HA 5 – Solutions

1) a) $x = R \cdot u$, $x' = R'u + Ru' = R\,Au$

$$R' = |x|' = \frac{\langle x, x' \rangle}{|x|} = \frac{R^2 (u, Au)}{R} = R\,(u, Au)$$

$$u' = Au - (u, Au)\,u$$
$$R' = R\,(u, Au)$$

eq$^!$: $Au - \underbrace{(u, Au)}_{\lambda} u = 0 \Rightarrow Au = \lambda u$ for some λ

$\Rightarrow u$ eigenvector $\Rightarrow u_i^{\pm} = \pm(0, 0, \dots, 1, 0, \dots 0)$ — ith entry

(b) $-\frac{1}{2}\frac{d}{dt}(Au, u) = -(Au, u') = -\left(Au, Au - (u, Au)u\right)$

$$= -|Au|^2 + |(u, Au)|^2$$
$$\leq -|Au|^2 + \left|\cancel{|u|} \cdot |Au|\right|^2 = 0$$

w/ equality when u, Au colinear $\Leftrightarrow$ eq$^!$.

linearize at eq' u w $Au = \lambda_1 u$

$\hookrightarrow$ $v' = Av - \lambda_1 v - (v, \lambda_1 u)u - (u, Av)u$

$= Av - \lambda_1 v$, since

$\langle v, u \rangle = 0$ for $v \in T_u S^{n-1}$, tangent space

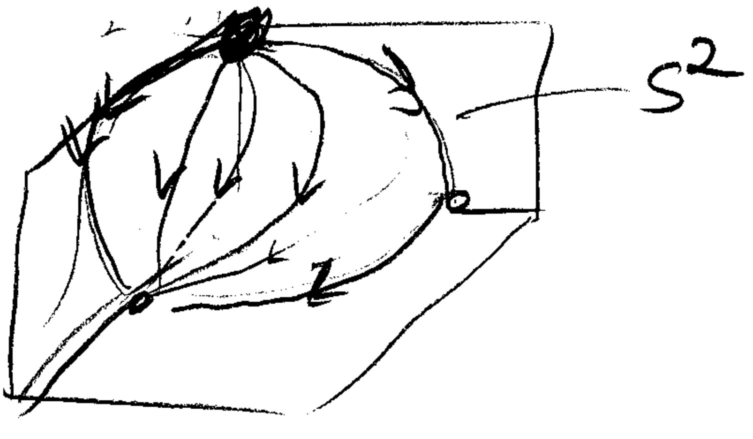
$\hookrightarrow$ eigenvalues $\lambda_j - \lambda_1$, $j \neq 1$

$\Rightarrow$ stable eq' only w/ λ_1 ! , $u = (1, 0, \ldots 0)$
$u = (-1, 0, \ldots 0)$

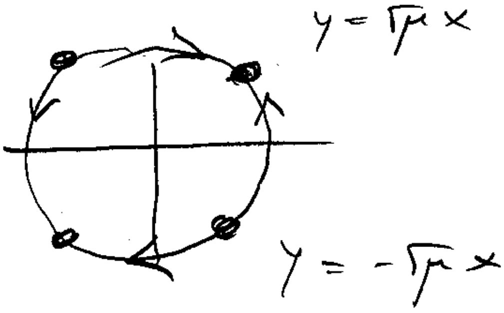
(c) S^{n-1} cpt $\Rightarrow$ ω- and α- limit sets cpt, $\neq \emptyset$; strict L-fct implies ω, α-limit sets consists of equilibria. Since eq' are isolated and α, ω connected, they consist of a single eq', each. Since L-fct is strict, $\alpha(u) \neq \omega(u)$, hence heteroclinic.

Heteroclinics from e_j to e_{j+k} fill hyperplane $\cap S^{n-1}$

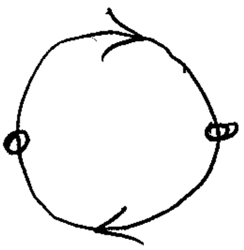
$\langle e_j, \dots, e_{j+k} \rangle \cap S^{n-1}$



