HA 7

Some solutions

4.3.1 plane $\left\{ x_1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \right\}$

$$\Big\| \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{b} - \underbrace{\begin{pmatrix} 1 & 0 \\ 2 & -1 \\ -1 & 3 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Big\| \longrightarrow \min$$

$$k = A^T A = \begin{pmatrix} 6 & -5 \\ -5 & 10 \end{pmatrix}, \quad f = A^T b = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$x = k^{-1} f = \begin{pmatrix} 22/35 \\ 6/7 \end{pmatrix}$$

$$Ax - b = \begin{pmatrix} -1 \\ 0.6 \\ 0.2 \end{pmatrix} / 7, \quad \|Ax - b\|^2 = 1/35$$

4.3.15 as 4.3.1, $K = \begin{pmatrix} 6 & 4 & -2 \\ 4 & 6 & -8 \\ -2 & -8 & 14 \end{pmatrix}$

$$b = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \quad x = \begin{pmatrix} 1/4 \\ 0 \\ 1/4 \end{pmatrix}$$

$$\|Ax - b\| = 3$$

4.3.15

d) $A^TA = \begin{pmatrix} 25 & 1 \\ 1 & 33 \end{pmatrix}$, $A^Tb = \begin{pmatrix} 7 \\ 11 \end{pmatrix}$

$$x = \begin{pmatrix} 0.267\ldots \\ 0.325\ldots \end{pmatrix}$$

5.1.2 (a) basis since $\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ not singular

not orthogonal $\left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle = 0$

$\left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle = 1 \neq 0$

not orthonormal

(b) basis: again, $\begin{pmatrix} -4/13 & 12/13 & 3/13 \\ 3/5 & 0 & 4/5 \\ -\frac{48}{65} & -5/13 & \frac{36}{65} \end{pmatrix}$ not singular

orthogonal $\left\langle \begin{pmatrix} -4/13 \\ 3/5 \\ -48/65 \end{pmatrix}, \begin{pmatrix} 12/13 \\ 0 \\ -5/13 \end{pmatrix} \right\rangle = 0$ ✓

$\left\langle \begin{pmatrix} -4/13 \\ 3/5 \\ -48/65 \end{pmatrix}, \begin{pmatrix} 3/13 \\ 4/5 \\ 36/65 \end{pmatrix} \right\rangle = 0$

$\left\langle \begin{pmatrix} 12/13 \\ 0 \\ -5/13 \end{pmatrix}, \begin{pmatrix} 3/13 \\ 4/5 \\ 36/65 \end{pmatrix} \right\rangle = 0$

orthonormal: $\left\| \begin{pmatrix} -4/13 \\ 3/5 \\ -48/65 \end{pmatrix} \right\| = 1$, $\left\| \begin{pmatrix} 12/13 \\ 0 \\ -5/13 \end{pmatrix} \right\| = 1$, $\left\| \begin{pmatrix} 3/13 \\ 4/5 \\ 36/65 \end{pmatrix} \right\| =$

c) basis ✓

not orthogonal $\left\langle \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \right\rangle \neq 0$

So not orthonormal

5.1.4 $(e_1, e_2) = 1.0 + 2(0.1) + 3.(0.0) = 0$

$(e_1, e_3) = 0, \ (e_2, e_3) = 0$

$\|e_1\| = 1$ ✓

$\|e_2\| = 2 \quad \leadsto \quad e_1, \ e_2/\sqrt{2}, \ e_3/\sqrt{3}$

$\|e_3\| = 3$

form orthonormal basis

5.1.6 $\left\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\rangle = -a + 2b \stackrel{!}{=} 0$

$a = 2b > 0$

5.1.21 (a) ~~on~~ see 5.1.2(b)

(b) $v = \frac{7}{5} v_1 + \frac{11}{13} v_2 - \frac{72}{65} v_3$

5.2.2 orthogonal basis, first

$W_1 \quad v_1 \quad \begin{pmatrix}1\\0\\1\\0\end{pmatrix}/\sqrt{2}$

$$v_1 = \begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

$$\langle W_2, v_1\rangle = 0$$

$$v_2 = w_2 - \frac{\overbrace{\langle w_2, v_1\rangle}^{=0}}{\langle v_1, v_1\rangle}\cdot v_1 = W_2 = \begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$$

$$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2}\cdot v_1 - \frac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2 =$$

$$= \begin{pmatrix}1\\0\\0\\1\end{pmatrix} - \frac{1}{2}\cdot\begin{pmatrix}1\\0\\1\\0\end{pmatrix} - \frac{-1}{2}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix} = \begin{pmatrix}1/2\\1/2\\-1/2\\1/2\end{pmatrix}$$

$$v_4 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix} - \cancel{\frac{2}{2}}\cdot\begin{pmatrix}1\\0\\1\\0\end{pmatrix} - 0\cdot\begin{pmatrix}0\\1\\0\\-1\end{pmatrix} - \frac{1}{1}\cdot\begin{pmatrix}1/2\\1/2\\-1/2\\1/2\end{pmatrix}$$

$$= \begin{pmatrix}-1/2\\1/2\\1/2\\1/2\end{pmatrix}$$

$$\hat{v}_1 = \begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\\0\end{pmatrix}, \hat{v}_2 = \begin{pmatrix}0\\1/\sqrt{2}\\0\\-1/\sqrt{2}\end{pmatrix}, \hat{v}_3 = v_3, \hat{v}_4 = v_4$$

5.2.6 ?

