|^2 \tag{1.2}$$

with $' = \frac{d}{dr}$, $a \in \mathbb{C}$ and the complex parameter λ being close to zero. The spiral wave solution to the original reaction diffusion equation is then approximately given by an expression of the form $U(r,\varphi,t) = u(r)e^{ik\varphi}e^{ict}$; see [8]. The speed of rotation c of the spiral wave – whose value must be found as a part of the problem – determines the imaginary part of the parameter λ. Indeed $d\lambda_I/dc \neq 0$, which allows us to control the imaginary part λ_I by the wave speed c.

The equation (1.2) has been studied for small imaginary parts of the parameters λ and a in the case when $\lambda_R = -1$ and $a_R = 1$; see [2, 3, 6]. As $\lambda_R < 0$ corresponds to an unstable zero state in the reaction-diffusion system (1.1), this case can be interpreted as a supercritical bifurcation. Here we address our attention to the case of a subcritical bifurcation, that is we suppose throughout this work that

$$a_R = -1 \quad \text{and} \quad \lambda_R = 1.$$

Moreover we assume that the imaginary part of a is small, quite as in the quoted references on the supercritical bifurcation.

Solutions to (1.2) which are bounded at $r = 0$ actually satisfy the expansion $u(r) = \alpha r^k + \mathrm{O}(r^{k+1})$. We are interested in localized solutions: we require that $u(r)$ decays to zero as $r \to \infty$.

The next propositions state that in the limit $a_I = \lambda_I = 0$, there are countably many solutions of this type.

Proposition 1 *Suppose $a_I = \lambda_I = 0$. Then for all $k \in \mathbb{N}$, (1.2) possesses a solution $u_0(r)$ such that $u_0(r) > 0$ for all $r \in (0,\infty)$ and $u_0(0) = u_0(\infty) = 0$.*

Proposition 2 *Suppose $a_I = \lambda_I = 0$. Then for all $k, n \in \mathbb{N}$, (1.2) possesses a solution $u_n(r)$ such that $u_n(0) = u_n(\infty) = 0$ and $u_n(r)$ possesses exactly n simple zeroes in $(0,\infty)$.*

The proofs are carried out in the next section exploiting the nodal structure of equation (1.2) and of its linearization

$$v'' + \frac{1}{r}v' - \frac{k^2}{r^2}v = v - 3u^2 v, \tag{1.3}$$

restricted on the real subspace $u_I = u_I' = 0$, In section 5 we give a completely different proof using variational arguments. In order to be able to consider a_I nonzero we need more detailed information on the solution:

Proposition 3 *The solutions $u_j(r)$, $j = 0, 1, 2, \ldots$, are transverse in the real subspace. The variational equation (1.3) with $u = u_j(r)$ does not possess a bounded solution on $(0, \infty)$.*

Our last result concerns the existence of localized solutions when a_I is close to zero.

Proposition 4 *Fix $k, n \in \mathbb{N}$ and a neighborhood of the solution $u_n(r)$. Then there is a smooth function $\lambda_I(a_I)$ with $\lambda_I(0) = 0$ such that eq. (1.2) possesses a bounded solution $u_n(r, a_I)$ with $u_n(0, a_I) = u_n(\infty, a_I) = 0$. Moreover this solution is unique in the fixed neighborhoods of $u_n(r)$ up to multiplication with $e^{i\varphi}$, $\varphi \in \mathbb{R}$.*

In the next two sections we proof our propositions for the positive solution $u_0(r)$. We then outline the necessary modifications for the solutions $u_j(r)$. In section 5 we then give an alternative proof using variational methods. We conclude with a brief discussion on the implication of the results presented here.

2 Proof of Propositions 1 and 3

The real system is given after a suitable rescaling by

$$u'' + \frac{1}{r}u' - \frac{k^2}{r^2}u = u - u^3. \tag{2.1}$$

Any solution which is bounded at $r = 0$ is of the form $u(r) = \alpha r^k + \mathrm{O}(r^{k+1})$.

We study in detail the variational equation of (2.1)

$$v'' + \frac{1}{r}v' - \frac{k^2}{r^2}v = v - 3u^2 v. \tag{2.2}$$

Any solution of this equation which is bounded at $r = 0$ is proportional to the derivative $\partial u/\partial \alpha$ and $u(r; \alpha)$ is the unique bounded solution of (2.1) growing like αr^k for small r.