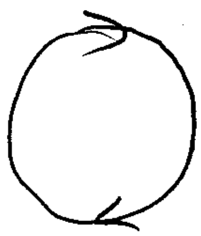
(d) $\mu > 0$



$\mu = 0$



$\mu < 0$



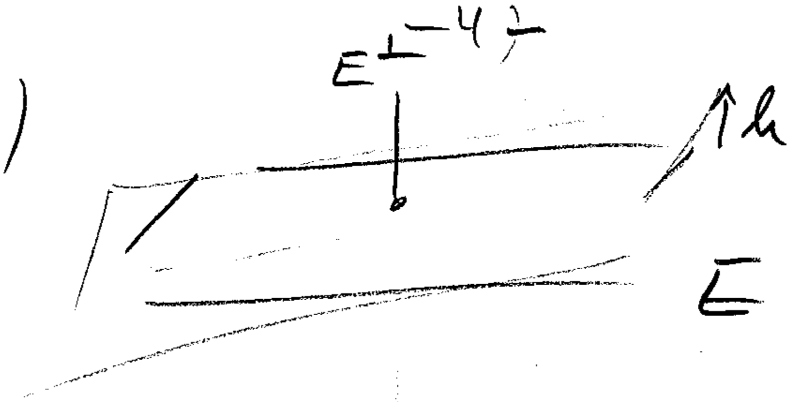
(e) (i) $E = \{ \sum_{j=1}^{k} \alpha_j x_j \, , \, \alpha_j \in \mathbb{R} \}$

$e^{At} E$ is again linear subspace.

Suppose $\dim(e^{At}E) < k$

$\Rightarrow \dim e^{-At} (\underbrace{e^{At} E}_{\text{Rank} < k}) < k$ ↯

(ii)



really a definition, could go through chart changes:

$$F = \text{graph}\,(h : E \to E^\perp)$$
$$= \text{graph}\,(h' : E' \to E'^\perp)$$

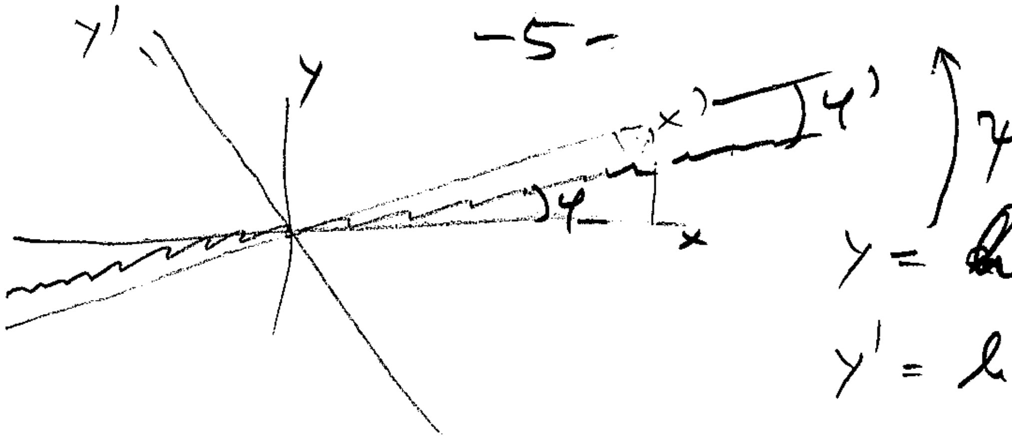
Charts: $\varphi : \{h : E \to E^\perp\} \to V_k$, k-dim subspace

$\varphi' : \{h' : E' \to E'^\perp\} \to V_k$

Pick coordinates in $E, E', E^\perp, E^{\perp'}$

$E, E' \simeq \mathbb{R}^k$, $E^\perp, E^{\perp'} \simeq \mathbb{R}^{n-k}$

and show $\varphi^{-1} \circ \varphi'$ smooth...



$y = h \cdot x$

$y' = h' \cdot x'$

$$h = \tan\varphi \qquad h' = \tan\varphi'$$

$\varphi' = \varphi - \psi$, ψ change of reference, fixed

$$h' = \underbrace{\tan\left(\arctan(h) - \psi\right)}_{\text{smooth in } h \text{ locally}}$$

(iii) P projection on E, $P^T = P$

$h : E \to E^\perp$, $x \in$ graph h

$\Rightarrow \quad x = (x_0, x_\perp) = (x_0, h x_0)$

w/ $x_0 = Px$, $x_\perp = (1-P)x$

$$x' = A(x_0 + h x_0)$$

$$x_0' = PA\,(x_0 + h\,x_0)$$

$$(Px_0)' = (1-P)A\,(x_0 + h x_0)$$

$$(h\,x_0)' = h'\,x_0 + hPA(x_0 + h\,x_0)$$
$$= (1-P)A\,(x_0 + h x_0)$$

$$h' = (1-P)A\circ(id_E + h) - hPA\circ(id_E + h)$$

If $PA = AP$,

$$h' = Ah - hA \quad ✓$$

(iv) equilibria are coordinate planes, linearization at these has ev

$$\sum_{\lambda\in\sigma(A|E^\perp)} \lambda_i - \sum_{\lambda\in\sigma(A|E)} \lambda\,,$$

the differences between eigenvalues outside of the invariant subspace and eigenvalues inside.

The stable equilibrium is the subspace $(\underbrace{*, *, \dots, *}_{1\dots k \text{ entries}}, 0, 0, \dots 0)$.

