HA 1 – Solutions

1) Suppose $G(x,y,\mu)=0$. Show $D_{(x,y,\mu)}G$ inv'ble $\Longleftrightarrow$ $\{\dim\ker D_x f=1,\ \langle\partial_\mu f, e^*\rangle\neq 0,\ \langle\partial_x^2 f[e,e], e^*\rangle\neq 0\}$

$$D_{(x,y,\mu)}G=\begin{pmatrix}\partial_x f & 0 & \partial_\mu f\\ \partial_{xx}f\cdot[y,\cdot] & \partial_x f & \partial_{x\mu}f\cdot y\\ 0 & y^T & 0\end{pmatrix}$$

"$\Rightarrow$" i) $e_1, e_2\in\ker\partial_x f \Rightarrow Rk\,(\partial_x f \mid 0 \mid \partial_\mu f)\leq n-1$

$\hookrightarrow$ not onto ↯

ii) $\langle\partial_\mu f, e^*\rangle=0 \Rightarrow Rk\,(\ \_\_\_\_\_\_\ )\leq n-1$

$\hookrightarrow$ not onto ↯

(iii) $\langle\partial_{xx}f(e,e), e^*\rangle=0 \Rightarrow$ solve w/ r.h.s. r_x, r_y, r_μ

$$DG\begin{pmatrix}\hat x\\ \hat y\\ \hat\mu\end{pmatrix}=\begin{pmatrix}r_x\\ r_y\\ r_\mu\end{pmatrix},\quad \text{project } \left\langle\begin{pmatrix}e^*\\0\\0\end{pmatrix},\cdot\right\rangle$$

and $\left\langle\begin{pmatrix}0\\e^*\\0\end{pmatrix},\cdot\right\rangle$ to find

$\langle e^*, \partial_\mu f\rangle\,\hat\mu = \langle e^*, r_x\rangle$

$\langle e^*, \partial_{x\mu}f y\rangle\,\hat\mu = \langle e^*, r_y\rangle$

which cannot be solved for all r_x, r_y.

"$\Leftarrow$" $DG \cdot \begin{pmatrix}\hat x\\ \hat y\\ \hat \mu\end{pmatrix} = 0 \Rightarrow (1): \hat\mu = 0$ since $\partial_\mu f \notin Rg(\partial_x f)$

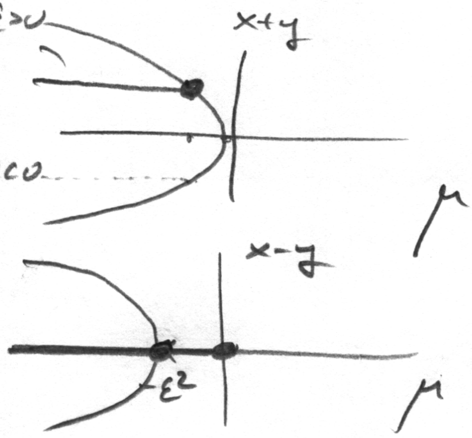
and $\hat x = \alpha e$; $\hat y = \beta e$, $\beta \neq 0 \Rightarrow$

$\alpha\beta\, \partial_{xx} f[e,e] + \partial_x f\, \hat y = 0_{xy}$, $\beta\langle e, \hat y\rangle = 0$

$\langle e^*, \cdot\rangle$ gives $\alpha = 0 \hookrightarrow \hat x = 0$, $\hookrightarrow \hat y = \gamma e$, $\gamma = 0$

