# HA - 10

8.2.1 (c) $p(\lambda) = \lambda^2 - 4\lambda + 4$, $\lambda_{1/2} = 2$

$$\begin{matrix} 1 & 1 \\ -1 & -1 \end{matrix} \leadsto \begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix} \quad u = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(g) $p(\lambda) = -\lambda^3 + 2\lambda^2 - 2\lambda$, $\lambda_1 = 0$, $\lambda_{2/3} = 1 \pm i$

$$u_1 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 3-2i \\ 3-i \\ 1 \end{pmatrix}, \quad u_3 = \overline{u_2}$$

(i) $p(\lambda) = p_1(\lambda) \cdot p_2(\lambda)$

$$p_1(\lambda) = \begin{vmatrix} 3-\lambda & 4 \\ 4 & 3-\lambda \end{vmatrix} = \lambda^2 - 6\lambda - 7$$

$$p_2(\lambda) = \begin{vmatrix} 1-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} = \lambda^2 - 6\lambda - 7$$

$\leadsto \lambda_1 = 7, \lambda_2 = -1$

$\lambda_1$: $u_{11} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $u_{12} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}$

$\lambda_2$: $u_{21} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $u_{22} = \begin{pmatrix} 0 \\ 0 \\ -3 \\ 2 \end{pmatrix}$

8.2.2 a) $p(\lambda) = \lambda^2 - 2\cos\theta + 1$, $\lambda_{1/2} = \cos\theta \pm i\sin\theta$

$u_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

(b) $\theta = 0$, $\theta = \pi$ $(\pm 1)$

8.2.6 $p(\lambda) = -(\lambda^3 + (a^2+b^2+c^2)\lambda)$, $\lambda_1 = 0$, $\lambda_{2/3} = \pm i\sqrt{a^2+b^2+c^2}$

$\lambda_1$: $u_1 = (a,b,c)^T$

$\lambda_2$: $u_2 = \begin{pmatrix} ab - ic\sqrt{a^2+b^2+c^2} \\ -(a^2+c^2) \\ bc + ia\sqrt{a^2+b^2+c^2} \end{pmatrix}$, $u_3 = \overline{u_2}$

8.2.8 a) $p(\lambda) = 0 \to \lambda = 0$, any vector u is an ev

b) $p(\lambda) = (1-\lambda)^n \to \lambda = 1$, any vector u is an ev

8.2.9 $A\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = n \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \to \lambda = n$ eigenvalue

$A\begin{pmatrix} -1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = 0$, $A\begin{pmatrix} 1 \\ -1 \\ \vdots \\ 1 \end{pmatrix} = 0 \ldots A\begin{pmatrix} 1 \\ \vdots \\ -1 \\ 1 \end{pmatrix} = 0$

$\lambda = 0$ eigenvalue, mult. $= n-1$

8.2.20

$\lambda$ eigenvalue, ex. $u \neq 0$: $Au = \lambda u$

$\Rightarrow A(Au) = A(\lambda u) = \lambda Au = \lambda \lambda u$
$= \lambda^2 u$ ✓

8.2.21 (a) false $\lambda = 1$ is eigenvalue of
$A = I$, $B = I$, but not of
$A + B = 2 \cdot I$, which has ev's 2

(b) true $\left.\begin{array}{l} Av = \lambda v \\ Bv = \mu v \end{array}\right\} (A+B)v = (\lambda + \mu)v$

8.2.22 false: $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\uparrow$ $\lambda = 1$ $\quad$ $\uparrow$ $\mu = 1$ $\quad$ $\uparrow$ ev 0

8.2.24 (a) $Au = \lambda u$ (if $A^{-1}$ exists)

$\Rightarrow u = A^{-1} \lambda u$
$\Rightarrow u = \lambda A^{-1} u$
$\Rightarrow \lambda^{-1} u = A^{-1} u$

(b) $A^{-1}$ does not ex....

