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THE EVOLUTION OF EVOLUTIONARY EQUATIONS

"May you live in exciting times!" This traditional Chinese saying aptly describes the environment surrounding the basic developments in mathematics, physics, and chemistry over the past four centuries. From the founding of the European Academies of Sciences during the era of Peter the Great and Napoleon, to the founding of the National Science Foundation in the United States of America during the presidency of Harry Truman, governments have realized the importance of scientific research.[1] To trace the implications of this academic research, as it affects the evolution of evolutionary equations, we begin in 1601 in Prague with the appointment of Johannes Kepler (1571 - 1630) to the position of Imperial Mathematician of the Holy Roman Empire, after the death of his predecessor, Tycho Brahe (1546 - 1601).

Newtonian Modeling: It was Kepler's early work on planetary motion which attracted the attention and respect of Brahe, who in turn invited Kepler to join his research staff. As Brahe's successor, Kepler had access to Brahe's very extensive records and observations of planetary motion. Kepler's goal was to derive a good mathematical model for planetary motion in our solar system. He succeeded!

In his 1609 paper *Astronomia Nova,* he derives two laws of planetary motion: (1) each planet travels in an elliptical orbit with the sun at a focus and (2) each planet sweeps out equal areas in equal times when traveling in its orbit. Then in 1619 he published *Harmonice Mundi,* in which he presents a third law: (3) the 3:2 rule relating the mean distance between the planet and the sun with the period of the motion. These three Keplerian Laws gave astronomers a new and

[1] Also, the emigration of some major scientists, such as Daniel Bernoulli (1700 - 1782) and Leonhard Euler (1707 - 1783) to St Petersburg, and Albert Einstein (1879 - 1955) and John von Neumann (1903 - 1957) to Princeton, accelerated the growth and international impact of science during this period.

unexpected paradigm for the study of the motion of asteroids and comets, as well as planetary motion.

Why was this considered to be a "good" model? Before answering, we must emphasize that any model must be measured by the standards of its time. Brahe's observations preceded the invention of the telescope, so one cannot fault the Keplerian model for lack of better experimental data.

The Keplerian Laws, which are fully valid for the 2-body problem, are only approximations to the planetary motion in the celestial mechanical N-body model of the solar system. It is a fact that Kepler, like Brahe before him, was very meticulous in his work, so much so that one wonders whether he would have found the three Keplerian Laws if the astronomical data of Brahe had been obtained with the more accurate telescopic observations. Only with the telescope did astronomers have the technology to see that the planetary motions do in fact deviate somewhat from a true elliptical orbit. The major importance of the Keplerian Laws is the seminal role they played later in the time of Isaac Newton (1643 - 1727) with the birth of classical mechanics.

The idea of formulating mathematical models of the solar system, in terms of differential equations representing the laws of motion, began to take hold in the scientific community around the time of Galileo Galilei (1564 - 1642). The mathematicians [2] of that day were trying to understand the basic relationships between force, momentum, displacement, and mass. Because of his extensive experimental work with pendula and inclined planes, Galileo was instrumental in the development of what is now called classical mechanics. It is at this point that Newton, a professor at Trinity College in Cambridge, enters the scene.

Newton, like Galileo, was searching for universal principles which could be used to explain the physical world around him. In this process, he arrived at a set of three laws, the Newtonian Laws of Motion. The first law, which is a reformulation of the Galilean concept of uniform motion, states that a body in motion remains in motion until a force acts on the body. The second law, $F = ma$, equates the force with the rate of change of momentum; and the third law states that for each action there is an equal and opposite reaction. These three laws are as insightful as they are simple. Even today, in the aftermath of more recent mechanical theories, such as quantum mechanics and relativistic mechanics, the Newtonian Laws are very widely used. For example, the momentum equation, which arises in the Navier-Stokes model of fluid flow, is a reformulation of the second law of Newton.

It was Newton's belief that the Keplerian Laws could be derived from the Newtonian Laws of Motion and the existence of a force caused by a gravitational field acting between the sun and the planets. By using the three laws of motion and the Keplerian 3:2 rule in the case of a single planet revolving about the sun, Newton found that the centripetal force acting on the planet was given by the inverse-square law, i.e., the force is inversely proportional to the square of the distance between the planet and the sun. However, the great achievement of

[2] During the time of Galileo, and for a long time thereafter, the science of physics was viewed as a part of mathematics.

