HA 3 - Sol

1) (i)

$$T^{-1}AT = (id+\mu T_1)^{-1} A (id+\mu T_1) =$$

$$(id - \mu T_1 + \mathcal{O}(\mu^2)) A (id + \mu T_1) = A + \mu [A, T_1] + \mathcal{O}(\mu^2)$$

(ii)

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} - \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} t_{21} & t_{22} - t_{11} \\ 0 & -t_{21} \end{pmatrix}$$

(iii)

$$\begin{pmatrix} a & 0 \\ b & a \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix} = \begin{pmatrix} a+\alpha & \beta \\ b & a-\alpha \end{pmatrix} \text{ general matrix}$$

(iv)

Solve $\cancel{T}$

$$\begin{pmatrix} \mu a & 1 \\ \mu b & \mu a \end{pmatrix} = \begin{pmatrix} \mu a_{11} & 1 + \mu a_{12} \\ \mu a_{21} & \mu a_{22} \end{pmatrix} + \mu (A T_1 - T_1 A) + \mathcal{O}(\mu^2)$$

$$a = a_{11} + t_{21} + \mathcal{O}(\mu)$$

$$0 = a_{12} + t_{22} - t_{11} + \mathcal{O}(\mu)$$

$$b = a_{21} + \mathcal{O}(\mu)$$

$$a = a_{22} - t_{21} + \mathcal{O}(\mu)$$

$t_{11} = 0$, lin' in a, t_{21}, t_{22}, b is invertible

$\hookrightarrow$ IFT for sol'

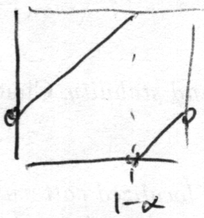
(v) $\langle \mathrm{ad}_1(A)U, V\rangle = \mathrm{tr}\big((AU-UA)V^T\big) = \mathrm{tr}\big(AUV^T - UAV^T\big)$

$\langle U, \mathrm{ad}_1(A^T)V\rangle = \mathrm{tr}\big(U(A^TV-VA^T)^T\big)$

$= \mathrm{tr}\big(UV^TA - UAV^T\big)$

$\mathrm{tr}(UV^TA) = \mathrm{tr}(AUV^T)$ since $\mathrm{tr}(abc) = \mathrm{tr}(cab)$

2) (i)



$x = 1-\alpha \Rightarrow k(x) = l+1$
(more iterates needed)

$x = 1 \Rightarrow k(x) = l$

$\underbrace{1 + l\cdot\alpha}_{l \text{ iterates}} \underbrace{-1}_{\text{mod } 1} \geq 1-\alpha$

$1 + (l-1)\alpha - 1 < 1-\alpha$

$\frac{1}{\alpha} > l > \frac{1}{\alpha} - 1 \Rightarrow l = [\frac{1}{\alpha}]$

(ii) Since $1 = (1-\alpha) + \alpha = R_\alpha(1-\alpha)$

(iii) $\Phi(1-\alpha) = (l+1)\alpha + (1-\alpha) - 1 = [\frac{1}{\alpha}]\cdot\alpha$

renormalize: subtract $1-\alpha$ $\hookrightarrow$ $[\frac{1}{\alpha}]\cdot\alpha - (1-\alpha)$

scale $\frac{1}{\alpha}$ $\hookrightarrow$ $[\frac{1}{\alpha}] - \frac{1}{\alpha} + 1$

reflect $1-\circ$ $\hookrightarrow$ $\frac{1}{\alpha} - [\frac{1}{\alpha}]$

(iv) $\alpha = \dfrac{1}{a_1 + \dfrac{1}{a_2 + \dots}}$ $\qquad a_1 = \left[\dfrac{1}{\alpha}\right]$,

$$\dfrac{1}{a_2 + \dots} = \dfrac{1}{\alpha} - \left[1/\alpha\right]$$

So iterating the map $\alpha \longmapsto 1/\alpha - [1/\alpha]$ gives the sequence of rotation nos, which are $\dfrac{1}{a_1 + \dfrac{1}{a_2 + \dots}}$, $\dfrac{1}{a_2 + \dfrac{1}{a_3 + \dots}}$, $\dfrac{1}{a_3 + \dfrac{1}{a_4 + \dots}}$

and $a_j = \left[\dfrac{1}{\alpha_j}\right]$, $\alpha_{j+1} = \dfrac{1}{\alpha_j} - \left[1/\alpha_j\right]$

(v) Clearly every orbit has to hit J_1 precisely once since J_1 has length α, before returning (or $\underset{l=1}{\text{not}}$) to J_0. The number l is $\left[\dfrac{1-\alpha}{\alpha}\right] = [1/\alpha] - 1$

(vi) The number of iterates for the return map jumps precisely at the "jump location", such that the length of the block encodes location relative to $1-\alpha_1$ in the renormalized map