$\hookrightarrow$ ker DG trivial ∎

3) Sol's (i) $x = y$, $\mu + x^2 = 0$



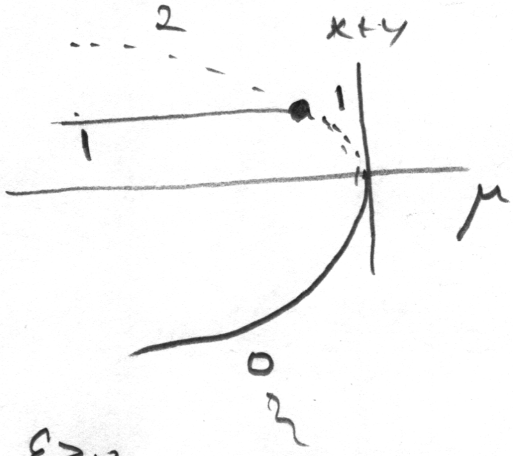
(ii) $x \neq y$: subtract

$\cancel{(x-y)}(x+y) - 2\varepsilon\cancel{(x-y)} = 0$

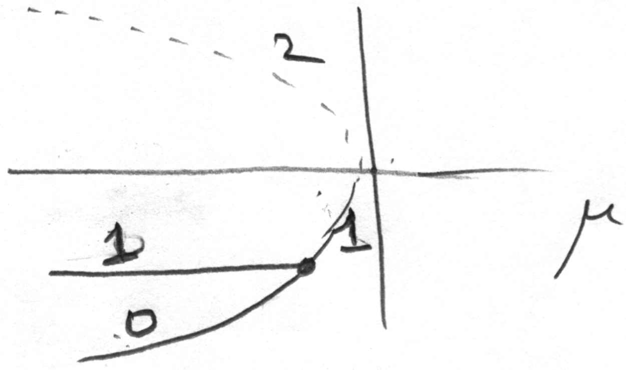
$x + y = 2\varepsilon$

<u>add</u> $(x+y)^2 + (x-y)^2 + 4\mu = 0$

$x - y = \pm 2\sqrt{-\mu-\varepsilon^2}$ for $\mu < -\varepsilon^2$



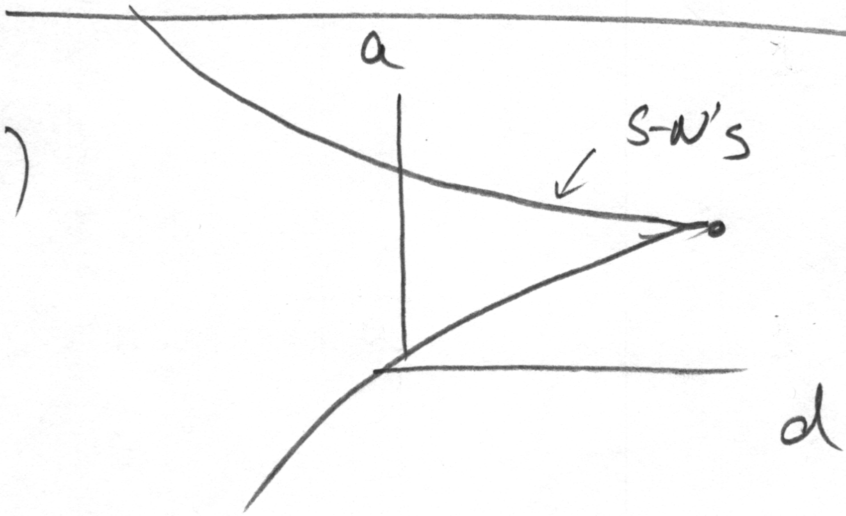
$\varepsilon > 0$



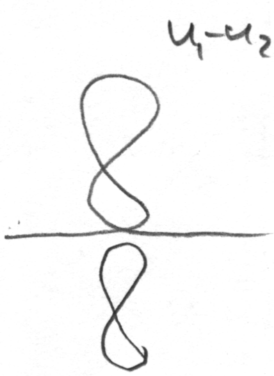
$\varepsilon < 0$

Morse index = # unstable ev

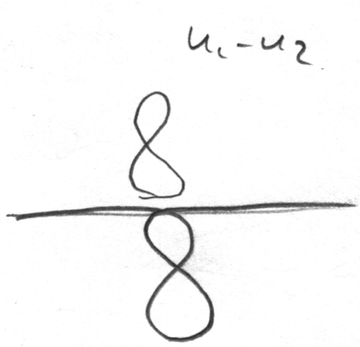
2)



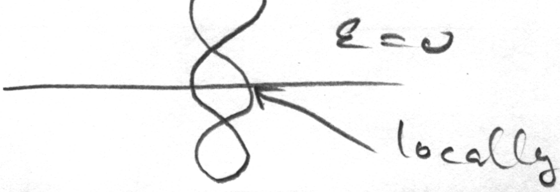
4)

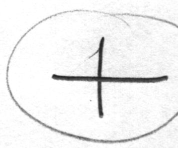


$\varepsilon < 0$



$\varepsilon > 0$





pitchfork
not
mfd