HA 4, some sol's

2.3.21 (f) too many, hence dependent

(g) $\begin{pmatrix} 4 & -6 \\ 2 & -3 \\ 0 & 0 \\ -6 & 9 \end{pmatrix} \to \begin{pmatrix} 4 & -6 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ — free variable   dependent

(h) $\begin{matrix} 2 & -1 & 5 \\ 1 & 3 & 1 \\ -1 & 1 & 2 \\ 3 & 0 & -3 \end{matrix} \to \begin{matrix} \cancel{3}^{1} & 0 & -\cancel{3}^{-1} \\ 2 & 1 & -5 \\ 1 & 3 & 1 \\ -1 & 1 & 2 \end{matrix} \to \begin{matrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 3 & 2 \\ 0 & 1 & 1 \end{matrix} \to \begin{matrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix}$

$\to$ independent

2.3.25 $\begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0 \\ 0&0&1 \\ 0&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0 \\ 0&0&1 \\ 1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&1 \\ 0&1&0 \\ 1&0&0 \end{pmatrix}$

are six vectors in $\mathbb{R}^9$, $P_1, \dots P_6$

Assume $c_1 P_1 + \dots + c_6 P_6 = 0$

Check upper left entry ~ first component ....

$$\begin{pmatrix} c_1+c_2 & c_3+c_4 & c_5+c_6 \\ c_3+c_5 & c_1+c_6 & c_2+c_4 \\ c_4+c_6 & c_2+c_5 & c_1+c_3 \end{pmatrix} \overset{!}{=} 0$$

$\Rightarrow c_1 \overset{1,1}{=} -c_2 \overset{2,3}{=} +c_4 = -c_6$
$c_1 \overset{3,3}{=} -c_3 = c_5$ } $c = (1,-1,-1,1,1,-1)^T$ solution

$(c_j = \det P_j)$ $\longrightarrow$ lin' dependent

2.3.26 false: zero is <u>always</u> in span

2.4.2. (c)
$$\begin{matrix} 0 & -1 & 1 \\ 4 & 0 & -8 \\ -1 & 1 & 1 \end{matrix} \rightsquigarrow \begin{matrix} 0 & -1 & 1 \\ 0 & 4 & -4 \\ -1 & 1 & 1 \end{matrix} \rightsquigarrow \begin{matrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{matrix}$$

not lin. indep'

(d) too many

(a) not enough

(b) is a basis...

2.4.3. (a) $0 \quad -2 \quad 1$

free variables — x, z

basis $x=1,\ z=0$

$x=0,\ z=1$

$$\Rightarrow \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\ \tfrac{1}{2}\\1\end{pmatrix}$$

(b) $1 \quad 2 \quad 1 \quad -1$

free variables
$$\begin{cases} y=1,\ z=0,\ w=0 \\ y=0,\ z=1,\ w=0 \\ y=0,\ z=0,\ w=1 \end{cases}$$

↳ compute x in each case

$$\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix} \quad \text{basis}$$

2.5.1.(d)

$$\begin{matrix} 1 & -1 & 0 & 1 & a \\ -1 & 0 & 1 & -1 & b \\ 1 & -2 & 1 & 1 & c \\ 1 & 2 & -3 & 1 & d \end{matrix} \rightsquigarrow \begin{matrix} 1 & -1 & 0 & 1 & a \\ 0 & -1 & 1 & 0 & b+a \\ 0 & -1 & 1 & 0 & c-a \\ 0 & 3 & -3 & 0 & d-a \end{matrix} \rightsquigarrow \begin{matrix} 1 & -1 & 0 & 1 & a \\ 0 & -1 & 1 & 0 & a+b \\ 0 & 0 & 0 & 0 & c-b-2a \\ 0 & 0 & 0 & 0 & d+3b+2a \end{matrix}$$

$$Rg(A) = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \;\middle|\; \begin{matrix} c = b+2a \\ d = 3b+2a \end{matrix} \right\} = \operatorname{span}\left(\begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 3 \end{pmatrix} \right)$$

$$\ker(A) = \left\{ \begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} \;\middle|\; \begin{matrix} x - y + u = 0 \\ -y + z = 0 \end{matrix} \right\} = \operatorname{span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

2.5.2.(d)

$$\begin{matrix} 1 & 2 & 5 & a \\ 0 & 4 & 8 & b \\ 1 & -6 & -11 & c \end{matrix} \rightsquigarrow \begin{matrix} 1 & 2 & 5 & a \\ 0 & 4 & 8 & b \\ 0 & -8 & -16 & c-a \end{matrix}$$

$$\rightsquigarrow \begin{matrix} 1 & 2 & 5 & a \\ 0 & 4 & 8 & b \\ 0 & 0 & 0 & c-a+2b \end{matrix}$$