Newton was to prove the truth of the opposite implication. That is, he succeeded in showing that, by using his Laws of Motion, together with the assumption of a gravitational force given by the inverse-square formula, one can derive the three Keplerian Laws as consequences. It was this theorem which gave birth to the Newtonian concept of the universal law of gravity. This work of Newton on mechanics appeared in 1687 in his masterpiece: *Philosophiae Naturalis Principia Mathematica,* or the Principia, for short.

It is very hard to overstate the importance of Newton's contributions to the advancement of science. The Principia, which includes the Newtonian model of mechanics (the three laws of motion, the law of universal gravity, and the inverse square law and the Keplerian model), as well as the beginnings of the differential and integral calculus, is probably the most important and the most significant treatise on mathematics ever written. While many other major successes in mathematical modeling were to follow, no other single achievement would have the same impact on the history of man's attempt to understand the world about us. It was this work of Newton, and the simultaneous discovery of the calculus by Newton and Gottfried Wilhelm Leibniz (1646 - 1716), that fully established the role of mathematics as the principal tool for modeling the laws of nature.

The Newtonian laws of motion for the N-body problem of celestial mechanics enable one to describe the dynamics of the problem in terms of the solutions of a system of ordinary differential equations. For the full problem, one has a three-dimensional (3D) position vector and a 3D velocity vector for each body. Thus for the three-dimensional problem, the equations of motion are described by a $6N$-dimensional system of ordinary differential equations. In the planar problem, where the N bodies are restricted to a plane, the equations of motion are described by a $4N$-dimensional system of ordinary differential equations.

However, there are some conservation laws for these problems which effectively reduce the dimension of the phase space for the equations of motion. In particular, the time derivatives of: the center of mass, the linear momentum, the angular momentum, and the energy are all zero. One has six conservation laws for the planar problem, and ten laws for the full three-dimensional problem. As a result, the reduced dimension for the planar problem is $4N - 6$; and for the full problem it is $6N - 10$, see Meyer and Hall (1992) and Siegel and Moser (1971). In particular, the 2-body problem is described by a system of ordinary differential equations in the plane $\mathbb{R}^2$.

In the celestial mechanical model of the solar system, where $N \geq 10$, the complexity of the equations of motion have defied all attempts at trying to find explicit formulae for the solutions, except in a few very special cases. Nevertheless, this model for the solar system is so good and the analysis of the solutions is so accurate that this has led to a very high degree of predictability of the position of the planets. As a matter of fact, on two occasions, this predictability has enabled astronomers to locate new planets. How did this happen?

The processes leading up to the discovery of Neptune (in 1846) and Pluto (in 1930) both began with the observations that the predicted positions of the then known planets were deviating from the actual positions in a way which could not be explained on the basis of the gravitational field of the sun and the known planets alone. This led in turn to the idea that one might postulate the existence of a new planet and then use its gravitational field to rederive the predicted positions. By adjusting the parameters (e.g., mass and position) of the new planet, one could try to reduce the deviations to zero. In other words, one seeks to use the deviations themselves to locate the unknown planet.

As it happens, the mass of the planet Pluto appears to be too small to explain fully the previously observed deviations between the predicted and actual orbits of Uranus and Neptune. Does that imply the existence of yet another planet, a Planet X? That is not known, and because of the long 248 Earth-year-period for Pluto's orbit, it may be too early to answer this question. However, the methodology for finding a tenth planet is now in place. Time will tell.

Birth of Dynamical Systems: The 3-body problem, in particular, presented a major challenge to the mathematical world. Of special interest was the satellite problem, for example, the Sun-Earth-Moon system, where the third body has a relatively small mass when compared to the two major bodies. As the efforts to find explicit formulae for the general solutions fell short, greater interest was placed on new qualitative methods for the analysis of the dynamics of the solutions. Furthermore, these new methods grew in importance as researchers turned to the issues of longtime dynamics, such as the stability of the solar system.

