Math 8501, Fall 2018

Theory of Ordinary Differential Equations

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— *Homework 5* —

(1) Consider the linear equation $\dot{x} = Ax$, $A = \mathrm{diag}\,(\lambda_j)$, $\lambda_1 > \lambda_2 > \ldots > \lambda_n$.

(a) Derive an equation for the projectivized flow, that is, write $x = u \cdot |x|$ and derive an equation for $u \in S^{n-1}$. Find all equilibria of this flow on the sphere.

(b) Show that the Rayleigh quotient $V(u) = -\frac{1}{2}\langle Au, u\rangle$ is a strict Lyapunov function, that is, strictly decreasing for non-equilibrium stolutions. Which equilibria are stable?

(c) Conclude that all trajectories are heteroclinic and describe heteroclinic orbits.

(d) Describe equilibria and heteroclinic orbits for the flow on S^1 associated with the (non-self-adjoint) $A = \begin{pmatrix} 0 & 1 \\ \mu & 0 \end{pmatrix}$ for all $\mu \in \mathbb{R}$?

(e) *Optional supplement:*

i. Let $x_1, \ldots, x_k$ be k linearly independent vectors in $\mathbb{R}^n$ and denote by E the linear subspace spanned by those vectors. Show that the linear equation induces a flow on the set of k-dimensional subspaces of $\mathbb{R}^n$.

ii. Show that the space of subspaces is a smooth manifold (the Grassmannian) by locally writing subspaces as graphs: any subspace F "near" E can be written as a graph of a linear map $h(F) : E \mapsto E^\perp$.

iii. Show that the flow on subspaces is a smooth flow, that is, compute the vector field for $h' = f_A(h)$.

iv. Determine all equilibria of this flow. Find the (unique) stable equilibrium.

(2) Some facts on Lie groups:

(a) Show that $\Phi_t = \mathrm{e}^{At}$ is an orthogonal matrix for all $t \in \mathbb{R}$, that is $\Phi_t^{-1} = \Phi_t^T$, if and only if A is skew symmetric, that is, $A^T = -A$.

(b) For which matrices A are all e^{At}, $t \in \mathbb{R}$, self-adjoint?

(c) Find a real matrix $M \in \mathbb{R}^{n\times n}$ such that there does not exist a real matrix A with $M = \exp(A)$.

(d) For symmetric or orthogonal M, can you always write $M = \exp(A)$ for some real A?

(3) Compute a Jordan normal form transformation and the general solution for the differential equation $\dot{x} = Ax,\ x(0) = x_0$[1], where

[1] The idea is to not use computer algebra but that's not enforceable.

(a) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(b) $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

(d) $A = \begin{pmatrix} 0 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$

Also try to sketch phase portraits.

(4) We consider the coupled linear system $\dot{x} = Ax + D(y - x)$, $\dot{y} = Ay + D(x - y)$, with $x, y \in \mathbb{R}^n$, $D = \mathrm{diag}\,(d_1 \ldots, d_n)$, and $d_i > 0$. We assume throughout that the eigenvalues of A are contained in the open left complex half plane $\mathrm{Re}\,(\mathrm{spec\,A}) < 0$.

(a) Note that for initial conditions where $x(0) = y(0)$,the solution $(x(t), y(t))$ converges to zero as $t \to \infty$.

(b) Find matrices A, D such that $x(t) \to \infty$ for suitable initial conditions $(x(0), y(0))$ and $t \to +\infty$.

Try to interpret (see also A. Turing, The chemical basis of morphogenesis, *Phil. Trans. Roy. Soc. B* **237** (1952), 37–72)

(5) Set $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ and consider $x' = Ax$, $x(0) = (1, 0)^T, (1, 1)^T, (1, -1)^T$.

(a) Find the solutions $x(t)$ explicitly for the three initial conditions listed.

(b) Solve numerically using and compute $x(T)$, $t = 1, 3, 5, 10, 15, 20$; compare with the prediction (try Euler's method with $h = 10^{-3}$ and Matlab's ode45 with tolerance 10^{-8}).

(c) Take $x(T)$ as an initial condition for $x' = -Ax$ and solve for time T, such that the result should be $x(0)$. Compare the numerical results and explain why and when they drastically differ. How much does this depend on the numerical method?

(d) Find $\det(\mathrm{e}^{At})$ theoretically. Then find it numerically by computing $\det(x_1(t)|x_2(t))$ with initial conditions $x_1(0) = (1, 0)^T$ and $x_2(0) = (0, 1)^T$. Show how the numerical determinant agrees for moderate times and then deviates. Why?

Homework is due on Monday, October 29, in class.
For full score, 4 correct exercises!