Math 8501, Fall 2018

Theory of Ordinary Differential Equations

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— Homework 6 —

(1) Consider the feedforward chain,

$$x_j' = x_{j-1} - x_j \in \mathbb{R},$$

where at each lattice sit j the system attempts to "mimic" the left neighbor, that is, for fixed x_{j-1}, x_j will relax to x_{j-1} (why?).

(a) Take $1 \leq j \leq N$ and set $x_0 \equiv 0$. Write the system in vector form $x' = Ax$ and find the eigenvalues. Conclude asymptotic stability.

(b) Take $1 \leq j \leq N$ but now set $x_0 \equiv x_N$ in the equation for x_1 (the chain is a ring!). Compute the eigenvalues and conclude stability (but not asymptotic stability). *Hint: To find the eigenvalues, use discrete Fourier transform* $x_j = \mathrm{e}^{\mathrm{i}j\sigma}$ *with* σ *such that* $x_N = x_0$.

(c) Solve numerically for large N, $x_j(t = 0) \equiv 1$. Demonstrate and explain why the limits $N \to \infty$ and $t \to \infty$ do for the linear evolution do not commute, that is,

$$\lim_{N\to\infty} \lim_{t\to\infty} |\mathrm{e}^{At}| = 0, \qquad \lim_{N\to\infty} \lim_{t\to\infty} \lim_{N\to\infty} |\mathrm{e}^{At}| = 1.$$

(d) Consider an ε-feedback, $x_1' = \varepsilon x_N - x_1$. Find the eigenvalues and conclude that for ε small fixed, we find arbitrarily slow decay when N is large. How large does N have to be to create an eigenvalue $\lambda = -0.1$ when $\varepsilon = 10^{-8}$?

(2) Construct an example where $A(t) = A(t + 1)$ is asymptotically stable for every t but not the non-autonomous equation. *Hint: Try* $A(t)$ *piecewise constant, with* $A(t) = A_1$, $0 < t < 1/2$, *and* $A(t) = A_2$, $1/2 < t < 1$, *and choose* A_j *asymptotically stable but not commuting.*

Homework is due on Wednesday, November 7, in class.
This homework counts for extra credit.

Those who may, vote before you turn this in!