HA-4- Solutions

1) a) $y_0 \in \omega(x_0)$, give T, ε

$\rightsquigarrow \phi_{2T}(y_0) \in \omega(x_0)$ $\qquad T_1 = 2T$

$\rightsquigarrow$ ex. t, $|\phi_t(x_0) - \phi_{2T}(y_0)| < \varepsilon/2$

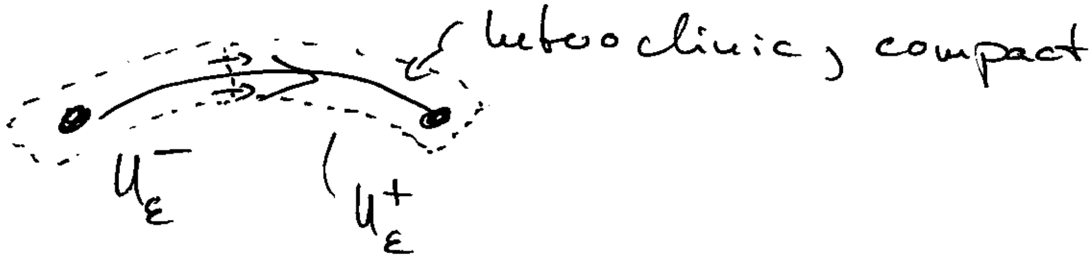
$\rightsquigarrow \phi_t(x_0) =: y_1$

$\rightsquigarrow |\phi_{t+2T+\tau}(x_0) - y_0| < \varepsilon$

for some $\tau > 0$ since $y_0 \in \omega(x_0)$

$\rightsquigarrow T_2 = \tau$ ✓

b)



$\phi_T(x_0)$ near x_+, then does not

trajectory from $\phi_T(x_0)$ to U_ε^-

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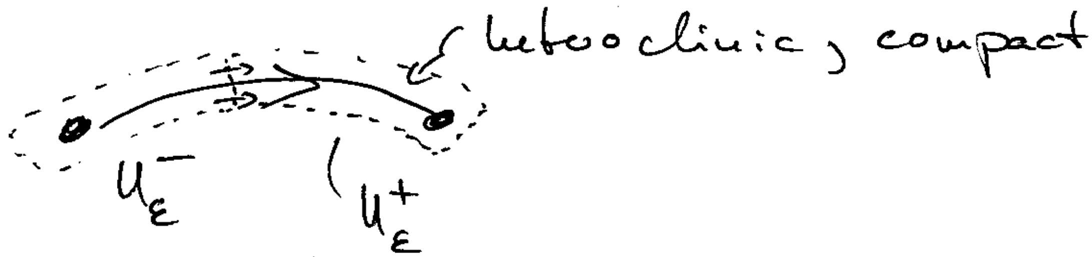
b) 

$\phi_T(x_0)$ near x_+, then does not ex. trajectory from $\phi_T(x_0)$ to U_ε^- contained in $U_\varepsilon^+ \cup U_\varepsilon^-$ $\Rightarrow$

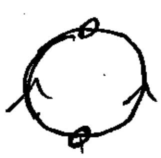
ex. accumulation point outside of $U_\varepsilon^\pm$

2) $\phi_h = \varphi_h$, e.g. $\phi_h = R_h$, rotation by h in the plane.

$\Rightarrow \omega(x_0 \neq 0) = \{$circle of radius $|x_0|\}$
for flow ϕ_h

but, for $h = \frac{2\pi}{n}$, $n \in \mathbb{N}$,
$\{\varphi_h^k x_0\}$ is finite $\Rightarrow \omega(x_0)$ finite.

3) $\mu = 0$



invariant sets
$x = 0$ unst.
$x = \pi$ st.
$S = [0, \pi]$ unst
$S = [\pi, 0]$ unst.
$S = S^1$ st.