Certainly among the most important advances in this area are two works of Henri Poincaré (1854 - 1912): his 1890 paper *Sur le problème des trois corps et les équations de la dynamique* and the 1892 treatise *Les Méthodes Nouvelles de la Mécanique Céleste* I-II-III. One of the most interesting features of these works was the realization of the possibility of an instability in the N-body problem (where $N \geq 3$) owing to intersections of the stable and unstable manifolds of a periodic orbit. Poincaré's works are highly significant. Not only did he win the prestigious King Oscar Prize, see Goroff (1993), but more importantly, Poincaré, along with Alexander Mikhailovich Lyapunov (1857 - 1918) and George David Birkhoff (1884 - 1944), emerged as a co-founder of a new area: dynamical systems.

The issue of stability arises, in one way or another, in all mathematical models. It is omnipresent and multifaceted. Whether a given dynamical feature is stable or not, depends on the context, or point of view. The different meanings of the word stability come from the point of view. In the van der Pol equation, for example, there is an (unstable) source, within the (stable) global attractor, and the source is a stable dynamical feature of the global attractor. The major work on stability theory appears in the 1892 paper by Lyapunov, *Problème géneral de la stabilité du mouvement.* This important study, which coincided with the advances in celestial mechanics noted above, is a very significant development in the evolution of evolutionary equations, for several reasons.

First, this is the work in which Lyapunov presented his theory of a generalized energy functional, now called a Lyapunov function, which can be used to study the stability of certain systems of differential equations without first solving for the solutions. This theory is a precursor of the LaSalle Invariance Principle and Morse structures for dissipative evolutionary equations. Unlike the N-body problem, in which the total energy is constant along solutions, in dissipative problems, the energy can vary along solutions, but it is typically ultimately bounded. This is a common feature of those dynamical systems that have a global attractor.

Second, a theory of characteristic exponents, now called Lyapunov exponents, for time-varying linear differential equations is developed in this work. Based on contributions of Lyapunov and his followers, it is now appreciated that the theory of Lyapunov exponents offers a good framework for finding upper bounds for the dimension of an attractor of an evolutionary equation. While there are now several theories of dimension which are applicable to this study, the theory of Lyapunov dimension plays a unique role because of the strong analytical tools it brings to the problem.

It is noteworthy that both of these theories, which Lyapunov had developed for applications to finite systems of ordinary differential equations, have meaningful extensions to the infinite dimensional world of dynamical systems. The work of Lyapunov is as important today as it was when it first appeared in 1892.

The concept of a dynamical system, as we know it today, was developed by G D Birkhoff in the early part of the twentieth century. His theory of minimal sets, recurrence, nonwandering sets, central motions, transitivity, and the foundations of Hamiltonian systems forms the basis of many advances. Much of this material appears in his 1927 book *Dynamical Systems.* An especially important contribution is his well-known ergodic theorem, see Birkhoff (1931a,b). This theorem, which is a pillar for the theory of statistical mechanics, also serves as a bridge for the use of related functional-analytic techniques in the theory of dynamical systems.

The issues studied by Poincaré, Lyapunov, and Birkhoff all fit within the general theory of finite dimensional systems of ordinary differential equations. At a later time, other researchers would show that some of the techniques developed by the Founders do extend to selected infinite dimensional problems. During the early period of dynamical systems there were, of course, other advances. Two of these are especially noteworthy. First, there are the extensions of the Birkhoff theory to the question of the existence of invariant measures and ergodic measures, for compact, invariant sets in a dynamical system, see, for example, Krylov and Bogoliubov (1937). Second, there are two methods for the construction of invariant manifolds for nonlinear problems: (1) the Hadamard (1901) method and (2) the Lyapunov (1892) - Perron (1928, 1930b) method.

