HA 11

8.4.4. If all eigenvalues are distinct, eigenvectors with norm 1 are unique, up to sign $\longrightarrow$ 2^n different bases.

If there are multiple ev, there's ∞-many.

e.g. $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow$ any orthonormal basis is an eigenbasis: $S^{-1}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} S = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

8.4.5. (a) $\lambda_{1/2} = \frac{a+d}{2} \pm \sqrt{\left(\frac{a+d}{2}\right)^2 - ad + bc}$

real if $(a-d)^2 \geqslant -bc$

(b) $b = c \Rightarrow$ r.h.s. is negative!

(c) $b = 1,\ c = 0,\ a = 0,\ d = 0$ $\quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

8.4.14 a) eigenvalues $-5, 5$

eigenvectors $\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\left(\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}/\sqrt{5}\right)^{-1} \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}/\sqrt{5} = \begin{pmatrix} -5 & 0 \\ 0 & 5 \end{pmatrix}$$

8.4.20 $\quad A = Q^{-1} \Lambda Q, \quad Q^T = Q^{-1}, \quad \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

$\overset{?}{\Rightarrow}$ A symmetric?

$A^T = Q^T \Lambda^T (Q^{-1})^T = Q^{-1} \Lambda Q = A$

YES ✓

8.4.40 $\quad q(x) = r^2 (y^T K y)$ with $y = x/r$,

$\|y\| = 1 \Rightarrow \max q(x) = r^2 \max_{\|y\|=1} y^T K y$

$= r^2 \lambda_1$

8.5.1. (b) $A^T A = I \Rightarrow \sigma_1 = \sigma_2 = 1$

(d) $A^T A = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \to \sigma_1 = 9, \sigma_2 = 4$

(f) $A^T A = \begin{pmatrix} 2 & -3 & 1 \\ -3 & 6 & -3 \\ 1 & -3 & 2 \end{pmatrix}, \quad \sigma_1 = 9, \sigma_2 = 1$

8.5.3 (a) $A^TA = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $\sigma_1 = \sqrt{\frac{3+\sqrt{5}}{2}}$, $\sigma_2 = \sqrt{\frac{3-\sqrt{5}}{2}}$

$$q_1 = \begin{pmatrix} \frac{1}{2}(-1+\sqrt{5}) \\ 1 \end{pmatrix} \Big/ \sqrt{\tfrac{1}{2}(5-\sqrt{5})}, \quad q_2 = \begin{pmatrix} \frac{1}{2}(-1-\sqrt{5}) \\ 1 \end{pmatrix} \Big/ \sqrt{\tfrac{1}{2}(5-\sqrt{5})}$$

$$Q = (q_1 \mid q_2), \quad \Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

$$p_j = A q_j / \sigma_j = \ldots$$

8.5.7 (a) $A = v^T \Rightarrow A^TA = vv^T$ has rank 1

$\Rightarrow$ one nonzero ev, the trace, which is $v^Tv = \|v\|^2 \Rightarrow \sigma_1 = \|v\|$

(b) eigenvector $q_1 = v/\|v\|$, since

$$(vv^T)\cdot v/\|v\| = v\, v^Tv/\|v\| = \|v\|\cdot v$$

$$A \cdot \frac{v/\|v\|}{\sigma_1} = v^Tv/\sigma_1^2 = 1$$

$$P = 1, \quad Q = v/\|v\|, \quad \Sigma = \sigma_1$$

8.5.8. $A = v \Rightarrow A^TA = v^Tv$

$A^T = P\Sigma Q^T \Rightarrow A = Q\Sigma P^T$

$= \frac{v}{\|v\|}\sigma_1$

from 8.5.7 ...

8.5.11 yes, from SVD as above

8.5.15 true, except for zero-eigenvalues...