For large α, solutions of (2.1) are approximated by an autonomous equation as follows. We set $\alpha \tilde{u} = u$, $\alpha r = s$ and obtain

$$\tilde{u}_{ss} + \frac{1}{s}\tilde{u}_s - \frac{k^2}{s^2}\tilde{u} = \frac{1}{\alpha^2}\tilde{u} - \tilde{u}^3,$$

which is close to the homogeneous equation

$$\tilde{u}_{ss} + \frac{1}{s}\tilde{u}_s - \frac{k^2}{s^2}\tilde{u} = -\tilde{u}^3.$$

Rescaling time $s = e^t$ and setting $e^t \tilde{u} = \hat{u}$ yields $\tilde{u}_{tt} - k^2 \tilde{u} = -e^{2t}\tilde{u}^3$ and

$$\hat{u}_{tt} - 2\hat{u}_t - (1 + k^2)\hat{u} = -\hat{u}^3. \tag{2.3}$$

Without the negative damping $-2\hat{u}_t$ the phase portrait of this system is the well-known double homoclinic loop to the origin, filled and surrounded by periodic orbits.

If $u(r)$ is bounded as $r \to 0$, then $u \sim \alpha r^k$, $\tilde{u} \sim r^k = (1/\alpha)^k s^k$, $\hat{u} \sim (1/\alpha)^k e^{(k+1)t} \to 0$ as $t \to -\infty$. Therefore $(\hat{u}, \hat{u}')$ belongs to the unstable manifold of the origin in (2.3) and different parameter values α now correspond just to a time shift.

Lemma 2.1 *Let α be sufficiently large. Then there exists $R_u(\alpha) > 0$ such that $u(r) > 0$ on $(0, R_u(\alpha))$ and $u(R_u(\alpha)) = 0$. Furthermore there is $R_v(\alpha) < R_u(\alpha)$ such that the solution of the variational equation (2.2) satisfies*

$$v(r) > 0 \text{ on } (0, R_v(\alpha)),\ \ v(r) < 0 \text{ on } (R_v(\alpha), R_u(\alpha)] \text{ and } v' < 0 \text{ on } [R_v(\alpha), R_u(\alpha)].$$

Moreover, this implies that $dR_u/d\alpha < 0$.

Proof. We solve the scaled autonomous equation (2.3). Global existence is ensured as the function $2\hat{u}_t^2 + \hat{u}^4$ can grow at most linearly with time. Level lines of the autonomous system, given by $\hat{u}_t^2 - (1+k^2)\hat{u}^2 + \frac{1}{2}\hat{u}^4 \equiv const$ are always crossed outwards. One can easily check that for positive energy values the solution $\hat{u}$ cannot stay positive (otherwise $\hat{u}$ would have to get unbounded but for large values of $\hat{u}$ the rotational component $-\hat{u}^3$ of the vector field becomes dominant). Shooting with the unstable manifold yields the desired positive solution $\hat{u}$ with some $R_u(\alpha)$, where $\hat{u}(t(R_u(\alpha))) = 0$ and $\hat{u}_t(\alpha) < 0$.

The transverse intersection with the axis $\hat{u} = 0$ persists for finite α when adding the perturbation term $\tilde{u}/\alpha^2$.

The claim on the sign of $v = \partial u/\partial\alpha$ is a claim on the sign of $\hat{u}_t$ in the limit $\alpha = \infty$ and an immediate consequence of the phase portrait.

The derivative of R_u is calculated from

$$0 = \frac{du(R_u(\alpha))}{d\alpha} = \frac{\partial u}{\partial\alpha}(R_u(\alpha)) + \frac{\partial u}{\partial r}(R_u(\alpha))\frac{dR_u(\alpha)}{d\alpha} = v + u'R'$$

and $v < 0$, $u' < 0$, because $\hat{u}_t < 0$ at $R_u(\alpha)$. ∎

For α close to zero, we can show that the solution $u(r)$ does not possess any zero.

Lemma 2.2 *Suppose α is sufficiently small. Then the solution $u(r)$ is strictly positive for all $r > 0$.*

Proof. We use a shooting argument. Let $\mathcal{M} \subset \mathbb{R}^2 \times \mathbb{R}_+$ denote the manifold of solutions $(u, u')(r; \alpha)$ bounded at $r = 0$. Then the tangent space of $\mathcal{M}$ along $u = u' = 0$ is calculated from the linear equation

$$v'' + \frac{1}{r}v' - (\frac{k^2}{r^2} + 1)v = 0, \qquad v(0) = 0.$$

The solutions are multiples of the modified Bessel functions of the first kind $I_k(r)$; see [10]. Asymptotic expansions for these functions yield $\dfrac{v'(r)}{v(r)} = 1 - \dfrac{1}{2r} + \mathrm{O}(\dfrac{1}{r^2})$. In particular, for large r, we see that $v'/v \nearrow 1$.