Infinite Dimensional Challenge: Not surprisingly, the theory of the long-time dynamical properties of solutions of infinite dimensional evolutionary equations generated by partial differential equations was slower in coming than the finite dimensional counterpart. Among other issues, the early researchers encountered additional difficulties, not seen on the theory of ordinary differential equations, in sorting out which partial differential equations problems had good solutions. A major step in resolving this was the concept of a well-posed problem proposed by Jacques Hadamard (1865 - 1963). It is very likely that in formulating this concept, Hadamard was influenced by the concurrent developments in the dynamics of ordinary differential equations, see, for example, Hadamard (1901). At a later time, the concept of a well-posed problem would play a central role in the definition of a <u>semiflow</u> generated by an evolutionary equation.

The development of the theories of the longtime dynamics for linear and nonlinear evolutionary equations generated by partial differential equations is one of the major triumphs of the area of functional analysis. This area of mathematics began with the works of Henri Lebesgue (1875 - 1941), who presented his new definition of the integral at the beginning of the twentieth century. Owing to the good limit theorems for the Lebesgue integral, this concept quickly replaced the Riemann integral in mathematical analysis. David Hilbert (1862 - 1943) used the theory of Lebesgue to analyze solutions of integral equations. In so doing, he built the basis for the abstract theory of Hilbert spaces, a term which was later coined by von Neumann (1930). In 1932, Stefan Banach (1892 - 1945) published a beautiful volume on *Théorie des Opérations Linéaries.* The concept of a Banach space is derived from this work.

Let us return to the concept of a well-posed problem in the context of a parabolic, or hyperbolic, partial differential equation. In each of these problems, unlike the case of an elliptic partial differential equation, one encounters an Initial Value Problem (IVP), or, as it is sometimes called, a Cauchy problem. This suggests that a time-varying solution of the IVP can be viewed as a trajectory, or curve, in some Banach space, which is the phase space for the problem. The equation of motion of this trajectory in the Banach space is given by a linear or nonlinear evolutionary equation. Loosely speaking, an evolutionary equation is an ordinary differential equation on a Banach space. This simple observation, by some researcher unknown to the authors, gave rise to the study of the dynamics of solutions of partial differential equations.

Nevertheless, the issue of the proper definition of a <u>solution</u> of an evolutionary equation is more complicated in the infinite dimensional setting. One might have a <u>strong</u> solution, which is an absolutely continuous function that satisfies the evolutionary equation almost everywhere in time; or one might have a <u>mild</u> solution, which is a solution given by an integral equation, the variation of constants formula. [3] In finite dimensions, these concepts are the same, but they differ in the infinite dimensional setting.

The raison d'être for Hilbert spaces and Banach spaces is the study of linear operators, both bounded and unbounded. Some of the early applications of this study were in the analysis of solutions of linear partial differential equations. In the case of linear evolutionary equations, an operator calculus is needed to study the solutions. Such a calculus was developed by means of a linear semigroup of bounded linear operators and its infinitesimal generator, see Hille and Phillips (1948, 1957). For some linear problems, such as the Stokes problem,

[3] See Sections 4.2, 4.6, and 4.7 for more details.

the semigroup is analytic. This permits one to introduce a tower of Banach spaces, which in turn offers a good framework for the analysis of the nonlinear problems.

For those evolutionary equations generated by a nonlinear system of partial differential equations, there are basically two approaches for the study of solutions: (1) a methodology based on the theory of mild and strong solutions of the nonlinear problem, and (2) a methodology based on a theory of weak and strong solutions. The mild-strong approach builds on the variation of constants formula

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}F(u(s))\,ds,$$

which defines the mild solution $u(t)$ in the Banach space H, where $u_0 \in H$, e^{-At} is a linear semigroup, and $-A$ is the infinitesimal generator. The nonlinear term $F = F(u)$ includes the nonlinearity in the underlying partial differential equation. In the case where F is a suitable Lipschitz continuous mapping of the phase space H into itself, then the proofs of the existence of mild solutions and the properties of these solutions follow the ordinary differential equation paradigm.

However, a serious complication occurs, as in the Navier-Stokes equations or the Cahn-Hilliard equation, when the nonlinear term in the underlying partial differential equation contains spatial derivatives of the unknown solution. In this case, the evolutionary equation is not well-defined on an L^2-space. One needs to set the problem in a space of functions with greater spatial regularity. However, since the image $v = F(u)$ will have less spatial regularity than u, there is another difficulty.

