HA-7 Solutions

1) Stable manifold is of the form

$$v = h^s(u) = h_2^s u^2 + \mathcal{O}(u^3)$$

$$h_2^s = -\frac{1}{2(2+s)} g_{uu}(0)$$

$\rightsquigarrow$ $u' = -u + f(u, h^s(u))$

on W^s. Since $f = \mathcal{O}(2)$,

$$u' = -u + \frac{1}{2} f_{uu}(0,0)\, u^2 + \mathcal{O}(3)$$

Ansatz $u = u_0 e^{-t} + u_2 e^{-2t} + \dots$ gives

$$-2u_2 = -u_2 + \frac{1}{2} f_{uu} u_0^2 \quad \text{at } \mathcal{O}(e^{-2t})$$

$\rightsquigarrow$ $u = u_0 e^{-t} - \frac{1}{2} f_{uu}(0)\, u_0^2 e^{-2t} + \dots$

$v = h_2^s u_0^2 e^{-2t}$

Now show that remainders are actually small!

Here, geometric proof:

$$\xi' = -\xi$$
$$u' = -u + \alpha u^2 + \mathcal{O}(3)$$

find $u = h(\xi)$ invariant manifold

$u_1 = u/\xi$ new variable

$$u_1'\xi + u_1\xi' = -u_1\xi + \alpha u_1^2\xi^2 + \mathcal{O}(\xi^3)$$

$$\begin{cases} u_1' = \alpha u_1^2 \xi + \mathcal{O}(\xi^2) \\ \xi' = -\xi \end{cases}$$

eq' $\xi = 0,\ u_1 = u_0$, lin $\begin{pmatrix} 0 & \alpha u_0^2 \\ 0 & -1 \end{pmatrix}$

$\Rightarrow$ ex. strong stable mfd

$$u_1 = u_0 - \alpha u_0^2 \xi + \mathcal{O}(\xi^2)$$

$$\Rightarrow u = u_0 e^{-t} - \alpha u_0^2 e^{-2t} + \mathcal{O}(e^{-3t})$$

3) $u' = -u + v$, $v' = -v + u^2$

$u = u_1$, $v = u_1 v_1$

$\boxed{u_1' = -u_1 + u_1 v_1}$, $v_1' u_1 + v_1 u_1' = -u_1 v_1 + u_1^2$

$\hookrightarrow v_1' + v_1(\cancel{-1} + v_1) = -v_1 + u_1$

$$\boxed{v_1' = u_1 + v_1^2}$$

$u_1 = 0$ invariant, $v_1' = v_1^2$

lin. at origin $\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$

$\rightsquigarrow$ strong stable mfd $v_1 = -u_1 + O(2)$

$v = -u_1^2 + O(3)$, tangent to $v = 0$. ⌧

4) a) $x' = -x$, $y' = -2y + x^2$

$y = h_2 x^2 + O(3)$

$2h_2 x \underbrace{x'}_{-x+\dots} = -2h_2 x^2 + x^2 + O(3)$

h_2-terms cancel! $0 = x^2 + O(3)$ ↯

b) $x(t) = x_0 e^{-t}$ $\qquad$ $x_0 = 1$ wlog.

$y(t) = y_0 e^{-2t} + x_0^2 t e^{-2t}$ $\qquad$ $t = -\log x$

$$\Rightarrow y = y_0 x^2 + (-\log x) x^2$$

$$\notin C^2$$

2) a)(i) $x=y=z=0$, lin. $(\ ' \ , \ )$, unstable

(ii) $x=y=z\neq 0$, $\quad x = (1+b+c)^{1/2}$ if $1+b+c>0$

lin' has eigespace $(1,1,1)^T$, eig'val' -2

→ Since A commutes w/ cyclic permutations, the two other eigenvalues are complex conjugate

$$\left(-2+b+c \pm \sqrt{3}\sqrt{-(b-c)^2}\right)/\underbrace{(1+b+c)}_{>0}$$

→ stable if $b+c-2<0$

unstable if $b+c-2>0$

(iii) $x \geq 0, y = z = 0 \to x = 1$,

$\lambda_1 = -2$, $(1,0,0)^T$

$\lambda_2 = 1-c$, $(0,1,0)^T$

$\lambda_3 = 1-b$, $(0,0,1)^T$

$\Rightarrow$ stable iff $c, b > 1$

(iv) $x = 0$, $y, z \neq 0$ $\quad \to y = \sqrt{\frac{1-b}{bc-1}}$

$\hookrightarrow 1 - y^2 - bz^2 = 0 \qquad z = \sqrt{\frac{1-c}{bc-1}}$

$1 - z^2 - cy^2 = 0$

if square roots are real

so $\boxed{bc > 1, \; b, c < 1}$ ($b, c < 0!$)

~~$bc>1$, $b,c<1$~~

$bc < 1$, $b, c > 1$ $\Downarrow$

one eigenvalue is $\frac{2(b-1)(c-1)}{bc-1} > 0$

hence always unstable.

(v) other eq': $x, y, z > 0$ and not all equal

$$\hookrightarrow \quad \begin{aligned} 1 - x^2 - by^2 - cz^2 &= 0 \\ 1 - y^2 - bz^2 - cx^2 &= 0 \\ 1 - z^2 - bx^2 - cy^2 &= 0 \end{aligned} \qquad \hookrightarrow \text{linear in } x^2, y^2, z^2$$

$\hookrightarrow$ unique sol' $(x^2 = y^2 = z^2)$ if $\det \neq 0$

$$\begin{vmatrix} -1 & -b & -c \\ -c & -1 & -b \\ -b & -c & -1 \end{vmatrix} = 0 \iff \begin{array}{l} \text{I) } b + c = -1 \quad \text{or} \\ \text{II) } b = c = 1 \end{array}$$

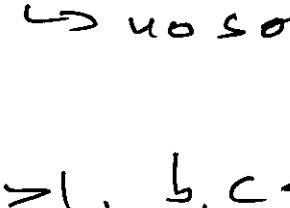
II) all eq' on $S^2 = \{x^2 + y^2 + z^2 = 1\}$, not hyperbolic.

I) kernel 1d, spanned by $(1,1,1)^T$, also cokernel $\Longrightarrow$ this $(1,1,1)$ not in range

$\hookrightarrow$ no sol'.

(b) not: $bc > 1$, $b, c < 1$,
need: $b + c > 2$ $\Big\}$ so $\boxed{b > 1 > c,\ bc < 1,\ b + c > 2}$

(c)



solutions spiral out towards a "triangle" in ~~the~~ coordinate planes $\cap S^2$

(d') The stable manifold of $\{x>0, y=z=0\}$ is two-dimensional by the linearization. restricting to a coordinate plane, we find that the eq' is stable inside the coordinate plane. By invariance of the coordinate planes, W^s (which is unique) is therefore contained in the plane. Similarly, the stable mfd of $(x=y=z>0)$ is contained in $\{x=y=z\}$.