Next we construct a forward invariant region close to $r = \infty$ where $u \geq 0$ and we show that a part of $\mathcal{M}$ gets trapped in this region, excluding zeroes of the corresponding solutions.

We consider the original equation, extended by the equation $\beta' = -\beta^2$ with $\beta = 1/r$ to make out of it an autonomous equation. At $\beta = 0$, $u = u' = 0$, we have an equilibrium with a uniquely defined two-dimensional center-stable manifold. The intersection of this manifold with the plane $\beta = 0$ is the homoclinic curve $q(r) > 0$ (and its symmetric). The tangent space of the center-stable manifold along this solution evolves under the linearized equation

$$\begin{aligned} u'' - u + 3q^2(r)u + \beta q'(r) &= 0 \\ \beta' &= 0. \end{aligned}$$

The positive damping term $\beta q'(r)$ forces solutions to the equation for fixed $\beta > 0$, which are bounded for $r \to \infty$, to cut the axis $u = 0$ at $u' > 0$ at a finite time $r = R(\beta)$.

The invariant region S we were looking for is now constructed as being bounded by:

- the interior of the homoclinic (q, q') in $\beta = 0$,
- the plane $u = 0$,
- the center-stable manifold and
- the plane $\beta = \beta_0 > 0$ sufficiently small.

All boundaries are flow invariant, except the planes $u = 0$ and $\beta = \beta_0$, where the vector field is pointing strictly inwards. By the calculations on the tangent spaces of the center-stable manifold and the shooting manifold $\mathcal{M}$, there are bounded trajectories $u(r)$ entering S. These trajectories do not possess any zeroes of u in S. Making α sufficiently small, we can guarantee that such a solution u is close to $I_k(r)$, the modified Bessel function of the first kind [10] as long as it stays outside of S and thereby does not possess any zeroes at all. ■

Now we want to decrease α, preserving the sign structure from Lemma 2.1. Suppose that v would achieve its minimum on $(0, R_u(\alpha))$ and suppose it would be negative. For α sufficiently large v does not achieve its minimum in the interior of the interval by the previous lemma. A minimum could appear in the interior of the interval if either at a point $r = R_0$ we had $v' = 0$ and $v'' = 0$ – which is excluded because then necessarily $v \equiv 0$ – or, a minimum could become negative – but then again $v' = v = 0$ would imply $v \equiv 0$ – or, alternatively, a minimum could enter through the boundary, at $R_u(\alpha)$. But then at $R_u(\alpha)$ we would have $v'' \geq 0$, $v' = 0$, $v < 0$ and $u = 0$. Using the equation for v this would imply

$$0 \leq v'' + \frac{1}{r}v' = (\frac{k^2}{r^2} + 1)v - 3u^2 v = (\frac{k^2}{r^2} + 1)v < 0,$$

a contradiction. Thereby v is strictly decreasing on the interval $(R_v(\alpha), R_u(\alpha))$ as long as $R_v(\alpha) < R_u(\alpha)$.

Lemma 2.3 *For all $\alpha > 0$ we have $R_v(\alpha) < R_u(\alpha)$ and $R_v(\alpha) \leq \bar{R}$ for all α with $R_u(\alpha) < \infty$. In particular there is $\alpha_0 > 0$ such that $R_u(\alpha_0) = \infty$ and $R_v(\alpha_0) < \infty$.*

Proof. We argue by contradiction. Suppose $R_v(\alpha) = R_u(\alpha) = R$. Then $u, v > 0$ on $(0, R)$ and $u = v = 0$ on $\{0, R\}$. As both $u, v \sim r^k$ at 0 and $u', v' < 0$ at R, there is a λ such that $\lambda v > u$ on $(0, R)$. We define

$$\lambda^* = \inf\{\lambda |\, \lambda v > u \text{ on } (0, R)\}.$$

This implies $\lambda^* v - u \geq 0$ for all $r \in (0, R)$ and at some $R_0 \in [0, R]$ we have $\lambda^* v(R_0) - u(R_0) = 0$, $\lambda^* v'(R_0) - u'(R_0) = 0$ and $\lambda^* v''(R_0) - u''(R_0) \geq 0$. But from (2.1) and (2.2) we can deduce that $(\lambda^* v'' - u'')(R_0) = -2u^3(R_0) \leq 0$ with strict inequality, in case $R_0 \in (0, R)$, which is thereby ruled out.