8.5.16 no, see 8.5.3. (a)

8.5.17 no: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to A^TA = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \to \sigma_1 = 1$

$A^2 = 0 \Rightarrow$ no singular values

8.5.18 no, $S = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\sigma_1 = 1$

$S^{-1}AS = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$

$\hookrightarrow \sigma_1 = \frac{1}{4}$

8.6.6 (a) eigenvalues $\lambda_1 = 2$

eigenvector $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$(A-\lambda)\, u_2 = u_1$ for $u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(b) eigenvalues $\lambda_1 = -3, \lambda_2 = 6$

eigenvectors $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

no chains

(c) $\lambda = 1$, eigenvectors $u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$(A-\lambda)\, u_3 = u_2$, $u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\left(\begin{array}{c|cc} 1 & & \\ \hline & 1 & 1 \\ & 0 & 1 \end{array}\right)$$

(d) $\lambda = 0$, one J-block only,

$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $(A-\lambda)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$(A-\lambda)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

(e) $\left(\begin{array}{c|cc|c} 4 & 0 & & 0 \\ \hline 0 & 3 & 1 & 0 \\ & 0 & 3 & \\ \hline 0 & 0 & & 2 \end{array}\right)$ $\lambda_1 = 4 \quad u_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

$\lambda_2 = 3 \quad u_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \; u_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$

$\lambda_3 = 2 \quad u_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

8.6.7 $\begin{pmatrix} 2 & & O & \\ & 2 & & \\ & & 2 & \\ O & & & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & & 0 \\ & 2 & & \\ 0 & & 2 & \\ & & & 2 \end{pmatrix}, \begin{pmatrix} 2 & & & \\ & 2 & 1 & \\ & & 2 & \\ & & & 2 \end{pmatrix}$

$\begin{pmatrix} 2 & & & \\ & 2 & & \\ & & 2 & 1 \\ & & & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & & \\ & 2 & 1 & \\ & & 2 & \\ & & & 2 \end{pmatrix}, \begin{pmatrix} 2 & & & \\ & 2 & 1 & \\ & & 2 & 1 \\ & & & 2 \end{pmatrix}$

$\begin{pmatrix} 2 & 1 & & \\ & 2 & & \\ & & 2 & 1 \\ & & & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & & \\ & 2 & 1 & \\ & & 2 & 1 \\ & & & 2 \end{pmatrix}$

8.6.9 (a) $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1/3\end{pmatrix}$

(b) $\lambda = -3$ $\begin{pmatrix}1\\2\end{pmatrix}$ eigenvector

$\begin{matrix} 2 & -1 & 1 \\ 4 & -2 & 2 \end{matrix} \sim \begin{matrix} 2 & -1 & 1 \\ 0 & 0 & 0 \end{matrix}$ $\begin{pmatrix}0\\-1\end{pmatrix}$ gen' ev

(c) eigenvalues 1, eig'vector $\begin{pmatrix}1\\0\\0\end{pmatrix} = u_1$

$$\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \rightarrow \quad u_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \qquad u_3 = \begin{pmatrix}0\\-1\\1\end{pmatrix}$$

(d) eigenvalues -3

$$\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{array} \rightarrow \quad u_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}$$

$(A+3I)u_j = u_{j-1}$

$j = 2, 3.$

$$\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & +1 \end{array} \rightarrow \quad u_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \end{array} \qquad u_3 = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

8.6.10 $\begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}$ $\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$ $\to$ $\begin{matrix} 1 & \frac{1}{\lambda} & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{matrix}$ $\begin{matrix} \frac{1}{\lambda} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$

$\to \dots \to$ $\begin{matrix} 1 & \frac{1}{\lambda} & 0 \\ 0 & 1 & \frac{1}{\lambda} \\ 0 & 0 & 1 \end{matrix}$ $\begin{matrix} \frac{1}{\lambda} & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & \frac{1}{\lambda} \end{matrix}$ $\to \dots \to$ $\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$ $\begin{matrix} \frac{1}{\lambda} & -\frac{1}{\lambda^2} & 0 \\ 0 & \frac{1}{\lambda} & -\frac{1}{\lambda^2} \\ 0 & 0 & \frac{1}{\lambda} \end{matrix}$

$\to \begin{pmatrix} \frac{1}{\lambda} & -\frac{1}{\lambda^2} & & 0 \\ & \ddots & \ddots & \\ & & & -\frac{1}{\lambda^2} \\ 0 & & & \frac{1}{\lambda} \end{pmatrix}$

8.6.12 No $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

are both in Jordan canonical form, and both have geometric mult' 2, alg. mult. 4

8.6.14 No, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ not in canonical form