Rg: $c - a + 2b$!

$$\rightsquigarrow = \operatorname{span}\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right)$$

ker (A) : c free, ker (A) = span $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ is a line

(f) $\begin{matrix} 1 & -2 & 3 & a \\ -3 & 6 & -9 & b \\ -2 & 4 & -6 & c \\ 3 & 0 & -1 & d \end{matrix} \rightsquigarrow \begin{matrix} 1 & -2 & 3 & a \\ 0 & 0 & 0 & b+3a \\ 0 & 0 & 0 & c+2a \\ 0 & 6 & -10 & d-3a \end{matrix}$

Range: $\left.\begin{matrix} b+3a=0 \\ c+2a=0 \end{matrix}\right\}$ c, d free variables

$c=1, d=0 \rightsquigarrow \begin{pmatrix} -1/2 \\ 3/2 \\ 1 \\ 0 \end{pmatrix}$; $c=0, d=1 \rightsquigarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\longrightarrow$ Rg(A) = span$\left(\begin{pmatrix} -1/2 \\ 3/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$

kernel: z free variable $\quad \begin{matrix} 6y-10z=0 \\ x-2y+3z=0 \end{matrix}$

$z=1, y=5/3, x=1/3$

$\longrightarrow$ ker (A) = span $\begin{pmatrix} 1/3 \\ 5/3 \\ 1 \end{pmatrix}$

2.5.21 (c)

$$\begin{array}{cccccc} 1&1&2&1&a \\ 1&0&-1&3&b \\ 2&3&7&0&c \end{array} \rightsquigarrow \begin{array}{cccccc} 1&1&2&1&a \\ 0&-1&-3&2&b-a \\ 0&1&3&-2&c-2a \end{array} \rightsquigarrow \begin{array}{cccccc} 1&1&2&1&a \\ 0&-1&-3&2&b-a \\ 0&0&0&0&c+b-3a \end{array}$$

$$\mathrm{Rg}(A) = \{c+b-3a=0\} \overset{a,b \text{ free}}{=} \mathrm{Span}\left(\begin{pmatrix}1\\0\\3\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}\right)$$

$$\mathrm{ker}(A) \overset{z,u \text{ free}}{=} \mathrm{Span}\left(\begin{pmatrix}1\\-3\\1\\0\end{pmatrix}, \begin{pmatrix}-3\\2\\0\\1\end{pmatrix}\right) > \text{backward substitution}$$

canonical basis in free variables

transpose:

$$\begin{array}{cccc} 1&1&2&x \\ 1&0&3&y \\ 2&-1&7&z \\ 1&3&0&u \end{array} \rightsquigarrow \begin{array}{cccc} 1&1&2&x \\ 0&-1&1&y-x \\ 0&-3&3&z-2x \\ 0&2&-2&u-x \end{array} \rightsquigarrow \begin{array}{cccc} 1&1&2&x \\ 0&-1&1&y-x \\ 0&0&0&z+x-3y \\ 0&0&0&u+2y-3x \end{array}$$

$$\mathrm{ker}(A^T) = \mathrm{span}\begin{pmatrix}-3\\1\\1\end{pmatrix} \quad (z \text{ free variable})$$

$$\mathrm{Rg}(A^T): \left.\begin{array}{l} z+x-3y=0 \\ u-2x+2y=0 \end{array}\right\} \quad x, y \text{ free variables}$$

$$\mathrm{Rg}(A^T) = \mathrm{Span}\left(\begin{pmatrix}1\\0\\-1\\3\end{pmatrix}, \begin{pmatrix}0\\1\\3\\-2\end{pmatrix}\right)$$

2.5.29 check v_i, w_i linearly independent ...

show $w_j \in \text{span}(v_1, v_2, v_3)$ for all j !

Determine $Rg \begin{pmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & -4 \\ -1 & -1 & 3 \end{pmatrix}$

$$\rightsquigarrow \begin{array}{cccc} 1 & -3 & 2 & a \\ 0 & 7 & -4 & b-2a \\ 0 & 1 & -4 & c \\ 0 & -4 & 5 & d+a \end{array} \quad \rightsquigarrow \quad \begin{array}{cccc} 1 & -3 & 2 & a \\ 0 & 1 & -4 & c \\ 0 & 0 & 24 & b-2a-7c \\ 0 & 0 & -11 & d+a+4c \end{array}$$

$\rightsquigarrow$ Rg when $24(d+a+4c) + 11(b-2a-7c) = 0$!

$$2a + 11b + 19c + 24d = 0$$

now substitute w_j for $(a,b,c,d)^T$

and verify that equality holds ✓