2) (a) '$\Leftarrow$' $A^T = -A$

Taylor series

$$\Rightarrow \phi_t^{-1} = e^{-At} = e^{A^T t} \overset{\downarrow}{=} \left(e^{At}\right)^T$$

$$= \phi_t^T$$

'$\Rightarrow$' $\phi_t^{-1} = \phi_t^T \xrightarrow{\frac{d}{dt}\big|_{t=0}} -A = A^T$ ✓

(b) $e^{At} = \left(e^{At}\right)^T \xRightarrow{\frac{d}{dt}\big|_{t=0}} A = A^T$

$A = A^T \underset{\text{Taylor}}{\Longrightarrow} e^{At} = \left(e^{At}\right)^T$ ✓

(c) $M = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$ Since $\operatorname{spec}(M)$

$= \exp(\operatorname{spec}(A))$

(d) symmetric: no, see (c)

orthogonal: det $= -1$ ↯

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

if det $= 1$: Floquet theory says yes

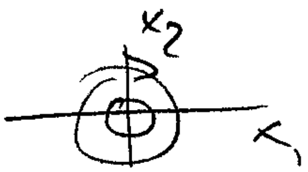
if $\lambda = -1$ is geom' mult' even

since alg' simple $\Rightarrow$ OK since

det > 0, so yes

$M = \exp(A)$, if $M^T = M^{-1}$, det $M > 0$

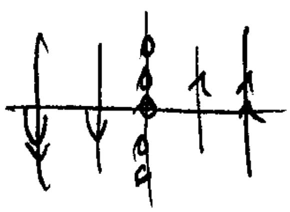
3) a) $u = x_1 + i x_2$, $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$



$x_1(t) = x_1^0 \cos t + x_2^0 \sin t$

$x_2(t) = -x_1^0 \sin t + x_2^0 \cos t$

b) $u = x_2$, $v = x_1$ $\quad \begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$



$x_1(t) = x_1^0$

$x_2(t) = x_2^0 + x_1^0 t$

c) ...

4)

a) $x = y \Rightarrow \dot{x} = Ax$, $|e^{At}| \leq Ce^{-\gamma t}$ ✓

b) $x = -y \Rightarrow \dot{x} = Ax + 2Dx$

$\text{spec}(A) < 0$, $\text{spec}(2D) < 0$, $\text{spec}(A+2D) \not< 0$

e.g. $\tilde{A} = \begin{pmatrix} -\varepsilon & 1 \\ 0 & -\varepsilon \end{pmatrix}$, $2\tilde{D} = \begin{pmatrix} -\varepsilon & 0 \\ 1 & -\varepsilon \end{pmatrix}$.

$\hookrightarrow \tilde{A} + 2\tilde{D} = \begin{pmatrix} -\varepsilon & 1 \\ 1 & -\varepsilon \end{pmatrix}$ w/ ev $\lambda = \pm 1 + O(\varepsilon)$

Now diagonalize $\tilde{D}$.

5) a) $x = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + x_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$

$\hookrightarrow x_1 = x_2 \Rightarrow x(t) = \frac{1}{2} \begin{pmatrix} e^{4t} + e^{2t} \\ e^{4t} - e^{2t} \end{pmatrix}$

$x_1 = 1$ $x_2 = 0 \Rightarrow x(t) = \begin{pmatrix} e^{4t} \\ e^{4t} \end{pmatrix}$

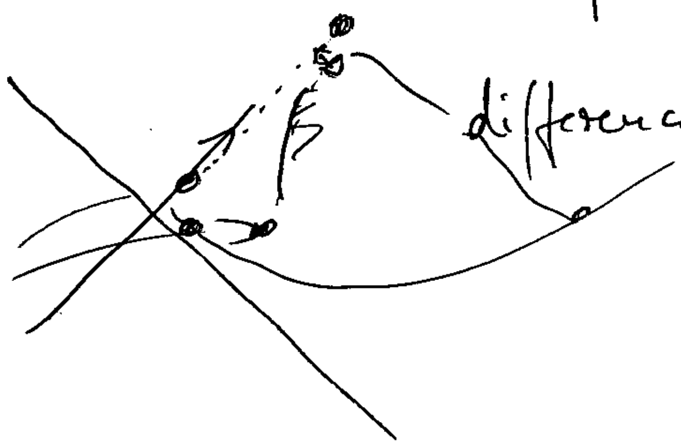
$x_2 = 1$, $x_1 = 0 \Rightarrow x(t) = \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix}$

(b-d) for $t \gg 1$,
$$x = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + x_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$$
$$= x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} \left(1 + \mathcal{O}(e^{-2t}) \right)$$

$$1 + \mathcal{O}(e^{-2t}) \mathrel{\hat{=}} 1$$

at machine precision

two i.c.



difference is large, but small compared to overall magnitude.