Math 8501, Fall 2018

Theory of Ordinary Differential Equations

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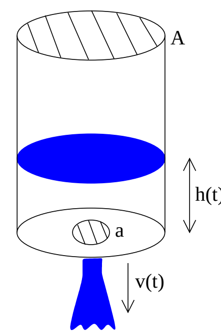
— *Examples and exercises* —

1) [From: Hubbard & West (1991)]

Consider a water bucket with a hole in the bottom. Let $h(t)$ be the height of the water remaining in the bucket at time t. The cross-sectional area of the bucket is denoted by A, while a is the area of the hole in the bucket. Lastly, $v(t)$ is the velocity of the water passing through the hole.



(i) Show that

$$av(t) = A\frac{\mathrm{d}h}{\mathrm{d}t}(t)$$

What physical law are you invoking?

(ii) We need an additional equation that relates the exit velocity v and the height h of the water in the bucket. Use conservation of energy to derive this equation. If we assume that potential energy is converted only into kinetic energy, then show that

$$\frac{1}{2}mv^2 = mgh.$$

(iii) Hence, combining the results obtained so far, we see that the height $h(t)$ satisfies the equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{\sqrt{2g}a}{A}\sqrt{h}.$$

(iv) Given that $h(0) = 0$, so that the bucket is empty at time $t = 0$, show that the solution $h(t)$ is non-unique in backwards time, i.e. for $t < 0$.

2) Show that the product of two functions that are locally Lipshitz is locally Lipshitz.

3) Let

$$P : C^0([0,1]) \to C^0([0,1]), \qquad f(\cdot) \mapsto (f(\cdot))^k.$$

Show that P is well-defined, continuous, and differentiable. What is the derivative?

4) *(Existence "by hand", without Banach)* Suppose f is globally Lipshitz with constant L. Consider the Picard iteration sequence

$$u_{k+1}(t) = u_0 + \int_0^t f(u_k(s))\mathrm{d}s, \qquad u_0(t) \equiv u_0,$$

and show that it converges uniformly on $|t| \leq \delta$. Conclude the existence of a solution to the ODE $\dot{u} = f(u), u(0) = u_0$.

5) Consider the following normed vector spaces, all subsets of the set of functions $f : \mathbb{R} \to \mathbb{R}$, and decide if they are Banach spaces:

   (i) the bounded functions $\mathcal{B} := \{f| \; \|f\|_\infty < \infty\}$ with $\|f\|_\infty = \sup_x |f(x)|$;

   (ii) $\{f| \; f$ bounded and locally Lipshitz$\}$ with norm $\|f\|_\infty$;

   (iii) $C^0_{\text{unif}} := \{f| \; f$ uniformly continuous$\}$ with norm $\|f\|_\infty$;

   (iv) the continuous functions with weighted norm $\|f\|_\omega := \|f(\cdot)\omega(\cdot)\|_\infty$, for some fixed, continuous, positive function $\omega(x) > 0$.

6) Decide for which values of α the function $f_\alpha(x) = x^\alpha$ is locally Lipshitz or globally Lipshitz on $(0, 1]$ or $(0, \infty)$.