(b) Kernel: $\begin{matrix} 2 & 1 & 0 & -1 \\ 3 & 2 & -1 & 1 \end{matrix} \xrightarrow{\text{Row-Echelon}} \begin{matrix} 2 & 1 & 0 & -1 \\ 0 & \frac{1}{2} & \underbrace{-1 \quad \frac{5}{2}}_{\text{free}} \end{matrix}$

basis $\begin{pmatrix} 3 \\ -5 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$

Gram-Schmidt: $\begin{pmatrix} 3 \\ -5 \\ 0 \\ 1 \end{pmatrix} - \frac{(-13)}{6} \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5/6 \\ -4/6 \\ 13/6 \\ 6/6 \end{pmatrix}$

normalize $\begin{pmatrix} 3 \\ -5 \\ 0 \\ 1 \end{pmatrix} / \sqrt{35}$, $\begin{pmatrix} 5 \\ -4 \\ 13 \\ 6 \end{pmatrix} / \sqrt{246}$

(f) $(u, 1, -1, -1)^T$ ... $x + y - u - z = 0$

$\hookrightarrow \text{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right)$

or, $\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}}_{v_2}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$

$v_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

$$V_3 = \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \end{pmatrix} \hookrightarrow \hat{V}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \hat{V}_2 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$\hat{V}_3 = V_3$$

5.29(b) $W_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = V_1 \quad , \quad W_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$V_2 = W_2 - \frac{\langle V_1, W_2 \rangle}{\|V_1\|^2} V_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{(-1)}{4} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\|V_1\|^2 = 4 \quad , \quad \hat{V}_1 = \begin{pmatrix} 1/\|V_1\| \\ 0/\|V_1\| \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

$$\|V_2\|^2 = \frac{1}{16} \begin{pmatrix} 1 \\ 4 \end{pmatrix}^T \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} =$$

$$= \frac{3}{4} \rightarrow \hat{V}_2 = \begin{pmatrix} \frac{1}{4} / \sqrt{3/4} \\ 1 / \sqrt{3/4} \end{pmatrix} = \begin{pmatrix} 1/2\sqrt{3} \\ 2/\sqrt{3} \end{pmatrix}$$

5.3.1 Compute $Q^T Q \stackrel{?}{=} I$

compute $\det Q$: $\begin{cases} +1 \to \text{proper} \\ -1 \to \text{not prop} \end{cases}$

5.3.2 $\det R = -1$, $\det Q = 1$

$\Rightarrow$ RQ with Q.Q. proper orthogonal

5.3.7 $\det(Q_1 Q_2) = \det(Q_1)\det(Q_2) = 1$

Q_1, Q_2 proper $\to$ $Q_1 Q_2$ proper

$Q_1 Q_2$ improper $\to$ $Q_1 Q_2$ proper

otherwise, $Q_1 Q_2$ improper

5.3.11 $Q = \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}$, there are 2^n such matrices

5.3.12 $\begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}$ columns orthogonal implies that matrix is diagonal!

5.3.15 (a) since every row and every column has precisely one non-zero entry, which is one, the columns are orthonormal.

(b) there are $n!$ permutations, half of them have positive determinant:

$$Q = \left(\begin{array}{cc|ccc} 0 & 1 & & 0 & \\ 1 & 0 & & & \\ \hline & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{array}\right), \quad P \text{ any [illegible]}$$

$$\begin{array}{ccc} P \text{ permutation}, & \longleftrightarrow & QP \text{ permutation} \\ \det P = 1 & & \det(QP) = -1 \end{array}$$

bijection between proper and non-proper permutations

5.3.27 Compare Exaple 5.26

(c) $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ -1 & -1 & 1 \end{pmatrix}$ $\quad r_{11} = \sqrt{5}$

$\rightarrow \begin{pmatrix} 2/\sqrt{5} & 1 & -1 \\ 0 & 1 & 3 \\ -1/\sqrt{5} & -1 & 1 \end{pmatrix}$ $\quad r_{12} = 3/\sqrt{5}$, $\quad r_{13} = -3/\sqrt{5}$

$\rightarrow \begin{pmatrix} 2/\sqrt{5} & 1-6/5 & -1+6/5 \\ 0 & 1 & 3 \\ -1/\sqrt{5} & -1+3/5 & 1-3/5 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{5} & -1/5 & 1/5 \\ 0 & 1 & 3 \\ -1/\sqrt{5} & -2/5 & 2/5 \end{pmatrix}$

$r_{22} = \sqrt{6/5}$

$\rightarrow \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{30} & 1/5 \\ 0 & 5/\sqrt{30} & 3 \\ -1/\sqrt{5} & -2/\sqrt{30} & 2/5 \end{pmatrix}$

$r_{23} = \frac{1}{\sqrt{30}} \cdot \frac{1}{5} \cdot (-1+75-4)$

$= 14/\sqrt{30} = \sqrt{15/2}$

$$\hookrightarrow \begin{matrix} 2/\sqrt{5} & -1/\sqrt{30} & 2/3 \\ 0 & 5/\sqrt{30} & 2/3 \\ -1/\sqrt{5} & -2/\sqrt{30} & 4/3 \end{matrix} \qquad r_{33} = \frac{\sqrt{24}}{3} = \sqrt{8/3}$$

$$Q = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{30} & 1/\sqrt{6} \\ 0 & 5/\sqrt{30} & 1/\sqrt{6} \\ -1/\sqrt{5} & -2/\sqrt{30} & 2/\sqrt{6} \end{pmatrix}.$$

$$R = \begin{pmatrix} \sqrt{5} & 3/\sqrt{5} & -3/\sqrt{5} \\ 0 & \sqrt{6/5} & 14/\sqrt{30} \\ 0 & 0 & \sqrt{8/3} \end{pmatrix}$$