$\mu = 1$

$x = \pi/2$ unst.
$S = S^1$ stable

$\mu = 2$

$S = S^1$ stable

4) a) $\dot{r} = r - r^3$

$\varphi' = 1$

b) $z_0 = 0 \rightarrow \omega(z_0) = \{0\}$

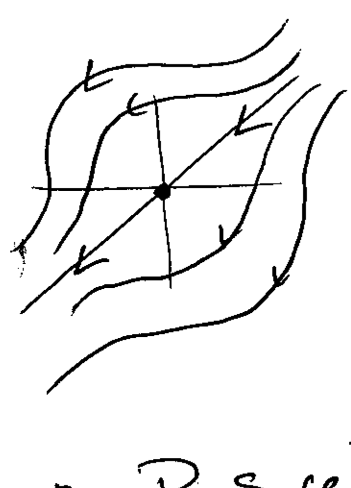
$0 < |z_0| \cancel{\leq} \rightarrow \omega(z_0) = \{|z| = 1\}$

c) $\omega(z_0) = \partial(\text{strip})$ after transformation, not connected.

5) a) $H = -\frac{1}{3}y^3 + \frac{1}{3}x^3$, $\quad x' = \partial_y H$

$y' = -\partial_x H$

$H \equiv$ const



b) $R' c - R s \varphi' = -R^2 s^2$

$R' s + R c \varphi' = -R^2 c^2$

$$R' = -R^2(s^2c + sc^2)$$

$$\varphi' = -R(c^3 - s^3)$$

d)

