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10.1.18 (b) eigenvalues $\lambda^2 - 5\lambda + 6 = 0 \Leftrightarrow \lambda_1 = 2$, $\lambda_2 = 3$

$$u_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$S = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$$

$$A^k = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2^k & 0 \\ 0 & 3^k \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} -2^k + 2\cdot 3^k & -2^k + 3^k \\ 2\cdot 2^k - 2\cdot 3^k & 2\cdot 2^k - 3^k \end{pmatrix}$$

10.1.37 (a) $A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$, $u_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$S = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 1/3 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^k = S \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}^k S^{-1} = S \begin{pmatrix} 2^k & k 2^{k-1} \\ 0 & 2^k \end{pmatrix} S^{-1}$$

$$= \begin{pmatrix} 2^k & 3k\, 2^{k-1} \\ 0 & 2^k \end{pmatrix}$$

10.2.3 (a) $\lambda^2+4\lambda+7=0 \Rightarrow \lambda_{1/2}<0$, asy. stable

(b) $\lambda_{1/2}=\frac{1}{2}\pm\frac{1}{2}i$, $|\lambda_{1/2}|=\frac{1}{\sqrt{2}}<1$, asy stable

(c) $\det=-1 \Rightarrow$ at least one λ with $|\lambda|\geq 1$
since $\lambda_1\lambda_2\lambda_3=\det \Rightarrow$ not asy stable

(f) trace $=\lambda_1+\lambda_2+\lambda_3=4 \Rightarrow$ at least one λ with
$|\lambda|\geq 1$, not asy stable

10.2.19 (a) $T^k \to A$

$\Rightarrow T^k \cdot T^k \to A\cdot A = A$

(b) only eigenvalue $\lambda=0, \lambda=1$,
in JNF only blocks of length 1
$\to$ semi-simple $\Rightarrow$ Projection

(c) eigenvalues $|\lambda_j|<1$ or
$\lambda_j=1$, if equal 1 then semi-simple

10.2.25 (a) $(A-I)u=0 \leadsto \begin{pmatrix}-.3 & .3\\ .2 & -.2\end{pmatrix}u=0$

$u=(1\ 1)^T\cdot\alpha$, $\alpha\in\mathbb{R}$

(b) $\begin{pmatrix}-.4 & 1\\ .3 & -1.7\end{pmatrix} \Rightarrow u=0$

10.2.25 (d)

$$\begin{matrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ -1 & -1 & 1 \end{matrix} \rightsquigarrow \begin{matrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

$$\Rightarrow \quad u_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \; u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \rightsquigarrow u = \alpha \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

10.2.26 (a) $\lambda_1 = 1$, $\lambda_2 = \mathrm{tr}(A) - \lambda_1 = 0.5$

$\Rightarrow$ fixed pt asympt. stable

(b) $\lambda_1 = 0.8$, $\lambda_2 = -0.9$ $\Rightarrow$ asy' stable

(d) $\lambda_1 = \lambda_2 = 0$, geom. double, $\lambda_3 = \mathrm{tr}(A) - 2 = 5$

$\Rightarrow$ unstable

10.4.5

From Atlanta, he goes to Boston (always)

From Boston, he goes to Atlanta or Chicago with equal probability.

From Chicago he goes to Atlanta or Boston with equal probability.

$$\begin{matrix} -1 & .5 & .5 \\ 1 & -1 & .5 \\ 0 & .5 & -1 \end{matrix} \rightsquigarrow \begin{matrix} -1 & .5 & .5 \\ 0 & -.5 & 1 \\ 0 & 0 & 0 \end{matrix} \qquad \begin{pmatrix} +1.5 \\ 2 \\ 1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 3/9 \\ 4/9 \\ 2/9 \end{pmatrix} \text{ (normalized to 1)}$$

10.5.6. (a) $D = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$, $L = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$, $U = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$

$$T = -D^{-1}(L+U) = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/5 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1/3 \\ 1/5 & 0 \end{pmatrix}$$

$$c = D^{-1}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/5 \end{pmatrix}$$

$u_0 = 0$

$u_1 = Tu_0 + c = \begin{pmatrix} 2/3 \\ 1/5 \end{pmatrix}$

$u_2 = Tu_1 + c = \begin{pmatrix} 1/15 \\ 2/15 \end{pmatrix} + \begin{pmatrix} 2/3 \\ 1/5 \end{pmatrix} = \begin{pmatrix} 11/15 \\ 5/15 \end{pmatrix}$

$u_3 = Tu_1 + c = \begin{pmatrix} 5/45 \\ 11/75 \end{pmatrix} + \begin{pmatrix} 2/3 \\ 1/5 \end{pmatrix} = \begin{pmatrix} 7/9 \\ 26/75 \end{pmatrix} = (\ldots$

10.6.4. Compare $u^{(k)}$ to $w^{(k)}$ obtained by

$$w^{(k+1)} = Aw^{(k)} : \quad u^{(k)} = \frac{w^{(k)}}{\|w^{(k)}\|} \quad !$$

We already saw $w^{(k)} = \lambda^k c u_1 + \text{"small"}$

eigenvector to eig'n λ_1

So $u^{(k)} \to u_1$

Sign is obtained from $Au_1 = \lambda_1 u_1$

10.6.11 $\begin{pmatrix} 1 & 2 \\ 2 & 6 \end{pmatrix} \to$ $r_{11} = \sqrt{5}$, $r_{12} = 14/\sqrt{5}$, $r_{22} = \frac{2}{\sqrt{5}}$ $\begin{pmatrix} 1/\sqrt{5} & 2 \\ 2/\sqrt{5} & 6 \end{pmatrix} \to \begin{matrix} 1/\sqrt{5} & 2 - 14/5 = -\frac{4}{5} \\ 2/\sqrt{5} & 6 - \frac{28}{5} = \frac{2}{5} \end{matrix}$

$$Q_0 = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}, \quad R_0 = \begin{pmatrix} \sqrt{5} & 14/\sqrt{5} \\ 0 & 2/\sqrt{5} \end{pmatrix}$$

$$A_1 = R_0 Q_0 = \begin{pmatrix} 33/5 & 4/5 \\ 4/5 & 2/5 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} \frac{1481}{221} & \frac{8}{221} \\ \frac{8}{221} & \frac{66}{221} \end{pmatrix}, \quad R_2 = \begin{pmatrix} 6.701 & 0.0378 \\ 0 & 0.2984 \end{pmatrix}$$

$\lambda_1 = 6.701$

$\lambda_2 = 0.2984$

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10.1.18 (b) eigenvalues $\lambda^2 - 5\lambda + 6 = 0 \Rightarrow \lambda_1 = 2$, $\lambda_2 = 3$

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$$A^k = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2^k & 0 \\ 0 & 3^k \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} -2^k + 2\cdot 3^k & -2^k + 3^k \\ 2\cdot 2^k - 2\cdot 3^k & 2\cdot 2^k - 3^k \end{pmatrix}$$

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10.4.5

From Atlanta, he goes to Boston (always)

From Boston, he goes to Atlanta or Chicago with equal probability.

From Chicago he goes to Atlanta or Boston with equal probability.

$$\begin{matrix} -1 & .5 & .5 \\ 1 & -1 & .5 \\ 12 & .5 & -1 \end{matrix} \rightsquigarrow \begin{matrix} -1 & .5 & .5 \\ 0 & -.5 & 1 \\ 0 & 0 & 0 \end{matrix} \qquad \begin{pmatrix} +1.5 \\ 2 \\ 1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 3/9 \\ 4/9 \\ 2/9 \end{pmatrix}$$ (normalized to 1)

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10.6.11

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