8.2.26 A singular $\Leftrightarrow$ ker$(A) \neq \{0\}$

$\Leftrightarrow$ ex. $u \neq 0$ so that $Au = 0$

$\Leftrightarrow$ $\lambda = 0$ is an ev

8.2.38 $u = \underbrace{Pu}_{v} + \underbrace{(I-P)u}_{w}$

if $u = Pu \Rightarrow$ u is ev to eigenvalue 1

$0 = Pu \Rightarrow$ u is ev to eigenvalue 0

v satisfies $Pv = v$ since $Pv = P^2u = Pu = v$

w satisfies $Pw = 0$ since $Pw = P(I-P)w =$

$(P - P^2)w = 0$

so $\{0, 1\}$ are all eigenvalues

eigenvectors

$\lambda = 0$ : ker$\{P\}$

$\lambda = 1$ : Rg$\{P\}$

8.2.40 (a) $u = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \Rightarrow Au = u$

(b) $A^T u = u$ with $u$ as in (a), so

$\det(A^T - I) = 0$, so $\det(A - I) = 0$

so $\lambda = 1$ is ev.

8.2.41 (a) $\det(Q - \lambda) = 0$

$\Leftrightarrow \det(Q^T - \lambda) = 0$

$\Leftrightarrow \det(Q^{-1} - \lambda) = 0$

$\Leftrightarrow \lambda^{-1}$ ev by 8.2.24

(b) $(Q - \lambda)u = 0 \Rightarrow \|Qu\| = \|\lambda u\|$

$\|Qu\| = \|u\|$, so $|\lambda| = 1$

(c) $\left.\begin{aligned} Qx &= \lambda x - \mu y \\ Qy &= \mu x + \lambda y \end{aligned}\right\}$ $|x|^2 = \lambda^2 |x|^2 + \mu^2 |y|^2 - 2\lambda\mu\langle x, y\rangle$
$\langle x, y\rangle = \langle \lambda x - \mu y, \mu x + \lambda y\rangle$

$\Rightarrow \mu^2(|x|^2 - |y|^2) = -2\lambda\mu\langle x, y\rangle$

$\lambda^2 + \mu^2 = 1$ $\quad 2\mu^2\langle x, y\rangle = \lambda\mu(|x|^2 - |y|^2)$

$\mu \neq 0 \Rightarrow |x|^2 = |y|^2, \ \langle x, y\rangle = 0$

8.3.3 (a) yes, $\lambda_1 = 4$, $\lambda_2 = -2$, $u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

8.3.4. (b) no for real $\lambda_{1/2} = 1 \pm 3i$

yes for complex $u_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$, $u_2 = \overline{u_1}$

(c) no, $\lambda_1 = 1$, $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(f) yes, $\lambda_1 = -2$, $\lambda_2 = 2$, $\lambda_3 = 0$

$u_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 8 \\ 5 \\ 7 \end{pmatrix}$, $u_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

(h) no $\mathbb{R}$, yes $\mathbb{C}$ $\quad p(\lambda) = \lambda^4 + 2\lambda^3 + \lambda^2 - 2\lambda - 2$

$\lambda_1 = -1+i$, $\lambda_2 = -1-i$, $\lambda_3 = -1$, $\lambda_4 = 1$

$u_1 = \begin{pmatrix} -1 \\ -i \\ i \\ 1 \end{pmatrix}$, $u_2 = \overline{u_1}$, $u_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $u_4 = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 6 \end{pmatrix}$

8.3.6 (a) yes

(b) no, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

8.3.15 (a) $\lambda_1 = -3, \lambda_2 = 0$, $u_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, u_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$S = \begin{pmatrix} 3 & 3 \\ 2 & 1 \end{pmatrix}$, $S^{-1}AS = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}$

(d) $\lambda_1 = -2, \lambda_2 = 1, \lambda_3 = 3$

$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} -1 \\ -5 \\ 10 \end{pmatrix}$

$S = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -5 \\ 0 & 0 & 10 \end{pmatrix}$, $S^{-1}AS = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

8.3.16 $\lambda_1 = \frac{1}{2}(1+\sqrt{5}), \lambda_2 = \frac{1}{2}(1-\sqrt{5})$

$u_1 = \left(\frac{1}{2}(1+\sqrt{5}), 1\right)^T$

$u_2 = \left(\frac{1}{2}(1-\sqrt{5}), 1\right)^T$

$S = \begin{pmatrix} \frac{1}{2}(1+\sqrt{5}) & \frac{1}{2}(1-\sqrt{5}) \\ 1 & 1 \end{pmatrix}$, $S^{-1}AS = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$