# HA 6 — Solutions

1) (a) $A = \begin{pmatrix} -1 & 0 & \cdots & & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & & \\ \vdots & & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix}$

A is lower triangular, hence ev are on the diagonal, $\lambda = -1$ w/ alg. mult. $N$, geom. mult. 1

$\Longrightarrow$ asy. stable

(b) $A = \begin{pmatrix} -1 & 0 & \cdots & & 1 \\ 1 & -1 & \cdots & & 0 \\ & \ddots & \ddots & & \\ & & & 1 & -1 \end{pmatrix}$

Plan! Fourier: $x = \left(e^{2\pi i j/N}, e^{2\pi i 2j/N}, \dots\right)$

are eig. vectors w/ eig. value

$e^{-2\pi i j/N} - 1$, for all $0 \le j \le N-1$

$\Rightarrow$ $j=0 \rightsquigarrow \lambda = 0$ eig'val', not asy'stable

<u>Plan2</u> char' pol'

$$\underbrace{(-1-\lambda)^N}_{\text{diagonal}} - \underbrace{(-1)^N}_{\text{from upper right entry}} = 0$$

$\hookrightarrow \lambda = -1 + e^{2\pi i j/N}$

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$\lambda = 0$ geom' simple, alg simple $\Rightarrow$ stability

(d) char pol': $(-1-\lambda)^N - \varepsilon(-1)^N = 0$

$$\lambda_0 = -1 + \varepsilon^{1/N} \overset{!}{\sim} -0.1$$

$$\rightsquigarrow \frac{1}{N}\log\varepsilon \sim \log 0.9$$

$$\underset{\varepsilon = 10^{-8}}{\rightsquigarrow} N \sim 175$$

(c) → $\boxed{\text{chain1d.m}}$ in google drive

Non commuting limits are first examined numerically by looking at

$N \gg T \gg 1$ vs

$\downarrow$
$\infty$

$T \gg N \gg 1$

$\downarrow$
$\infty$

2) $A_1 = \begin{pmatrix} -\varepsilon & 2 \\ 0 & -\varepsilon \end{pmatrix}$, $A_2 = \begin{pmatrix} -\varepsilon & 0 \\ 2 & -\varepsilon \end{pmatrix}$

$$e^{A_1/2} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \mathcal{O}(\varepsilon), \quad e^{A_2/2} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \mathcal{O}(\varepsilon)$$

$$e^{A_2/2} e^{A_1/2} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} + \mathcal{O}(\varepsilon)$$

$\operatorname{Spec}(A_j) = -\varepsilon < 0$

$\operatorname{spec} e^{A_1/2} e^{A_2/2}$ not in $\{|\lambda| \le 1\}$.