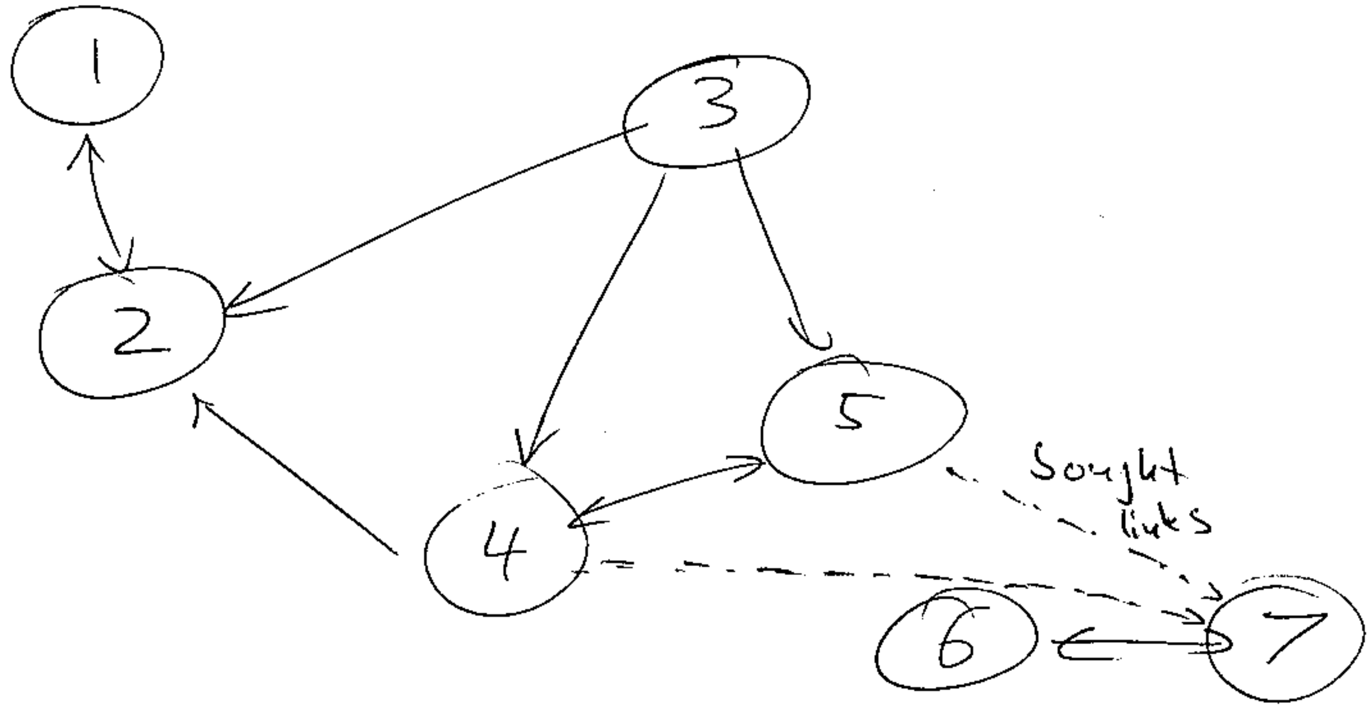
Page Rank



$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & & \\ 1 & 0 & \frac{1}{3} & \frac{1}{2} & 0 & & \\ 0 & 0 & 0 & 0 & 0 & & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 1 & & \\ 0 & 0 & \frac{1}{3} & \frac{1}{2} & 0 & & \\ & & 0 & & & 0 & 1 \\ & & & & & 1 & 0 \end{pmatrix}$$

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⇝ Look for equilibrium distribution

$\underline{x}$: $A\underline{x} = \underline{x}$

"start with probability x_j at node j $\forall j$

⇒ ~~stay~~ preserve probability on average"

Sometimes

"or, start an agent and let it hop around the graph for a long time,

x_j = avg time spent at node j

For instance $x_6 = x_7 = \frac{1}{2}$, $x_1 = \ldots = x_5 = 0$

Problem: · how do we distinguish starting at ①...⑤ vs ⑥...⑦

Add random jumps, probability ε

$$\varepsilon \begin{pmatrix} 1/N & \cdots & 1/N \\ \vdots & & \vdots \\ 1/N & \cdots & 1/N \end{pmatrix}, \quad N = 7, \text{ here}$$

$$\underbrace{\qquad\qquad}_{\frac{1}{N}\cdot \underline{1}}$$

$$A_{new} = (1-\varepsilon)A + \varepsilon/N \cdot \underline{1}$$

Solve $A_{new}x = x$!

Take $\sum x_j = 1$ (probability vector!)

$$(1-\varepsilon)Ax + \underbrace{\varepsilon/N\, \underline{1}\cdot x}_{= \varepsilon/N \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}} = x$$

$$\hookrightarrow \quad Sx = ((1-\varepsilon)A - I)x = -\varepsilon/N \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = r$$

inhomogeneous equation

Since $\frac{\varepsilon}{N} \mathbb{1} \cdot x = \frac{\varepsilon}{N} \begin{pmatrix} \sum x_i \\ \vdots \\ \sum x_i \end{pmatrix} = \frac{\varepsilon}{N} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

In our example

$$S = (1-\varepsilon) A - I =$$

$$= \begin{pmatrix} -1 & 1-\varepsilon & 0 & 0 & 0 & & \\ 1-\varepsilon & -1 & \frac{1-\varepsilon}{3} & \frac{1-\varepsilon}{2} & 0 & 0 & \\ & & -1 & 0 & 0 & & \\ & 0 & \frac{1-\varepsilon}{3} & -1 & 1-\varepsilon & & \\ & & \frac{1-\varepsilon}{3} & \frac{1-\varepsilon}{2} & -1 & & \\ & & & & & -1 & 1-\varepsilon \\ & & 0 & & & 1-\varepsilon & -1 \end{pmatrix}$$