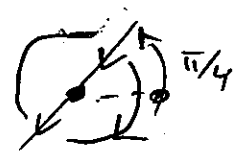
c) $$\mathring{R} = -R(s^2c + sc^2)$$

$$\mathring{\varphi} = -(c^3 - s^3)$$

eq' $c = s \Rightarrow \varphi = \pi/4,\ 5\pi/4$

where $\mathring{R} = -R\, 2^{-3/2}$

$$R' = -R^2\, 2^{-3/2}$$



d) $$\boxed{x_1' = -x_1^2 y_1^2}$$

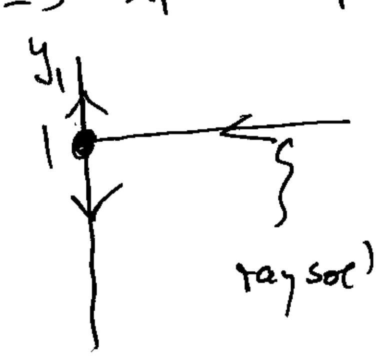
$$y_1' x_1 + x_1' y_1 = -x_1^2$$

$$\boxed{y_1' = -x_1 + x_1 y_1^3}$$

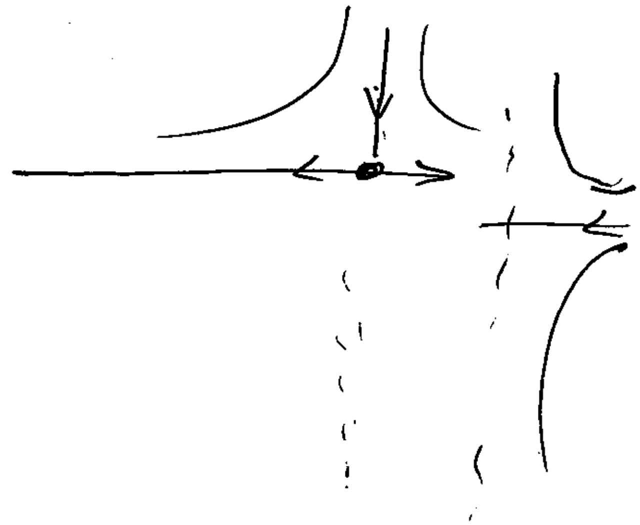
$$\mathring{x}_1 = -x_1 y_1^2$$

$$\mathring{y}_1 = -1 + y_1^3$$

$y_1 = 1 \Rightarrow \mathring{x}_1 = -x_1$



$$\dot{y}_1 = -y_1 x_1^2$$
$$\dot{x}_1 = -1 + x_1^3$$

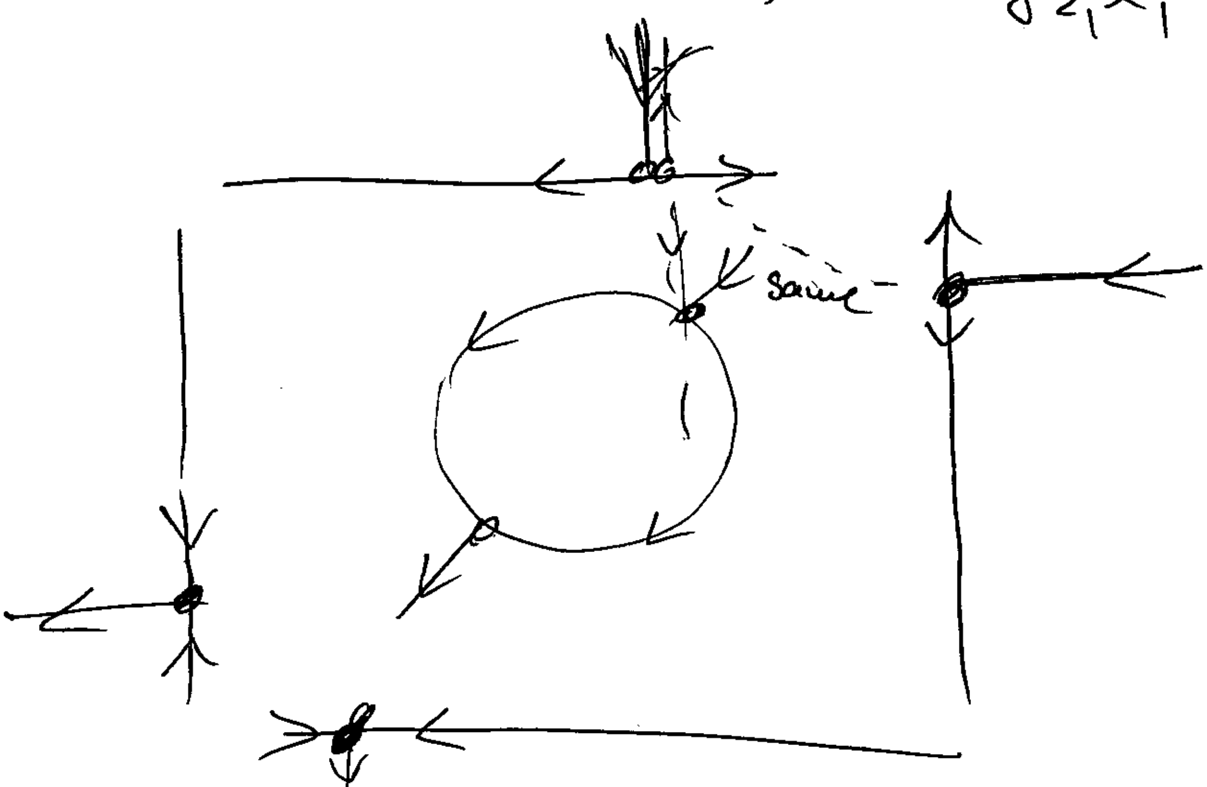


f) $y = y_2 \quad x = x_2 y_2$

$y = x_1 y_1 \quad x = x_1$

$x_2 = x/y_2 = 1/y_1 \quad \dots$ ← inversion at infinity

g) same pictures, but Euler multiplier $x_1 < 0$, $y_2 < 0$, so reverse arrows, reverse orientation of y_2, x_1



6) a) $\omega \notin \{0,1\}$

$$y = y_1 c + y_2 s, \quad c = \cos t, \; s = \sin t$$

$$x'' + \omega^2 x = y_1 c + y_2 s$$

$$x = x_1 c_\omega + x_2 s_\omega + \frac{1}{\omega^2-1} y_1 c + \frac{1}{\omega^2-1} y_2 c$$

$$c_\omega = \cos \omega t, \quad s_\omega = \sin \omega t$$

$\Rightarrow x$ bdd

$\omega = 0$: $\quad y = 0, \; x = x_1 t + x_0$ unbdd

$\omega = 1$ $\quad y = \cos t, \; x = \frac{t}{2} \cos t$ unbdd.

b) set $x = \hat{x} + \frac{1}{\omega^2+1} y$

$\hookrightarrow \hat{x}'' + \omega^2 \hat{x} = 0$

$\quad y'' + y = 0$

i) $(\hat{x}, \hat{x}') = 0, \; (y', y) \neq 0 \Rightarrow \Omega = S^1$

$(\hat{x}, \hat{x}') \neq 0, \; (y', y) = 0 \Rightarrow \Omega = S^1$

(ii) $(\tilde{x}, \tilde{x}') \neq 0$, $(y', y) \neq 0 \Rightarrow \Omega = \mathbb{T}^2$

(iii) $\hat{x} = \tilde{x}' = y = y' = 0 \Rightarrow \Omega = \{0\}$.