Math 8501, Fall 2018

Theory of Ordinary Differential Equations

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— *Homework 8* —

(1) Find a homeomorphism that conjugates the flow to $z' = (-1+\mathrm{i})z \in \mathbb{C}$ and the flow to $z' = -z \in \mathbb{C}$.

(2) Consider the equation

$$A' = Af(|A|^2) \in \mathbb{C}, \qquad f : \mathbb{R} \to \mathbb{C}.$$

Write the equation in polar coordinates and note that the zeros $r > 0$ of $\mathrm{Re}\, f$ give invariant circles, which are periodic orbits when $\mathrm{Im}\, f$ does not vanish at these r-values. Find the Floquet multipliers for these periodic orbits. In the case $\mathrm{Re}\, f(r) = (r-1)^2$, $\mathrm{Im}\, f(r) = r$, show that solutions with $0 < |A| < 1$ at $t = 0$ converge to the periodic orbit with $r = 1$ but not with asymptotic phase!

(3) The phase-amplitude oscillator

$$A' = (1 + \mathrm{i}\omega)A - (1 + \mathrm{i}\gamma)A|A|^2 \in \mathbb{C},$$

for some $\omega \neq \gamma \in \mathbb{R}$.

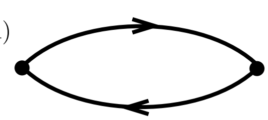
(a) Write the equation in polar coordinates (r, θ), find a periodic orbit and determine its frequency.

(b) Find the strong stable foliation *explicitly*! Therefore, solve the equation for r explicitly, treating the cases $r_0 > 1$ and $r_0 < 1$, separately, and solve the equation for θ using this solution for r. Note that the strong stable fibers are conjugate to each other by rotations in the plane.

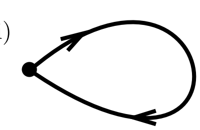
(c) Compute the strong stable foliation numerically using the method outlined in the script and overlay a plot of the theoretical explicit prediction.

(4) Which of the following invariant sets can occur as ω-limit sets of bounded trajectories to planar flows? Say yes/no and write a short reason why ((iv) and (vi) refer to continua of equilibria and perioric orbits, respectively).

(i)

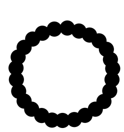


(ii)

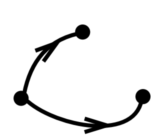


(iii)

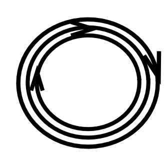
(iv)



(v)



(vi)



(5) Consider the innocent $u' = \sin(t)u^3$.

(a) Draw a phase portrait in $u, t \in \mathbb{R} \times S^1$.

(b) Find all initial conditions (t_0, u_0) such that the solution stays bounded for all times.

(c) Does the solution blow up, and does it blow up in finite or in infinite time, when it does not stay bounded?

(6) We study the discretized nonlinear pendulum

$$x_{n+1} = x_n + hy_n, \qquad y_{n+1} = y_n - h \sin x_{n+1}.$$

(a) Show that this method preserves volume in phase space.

(b) Explain how our discussion of averaging suggests that this method is (extremely) close to a time-h map of a planar vector field (one can show that this planar vector field is Hamiltonian).

(c) The Poincaré Bendixson theorem says that ω-limit sets in planar vector field are periodic orbits if they do not contain equilibrium points (see Chapter 8, below). Simulate the map and show that even for h moderately large, the ω-limit set does appear to lie on a smooth closed curve. Try for instance $x_0 = \pi$ and $y_0 = -1$, $h = 0.001, 0.01, 0.1$. Iterate for very long times, e.g. 10^7 and plot once all points are computed.

(d) Repeat the numerical experiments from the previous step with $h = 1.2$, 1, 0.8, 0.5, 0.3, 0.2, 0.1, ... When you see invariant curves, zoom in up to numerical accuracy (roughly 10^{-8} in matlab) and describe what you see. Explain how this is evidence that the averaging leaves an error that decreases extremely rapidly (in fact exponentially $\exp(-const/h)$).

Homework is due on Wednesday, December 12, in class.
For full score of 15, 6 correct exercises!