HA2 - Solutions

1) a) $\|\cdot\|_\eta$ is a norm :

$\|u\| = 0 \implies u = 0$ ✓

$$\|u+v\| = \sup_t e^{-\eta|t|} |u(t) + v(t)|$$
$$\leq \sup_t e^{-\eta|t|} \left(|u(t)| + |v(t)| \right)$$
$$\leq \sup_t e^{-\eta|t|} |u(t)| + \sup_t e^{-\eta|t|} |v(t)|$$
$$= \|u\| + \|v\|$$

$\|\lambda u\| = |\lambda| \, \|u\|$ ✓

C^0_η is complete, Cauchy seq / converge:

$$\lim_{\substack{k,l \geq N \\ N \to \infty}} \|u_k - u_l\| = 0$$

$\implies u_k(t) \longrightarrow u(t)$
locally uniformly
$\hookrightarrow$ limit $u(t)$ is continuous

$$
\begin{array}{ccc}
e^{-\eta|t|}|u_k(t)| & \le |u_k|_\eta & \le C \\
\big\downarrow{\scriptstyle k\to\infty} & & \\
e^{-\eta|t|}|u(t)| & & |u|_\eta
\end{array}
$$

$$\Rightarrow \sup e^{-\eta|t|}|u(t)| \le C,$$

hence limit $u \in C^0_\eta$ ✓

b) $(Tu)(t) = u_0 + \int_0^t f(u(s))\,ds$

$$|Tu|_\eta = \sup e^{-\eta|t|}\left|u_0 + \int_0^t f\right|$$

$$\le |u_0| + \sup_t e^{-\eta|t|}\left|\int_0^t f(u(s))\,ds\right|$$

$t \ge 0$ only; $t < 0$ similar.

Use $|f(u(s))| \le |f(0)| + L|u(s)|$

$$\sup_t e^{-\eta|t|} \left| \int_0^t f(u(s))\,ds \right|$$

$$\leq \sup_t \underbrace{e^{-\eta|t|}\, t\, |f(0)|}_{\leq C} + \sup_t L e^{-\eta t} \int_0^t e^{\eta s} |u|_\eta \, ds$$

$$\leq C + \frac{L}{\eta} \quad \checkmark \quad \text{so } Tu \in C^0_\eta$$

$$|Tu - Tv| \leq L \sup e^{-\eta t} \int_0^t |u(s) - v(s)|\,ds$$

$t \geq 0$ only

$$\leq L \sup \int_0^t e^{-\eta(t-s)} |u-v|_\eta \, ds$$

$$\leq \frac{L}{\eta} |u-v|_\eta \quad \checkmark$$

c) The unique fixed pt is a solution. Since by standard Picard the solution is unique on its maximal interval of existence, this also proves uniqueness, globally

2) a) $x_0 = 0 \Rightarrow x(t;0) = 0, y(t;0) = 0$

$a_1(t)$ solves linearized equ

$a_1'' = -a_1, \quad a_1(0) = 1, \quad a_1'(0) = 0$

$a_1 = \cos t$

a_2 solves second derivative in x_0

$x'' = -x - x^2, \quad x(0) = x_0$

$x_1'' = -x_1 - 2xx_1, \quad x_1(0) = 1$

$x_2'' = -x_2 - \cancel{2xx_2} - 2x_1^2, \quad x_2(0) = 0$

$x_3'' = -x_3 - 2xx_3 - 6x_1x_2, \quad x_3(0) = 0$

$$a_2'' = -a_2 - 2\cos^2 t \quad , \quad a_2(0) = 0$$

$$\rightsquigarrow a_2(t) = \tfrac{1}{6}\left(4\cos t - 4\cos^4 t - 9\sin^2 t - \sin t \sin(3t)\right)$$

bounded!

$$= -\tfrac{4}{3}\,(2+\cos t)\,\sin^2(t/2)$$

(c) ←

$$a_3'' = -a_3 - 6 a_1 a_2$$

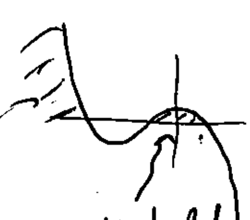
$$\rightsquigarrow a_3(t) = -\tfrac{1}{12}\sin(t/2)\left(-60\, t \cos\tfrac{t}{2} + 48 \sin\tfrac{t}{2} + 19\sin\left(\tfrac{3t}{2}\right) + 3\sin\left(\tfrac{5t}{2}\right)\right)$$

unbounded.

b) $\frac{1}{2}(x^2+y^2) + \frac{1}{3}y^3 \equiv H$

is locally, $H \sim 0$, $x \sim y \sim 0$, bounded

$$x = \sqrt{\underbrace{2H - y^2 - \tfrac{2}{3}y^3}}$$



y bdd

c) clearly derivatives predict $x(t)$ unbounded, which it is not.

d) periodic orbits are functions

$$x_*(\omega(x_0)t; x_0) \quad , \quad x_*(\varphi; x_0) = x_*(\varphi + 2\pi, x_0)$$

$$\frac{d}{dx_0} x_*(\omega(x_0)t; x_0) = \partial_{x_0} x_* + \partial_t x_* \cdot \omega'(x_0) \cdot t$$

so if $\partial_t x_* \neq 0$ (v) and $\omega'(x_0) \neq 0$,

the linearized solution grows exponentially

(confer $\sin(x_0 \cdot t)$ as example,

$\partial_{x_0} \sin(x_0 t) = t \cos(x_0 t)$)