One would be at an impass here, except for confluence of two very important developments. First, there is the notion of a tower of Banach spaces which arise in the case where the linear semigroup is analytic. Second, there are the imbedding theorems of Sergei L Sobolev (1908 - 1989) and the applications to the Sobolev spaces $W^{m,p}$, see Sobolev (1938, 1950). What follows from these two theories is a calculus for the study of the nonlinear terms appearing in many partial differential equations. For example, the inertial term $F = F(u)$ arising in the Navier-Stokes equations is a mapping $F : W^{2,2} \to W^{1,2}$ with two continuous Fréchet derivatives.

The alternate weak-strong approach for solutions, was discovered by Jean Leray (1906 - 1998) in his three papers on the Navier-Stokes equations, one in 1933 and two in 1934. The concept of a weak solution is based on the observation that any bounded set in H, where H is a Hilbert space or a reflexive Banach space, has compact closure in the weak topology on H.

A simplified description of Leray's approach is to begin with the construction of a sequence of approximate solutions for a given IVP for the Navier-Stokes equations. The next step is to use properties of the linear and nonlinear terms in the equations to show that the given sequence is in a bounded set and, therefore, that there is a subsequence that converges weakly. The limit of this subsequence is shown to be a weak solution. With a variation of this argument, one shows that if the initial datum has greater spatial regularity, then the weak solution is a strong solution, at least for time in some finite interval.

While the theory of Leray was formulated for unbounded domains in $\mathbb{R}^3$, Hopf (1951) showed that a similar theory was valid on suitable bounded domains in $\mathbb{R}^3$. Owing to the good properties of the Stokes operator on a bounded domain, the Hopf theory is especially important for the study of the longtime dynamics of the Navier-Stokes equations.

The study of differential equations with time delays, or more generally functional differential equations, is a rather recent development in the theory of evolutionary equations. The basic impetus for obtaining a good theory for these problems was the simple, yet insightful, observation by Hale (1963) and Krasovskii (1963) that the initial value problem is well-posed only when the initial condition is in a suitable function space (e.g., the space of continuous real-valued functions) defined over the delay interval. While it is then a straightforward issue to generate solutions, it should be noted that the resulting theory behind the dynamical features can have all the complexity seen in the case of partial differential equations, see Hale and Verduyn Lunel (1993).

Just the Beginning: By the 1930s, the basic theory of dynamical systems was well in place, and the basic studies, which at a later time would lead to a theory of flows and semiflows for the infinite dimensional evolutionary equations arising in partial differential equations, had begun. During the period 1930 - 1970 there were many major developments in the study of the longtime dynamics of systems of ordinary differential equations, including perturbation theory for invariant manifolds, bifurcation theory, exponential dichotomies and hyperbolic structures, the Pliss reduction principle (center manifold), the Kolmogorov-Arnold-Moser theory, skew products flows for nonautonomous problems, Morse-Smale dynamical systems, the structural stability program, the role of symmetries, and index theory.

By the 1970s, the dynamical theories for dissipative partial differential equations, such as reaction diffusion equations, the Navier-Stokes equations, and the Cahn-Hilliard equation, were coming to fruition. In this area and during the subsequent 30 years, one finds the development of existence theories and dimension theories for global attractors and inertial manifolds, the use of smooth and discrete-valued Lyapunov functions to find Morse-Smale structures and Poincaré-Bendixson theories, and the use of exponential trichotomies and hyperbolic structures for the perturbation theory of invariant manifolds, for example.

The year 1970 is an approximate date of the merger of finite dimensional and infinite dimensional dynamical systems. Since that time, this has become a united subject, the Dynamics of Evolutionary Equations. Other major developments in longtime dynamics which date from the time of this merger include the Melnikov method, singular perturbations, random dynamical systems, almost periodic and almost automorphic dynamics, and approximation dynamics. The subject of the Dynamics of Evolutionary Equations is only at its beginning.

While it is not possible to predict the future, we sincerely hope that this volume will be helpful for scholars working in these areas and in some of the newer areas of dynamics, such as global climate modeling, numerical simulation of longtime dynamics, and control theory in time-varying media.