Now suppose that $R_0 = R$. From the equation for $w = \lambda^* v - u$

$$w'' + \frac{1}{r} w' = (\frac{k^2}{r^2} + 1 - u^2) w - 2u^2 v,$$

and from the condition $w = w' = 0$ at $r = R$ we can get expansions of w at R, equating the lowest order terms $w'' = -2u^2 v$; then $w^{(5)} = -12(u')^3 > 0$ and, for $r < R$ but r close to R, w is negative which contradicts $\lambda^* v - u \geq 0$.

If $R = 0$, we can conclude that both, $\lambda^* v$ and u are at leading order given by αr^k for some $\alpha > 0$. Then $w'' = -2\alpha^3 r^{3k} < 0$ at leading order and thereby again $w(r) < 0$ for small enough r.

This proves $dR_u(\alpha)/d\alpha < 0$ for all $\alpha \geq \alpha_0 \geq 0$, and $R_u(\alpha_0) = \infty$. By Lemma 2.2, we know that $\alpha_0 > 0$.

In order to prove the lemma, we have to exclude that $R_v(\alpha_0) = \infty$. This follows for the same reasons as for the finite interval above. We can similarly define $\lambda^*_\infty = \inf\{\lambda | \, \lambda v > u \text{ on } (0, \infty)\}$ because u and v possess at leading order the same exponential decay property $\sim e^{-r}$ as $r \to \infty$. Then again $u \neq \lambda^*_\infty v$ on $[0, \infty)$ by the same arguments as above. The last possibility we have to rule out is that $R_0 = \infty$. Then actually $w(r)$ is given by the variation of constants formula

$$w(r) = \int_r^\infty \Phi(r, s) 2u^2(s) v(s) ds$$

with the linear evolution operator Φ given by

$$\Phi(r, s) = \frac{I_k(s) K_k(r) - I_k(r) K_k(s)}{I_k(s) Y_k'(s) - I_k'(s) Y_k(s)}.$$

Here the I_k and K_k are the modified Bessel functions of the first and second kind. The expression in the numerator is the Wronski determinant and strictly positive, whereas the expression in the denominator is negative for large r, as $I_k(r) \sim e^r / r^{1/2}$ and $K_k(r) \sim e^{-r} / r^{1/2}$; see [10] . Thereby again $w(r) < 0$, for large r, and we have reached a contradiction. This proves the lemma. ∎

Together with the previous lemmata the proof of Proposition 1 and Proposition 3 for $u_0(r)$ is now easy. We define $u_0(r) = u(r; \alpha_0)$ with $\alpha = \alpha_0 > 0$ from the previous lemma. This solution is bounded, converges to zero at infinity and is transverse, again by Lemma 2.3.

3 Proof of Proposition 2

We mimic the proof for $n = 0$. As in Lemma 2.1, we can guarantee that for large α there are solutions bounded at $r = 0$ with infinitely many zeroes, winding around the two homoclinic curves of the autonomous problem ($r = \infty$). They possess a sequence of non-degenerate zeroes $R_u^n(\alpha)$. Similarly the solution to the variational equation (1.3) possesses a sequence of non-degenerate zeroes $R_v^n(\alpha)$ and we have $R_v^0(\alpha) < R_u^0(\alpha) < R_v^1(\alpha) < R_u^1(\alpha) < \ldots$. We have to continue this pattern for decreasing α. As in the previous section, we can conclude that $\frac{dR_u^n}{d\alpha} < 0$ if we can ensure that $R_v^n(\alpha) < R_u^n(\alpha) < R_v^{n+1}(\alpha)$. Arguing as in the case

$n = 0$, we can show that $v(r;\alpha)$ cannot achieve its local minimum on $(R_v^n(\alpha), R_u^n(\alpha))$. It is therefore sufficient to prove an analogue of Lemma 2.3. Proceeding by induction on n, we compare u and v on $(R_u^{n-1}(\alpha), (R_u^n(\alpha))$. By the induction hypothesis, sign $v(R_u^{n-1}(\alpha)) = (-1)^{n+1}$. Now assume that $v(R_u^n(\alpha)) = 0$ and, to fix signs, $v \leq 0$ on $(R_u^{n-1}(\alpha), (R_u^n(\alpha))$ (n is supposed to be even, the case of n odd being similar). Then there is a λ^* such that $w = \lambda^* v - u \leq 0$ on $(R_u^{n-1}(\alpha), (R_u^n(\alpha))$ and $w(R) = 0$ for some $R \in (R_u^{n-1}(\alpha), (R_u^n(\alpha)]$. Then w achieves its local maximum in R which is however forbidden from (1.2) and (1.3), because $w'' > 0$ where $w' = w = 0$ and $v < 0$. Thus we have reached a contradiction showing that there is $\alpha_n > 0$ such that $R_u^n(\alpha_n) = \infty$. Arguing as above and in Lemma 2.3, it is easy to see that $R_v^n(\alpha_n) < \infty$, which shows that the solutions are transverse. This proves Proposition 2 and Proposition 3 for $u_n(r)$.

