Math 8501, Fall 2018

Theory of Ordinary Differential Equations

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— *Homework 2* —

(1) Assume that the vector field f is globally Lipshitz with Lipshitz constant L. We want to prove global existence directly. For any $\eta \in \mathbb{R}$, onsider therefore the space C^0_η of continuous functions on $\mathbb{R}$ that are bounded with respect to the norm

$$\|u\|_\eta = \sup_t \mathrm{e}^{-\eta|t|}|u(t)|.$$

(a) Show that C^0_η is a Banach space.

(b) Show that the operator T used for Picard iterations defines a contraction on C^0_η when $\eta > L$.

(c) Conclude global existence and uniqueness of solutions.

(2) Consider the solution $(x(t;x_0), y(t;x_0))$ to the pendulum equation

$$x' = y, \qquad y' = -x - x^2, \qquad x(0) = x_0, \quad y(0) = 0.$$

(a) Find the expansion of $x(t;x_0)$ for fixed t to second order in x_0, that is, find functions $a_j(t), j = 0, 1, 2$ such that

$$x(t;x_0) = a_0(t) + a_1(t)x_0 + a_2(t)x_0^2 + \mathrm{O}(x_0^3).$$

(b) Conclude from the Hamiltonian function that $x(t;x_0), y(t;x_0)$ are bounded functions for any fixed, small x_0, y_0.

(c) *Optional:* Compute the expansion to order 3,

$$x(t;x_0) = a_0(t) + a_1(t)x_0 + a_2(t)x_0^2 + a_3(t)x_0^3 + \mathrm{O}(x_0^4),$$

and find that $a_3(t)$ is linearly growing. Conclude that the Taylor approximation is not valid uniformly in t.

(d) *Optional:* Explain how linear growth in the expansion results from a change in period of the periodic orbits.

Homework is due on Monday, September 24, in class