Math 8501, Fall 2018

Theory of Ordinary Differential Equations

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— *Homework 4* —

(1) Recall the definition of the chain recurrent set CR for a flow Φ_t,

$$x \in R :\Leftrightarrow \text{ for all } \varepsilon, T > 0 \text{ ex. } \varepsilon\text{-pseudo orbit with } x_n = x,$$

where ε-pseudo orbits are piecewise orbits with at most ε-jumps, that is, there exist $T_j > T$, x_j, $0 \leq j \leq n-1$, $|\Phi_{T_j}(x_j) - x_{j+1}| < \varepsilon$.

- (a) Show that the ω-limit set is chain recurrent, that is, for $\gamma_+(x_0)$ bounded, show $\omega(x_0) \subset \mathbb{R}$.
- (b) Suppose $H = \{x(t), t \in \mathbb{R}\} \subset \mathbb{R}^n$ is a heteroclinic trajectory, $x(t) \to x_\pm$ for $t \to \pm\infty$, $x_- \neq x_+$. Conclude that $\bar{H}$ is not the ω-limit set for some initial condition y_0.

(2) Suppose our numerical method $\varphi_h = \Phi_h$ is exact for some $h > 0$. Are the ω-limit sets of φ_h the same as the ω-limit sets of the flow:

$$\{y|\text{ ex. } n_k \to \infty, n_k \in \mathbb{N},\ \varphi^{n_k} x = y\} = \{y|\text{ ex. } t_k \to \infty,\ t_k \in \mathbb{R},\ \Phi_{t_k}(x) = y\}\ ?$$

(3) Consider the vector field $x' = \mu + \sin(x)$ for parameters $\mu > 0$, with associated flow on the circle $x \in \mathbb{R}/2\pi\mathbb{Z}$. For the three cases $\mu = 0, 1, 2$, find all invariant sets and determine their stability.

(4) Consider the equation on $\mathbb{C} \sim \mathbb{R}^2$

$$z' = (1 + \mathrm{i})z - z|z|^2.$$

- (a) Draw the phase portrait after analyzing the equation in polar coordinates.
- (b) Find the ω-limit sets for all $z_0 \in \mathbb{C}$.
- (c) By the Riemann Mapping theorem, we can map the open unit disc in $\mathbb{C}$ to the strip $\{|\operatorname{Im} z| < 1\}$. Conclude that there exists a smooth flow such that the ω-limit set of a (unbounded) trajectory is not connected.

(5) We wish to analyze

$$x' = -y^2, \qquad y' = -x^2,$$

using various methods.

- (a) Show that the system is Hamiltonian, compute the Hamiltonian and plot the phase portrait (level sets of the Hamiltonian).

(b) Introduce polar coordinates $X = R\cos\varphi$, $y = R\sin\varphi$, and find the differential equation for R, φ.

(c) Use the Euler multiplier R to simplify the ODE. The equation for φ now decouples. What are the equilibria φ_j for the φ-equation? What are the dynamics on the invariant rays (R, φ_j)?

(d) Describe the dynamics outside of the rays.

(e) As an alternate coordinate system, introduce "projective coordinates"

$$x = x_1, \qquad y = x_1 y_1,$$

on $x > 0$, find the equation for x_1, y_1 and use the Euler multiplier x_1 to simplify the equation into a skew-product. Discuss"ray solutions" in these coordinates.

(f) Add a coordinate system

$$y = y_2, \qquad x = x_2 y_2,$$

in $y > 0$ and repeat the construction above. Describe the coordinate change between the (x_2, y_2) and the (x_1, y_1)-coordinates.

(g) Describe how you would treat the regions $x < 0$ and $y < 0$, respectively.

(6) Consider two linear coupled oscillators

$$x'' + \omega^2 x = y, \qquad y'' + y = 0.$$

(a) For which $\omega \in \mathbb{R}$ are all trajectories bounded?

(b) Suppose $\omega \notin \mathbb{Q}$. Describe the ω-limit sets of trajectories depending on the initial condition, that is, specify when ω is a point, a circle, or a torus, $\mathbb{T}^2$ or $\mathbb{T}^3$.

Homework is due on Monday, October 15, in class.
Four (4!) correct exercises required for full score.