Math 8501, Fall 2018

# Theory of Ordinary Differential Equations

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— *Homework 7* —

(1) Consider a saddle

$$u' = -u + f(u,v), \qquad v' = \rho v + g(u,v), \qquad \rho > 0,$$

with $f, g = \mathrm{O}(2)$ and smooth.

Let $(u,v)(t) \to 0$ be a solution in the stable manifold. Show that the solution possesses an exponential expansion, that is,

$$(u,v)(t) = (u_1,v_1)\mathrm{e}^{-t} + (u_2,v_2)\mathrm{e}^{-2t} + \mathrm{o}(\mathrm{e}^{-2t}),$$

(you will see that you can continue to higher orders).
*Optional: Compute the coefficients $u_j, v_j$ assuming that $u_1 = \delta \ll 1$ to order $\delta^2$, using the Taylor expansion of $f, g$.*

(2) Recall the amplitude equations for rotating convection

$$\begin{aligned} x' &= x(1 - x^2 - by^2 - cz^2), \\ y' &= y(1 - y^2 - bz^2 - cx^2), \\ z' &= z(1 - z^2 - bx^2 - cy^2). \end{aligned}$$

(a) Find all equilibria, determine their linear stability, and their nonlinear stability when the linearization is hyperbolic.

(b) Find the parameters $b, c$ such that there are no equilibria with $x = 0, yz \neq 0$ and such that all equilibria are unstable.

(c) Simulate the case $c = 3, b = 0$ in the octant $x, y, x \geq 0$. What do you see in the coordinate planes? What happens to solutions in $x, y, z > 0$?

(d) Show that in the parameter regime (b) (or, specifically, for the parameters in (c)), the $\omega$-limit set of a solution in $x > y > z > 0$ cannot be a single equilibrium.

(3) Consider $u' = -u + v$, $v' = -v + u^2$. Show that there exists a one-dimensional, smooth invariant manifold tangent to the eigenspace. Compute its quadratic expansion. *Hint: Use projection coordinates $v_1 = v/u, u_1 = u$ and find the strong stable manifold of the origin in the new coordinates.*

(4) Consider $x' = -x, y' = -2y + x^2$. The linear equation possesses a weak stable subspace $E^{\mathrm{s}} = \{y = 0\}$.

(a) Assume that there exists a $C^2$-invariant manifold tangent to this subspace and try to compute its quadratic terms at the origin: what goes wrong?

(b) Find solutions explicitly and show that manifolds tangent to $E^{\mathrm{s}}$ are not of class $C^2$.

*Homework is due on Monday, November 19, in class.*
*For full score, 4 correct exercises!*