4 Proof of Proposition 4

The proof requires a Melnikov type calculation. In the phase space extended by the equation for $\beta = 1/r$, our solutions are transverse intersections of a shooting manifold $\mathcal{M}$ (the set of solutions bounded at $r = 0$) and the center-stable manifold $W^{cs}(0)$ of the origin $u = u' = \beta = 0$. The intersection is transverse only when restricted to the real subspace $u = u' = 0$. The complex problem, $a_I \neq 0$, possesses an additional S^1-symmetry $(u, u') \to (e^{i\varphi}u, e^{i\varphi}u')$ for $\varphi \in \mathbb{R}$. Due to this symmetry, there is one direction orthogonal to the sum of the tangent spaces of $\mathcal{M}$ and $W^{cs}(0)$ at the heteroclinic orbits $u_n(r)$. This direction is orthogonal to the generator of the rotational symmetry at the heteroclinics $i(u_n(r), u_n'(r)$, and therefore given as $i(u_n'(r), -u_n(r))$. The derivative of the vector field with respect to λ_I, our perturbation parameter, points in the direction $(0, iu_n)$. The scalar product with the direction orthogonal to the sum of the tangent spaces has a definite sign for all r and gives a nonzero contribution to the Melnikov integral. In other words, the two manifolds intersect transversely if the phase space is extended by the equation $\lambda_I' = 0$. This transverse intersection persists at a point $\lambda_I(a_I)$ for small perturbations a_I. This proves Proposition 4.

5 Variational approach

Proposition 1 might be proved using variational methods. We have to consider

$$u'' + \frac{1}{r}u' - \frac{k^2}{r^2}u = u - u^3. \tag{5.1}$$

We want to apply mountain-pass lemma to a variational formulation of the equation

$$I(u) = \int_{\mathbb{R}_+} [u_r^2 + (\frac{k^2}{r^2} + 1)u^2 - \frac{1}{2}u^4] r\, dr, \quad u \in H^{1,2}(\mathbb{R}_+).$$

Of course, any critical point of the functional I gives a solution of the above equation on $\mathbb{R}_+$.

We next apply the mountain pass lemma to our functional. First of all zero is a non-degenerate local minimum: the kernel of the linearization are the Bessel functions of the first kind which however do not lie in $H^{1,2}(\mathbb{R}_+)$ as they grow exponentially at $r \to \infty$.

On the other hand, the functional decays to $-\infty$ along any ray $s \cdot u$, $u \in H^{1,2}(\mathbb{R}_+)$ fixed, $s \in \mathbb{R}_+$.

It remains to establish convergence of a Palais-Smale sequence, a non-trivial task due to non-compactness at 0 and $+\infty$. We do not carry out details here.

This would then establish the existence of a heteroclinic orbit as claimed in Lemma 1. Transversality does not follow from this construction.

We suspect that one could prove as well the existence of infinitely many critical points, using the $\mathbb{Z}_2$-symmetry of the functional, $u \to -u$, see for example [9, Chapter II, Theorem 6.5 and 6.6].

6 Discussion

As already pointed out in the introduction, the solutions proved to exist in Proposition 4 have an interesting interpretation as localized rotating wave solutions of reaction-diffusion equations. A particular equation undergoing a Hopf bifurcation and exhibiting such spatio-temporal phenomena was given in [8].

The solutions are, in contrast to the ones found for supercritical bifurcations, localized, that is, along rays emanating from the origin, the amplitude and derivative of the phase of the solutions decay exponentially to zero. In particular for the solutions $u_0(r)$, regions of constant phase form arcs which run from the origin to infinity, asymptotic to a straight line through the origin. The solutions with zeroes of the amplitude form more complicated patterns: there are n circles, where the amplitude gets close to zero. The phase changes sign, when crossing these circles.

We suspect that the localized solutions of Proposition 1 and Proposition 2 are unique as localized solutions with a prescribed number of zeroes.

We did not try to prove stability or instability of the solutions for the full reaction-diffusion system. The considerations on a variational approach in Section 5 suggest that all waves are unstable, with Morse index increasing with n (which is well defined because the continuous spectrum of the linearization is bounded away from the imaginary axis, see [7, Lemma 5.4]).

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