# HA 8

1) polar coordinates

$r' = -r$ vs $R' = -R$

$\varphi' = 0$ $\phi' = 1$

$\varphi \equiv \text{const}$ $\to \phi = -\log R$

transform $\varphi = \phi + \log R$ is continuous, homeo

2) $r' = r f_r(r^2)$, $\varphi' = f_i(r^2)$,

so $r' = 0$ if $f_r(r^2) = 0$ $\to$ inv' circles.

p.o. if $f_i(r^2) \neq 0$, $\varphi' \neq 0$.

for F-muld' solve lin' equ

$\dot{r}' = 2r_0^2 f_r'(r_0^2) \cdot r$, $\varphi' = 2 f_i'(r_0^2) r_0 \cdot r$

$\to$ const coeff', upper triangular matrix, so ev on diagonal

$\lambda = 2r_0^2 f_r'(r_0^2)$ and $0$ as Floquet exp,

and multipliers $e^{\frac{2\pi}{f_1(r_0^2)}} \cdot \lambda$.

In special case, $r' > 0$ in $0 < r < 1$

$\Rightarrow r(t) \nearrow 1$. In fact, near $r = 1$,

$$r' = r(r-1)^2 \Rightarrow r = 1+\xi, \quad \xi' = (1+\xi)\xi^2 = \xi^2 + \xi^3$$

$$\xi(t) = \frac{1}{\frac{1}{\xi_0} - t} + o(1/t)$$

(extra credit if justified) +1

$$\varphi(t) = \int_0^t (1+\xi(t))\,dt \sim t + \log\left|\tfrac{1}{\xi_0} - t\right|$$

does not converge

3) a) $r' = r - r^3$ $\rightarrow r = 1$

$\theta' = \omega - \gamma r^2$ frequency $\omega - \gamma$

$T = \frac{2\pi}{\omega - \gamma}$

b) $r(t) = \frac{e^t}{\sqrt{1/r_0^2 - 1 + e^{2t}}}$ for $r_0 < 1$

$$\theta(t) = \int_0^t \left(\omega - \gamma r(s)^2\right) ds$$

$r_p(t) = 1$

$\theta_p(t) = (\omega - \gamma) t$

$\theta(t) - \theta_p(t) - \theta_{asy} \xrightarrow{!} 0$

$\Rightarrow \theta_{asy} = \lim_{t \to \infty} \left(\theta(t) - \theta_p(t)\right)$

$\ldots = -\theta_0 + \ln(r_0) \sim \log r_0 - \theta_0$

4) (i) yes, e.g. $H = \frac{1}{2}y^2 \cancel{+} \cos x$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = J \nabla H - (H - H_0) \nabla H$$

w/ $H_0 = 1$

↳ $H$ converges to 1, where we have heteroclinic loop

(ii) yes, e.g. $x'' = x - x^2$ as above.

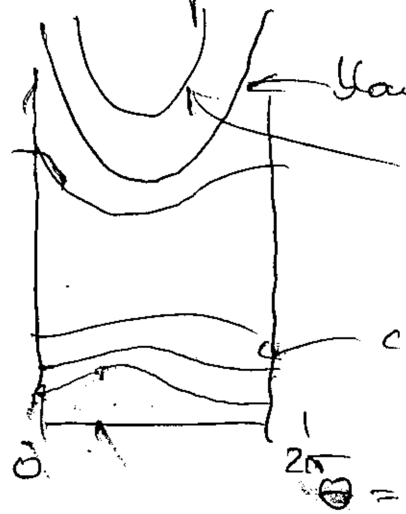
(iii) no, not connected

(iv) yes, e.g. example from (2) when $\theta' = r - 1$

(v) not chain recurrent (previous exercise)

(vi) contradicts P-B

5) (a) solve explicitly ... or as in (b)




6) $u' = \sin(\theta)\, u^3 \qquad u = 1/v$

$\theta' = 1$

$v' = -\sin\theta \; 1/v \qquad \left\{ \begin{array}{l} \text{Euler} \\ \longrightarrow \\ v \end{array} \right. \quad v' = -\sin\theta$

$\theta' = 1 \qquad\qquad\qquad\qquad \theta' = v$

$\Longrightarrow$ pendulum


$\frac{1}{2} v^2 - \cos\theta = 1$

$v < \sqrt{2(1+\cos\theta)}$

$\Rightarrow$ blow up finite time

6) a) derivation

$$\det \begin{pmatrix} 1 & h \\ -h\cos(x+hy_n) & 1-h^2\cos(x+hy_n) \end{pmatrix} = 1$$

b) this is the time-one map to a (Ham') vector field

$$\begin{pmatrix} x \\ y \end{pmatrix}' = J\nabla H\Big(\underbrace{t/h}_{1\text{-per}}, x, y\Big)$$

↳ averaging gives up to <u>any</u> order in $h$ (practically, an exponentially small remainder), a vector field

$$\begin{pmatrix} x \\ y \end{pmatrix}' = J\nabla \hat{H}(x,y)$$

with the <u>same</u> time-$h$-map, therefore the numerics lie on level sets (mostly circles) of $\hat{H}$.