MA5076
Mathematics of Options, Futures, and Derivative Securities II.

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Syllabus

Semester I

1. Mechanics of Futures Markets
2. Hedging Strategies Using Futures
3. Interest Rate Markets
4. Determination of of Forward and Future Prices
5. Interest Rate Futures
6. Swaps
7. Mechanics of Options Markets
8. Properties of Stock Option Prices
9. Trading Strategies Involving Options
10. Binomial Trees
11. Wiener Processes and Ito's Lemma
12. The Black-Scholes Model
13. Options on Stock Indices, Currencies and Futures
14. The Greek Letters
15. Volatility Smiles

Semester II

1. Review of Black-Scholes
2. Review of Greeks
3. Volatility Smiles
4. Basic Numerical Procedures
5. Value at Risk
6. Time Series
7. Estimation of Volatilities and Correlations
8. Credit Risk
9. Credit Derivatives
10. Exotic Options
11. Weather, energy, and insurance derivatives
12. More on models and numerical procedures
13. Martingales and measures
14. Interest Rate Derivatives
15. Convexity, timing, and quanto adjustments
16. Models of Short Rate
17. Heath-Jarrow-Morton
18. Swaps revisited
19. Real Options

Class Information

- Lecture: Mondays & Wednesdays 5:00PM–6:30PM.
- Lecture Room: Vincent Hall 20
- Office Hours: Mondays 3:00PM–4:30PM
- Office: Vincent Hall 112b
- Contact: spirn@math.umn.edu & 612-625-1349
- Textbook: *Options, Futures, and Other Derivatives*, 6th Edition, John Hull, Prentice Hall.
- Grade Information:
  - Homework: 50 %
  - Midterm: 20 %
  - Final: 30 %

What is a derivative?

What is a derivative?

A financial instrument whose value **derives** from the value of underlying variables.

or

Financial instruments whose price and value derive from the value of assets underlying them. [1]

or

Financial contracts whose value derive from the value of underlying stocks, bonds, currencies, commodities, etc.

Examples:

Future contract for orange juice. Not the orange juice itself.

Option to buy/sell a stock. Not the stock itself.

[1] Edmund Parker

Asset Derivatives

Examples

- Commodity Derivatives - Pork Bellies (Trading Places), Precious Metals
- Equity Derivatives - Stocks / Bonds
- Interest Rate Derivatives - Interest Rates
- Currency Derivatives - Currency Exchange Rates (Yen vs. Euro)
- Property Derivatives - Real Estate
- Other More Exotic Derivatives

Why trade derivatives?

- Used to protect assets from drastic fluctuations
- Covers many kinds of risk.
- Allows access to financial instruments with considerably larger risk than conventional assets (speculation).

Financial instruments used in derivative transactions

Forwards

Futures

Options

Swaps

Forward Contracts

Definition: A **forward contract** is an agreement to buy or sell an asset at a certain time for a certain price. Usually traded in the OTC market.

One party assumes a **long position** by agreeing to buy the underlying asset on a specified future date for a specified price.
The other party assumes a **short position** by agreeing to sell the underlying asset on the same date and at the same price.

Definition: A **spot contract** is an agreement to buy or sell an asset today.

Spot contracts are for immediate delivery of the asset.
Common among currency derivatives.

Futures Contracts

- Similar to a forward contract - it is an agreement between two parties to buy or sell an asset at a certain time in the future for a certain price.

- However, futures are usually traded in exchanges.

- Mechanisms by the exchange to *guarantee* that the contract will be honored.

- Example: Chicago Mercantile Exchange (CME) trades futures on
  - commodities such as pork bellies, orange juice, copper, sugar, etc.
  - financial assets such as stock indices, currencies, Treasury bonds, etc.

- Mechanisms for such exchanges will be explained next week.

Options Contracts

The third type of derivative we will discuss is an options contract. These are divided into two types:

- A call option entitles the holder the right to buy the underlying asset by a certain date for a certain price.

- A put option entitles the holder the right to sell the underlying asset by a certain date for a certain price.

Some more terminology:

- The price in the contract is the strike price or the exercise price.

- The date in the contract is known as the expiration date or maturity.

Types of options

- American options can be exercised at any time up to the expiration date.
- European options can be exercised only on the expiration date.

Mathematics involved in American options is more difficult than European options.

One contract is usually an agreement to buy or sell 100 shares.

Note an option is exactly that - an option to exercise the right to buy or sell.
The holder of the option does not need to exercise the option.

Participants in Options Markets

Four types of participants in the Options markets, as yet:

- Buyers of calls   a long position
- Sellers of calls   a short position
- Buyers of puts   a long position
- Sellers of puts   a short position

Selling an option is also known as writing the option.

Types of derivative traders in the market

- Hedgers - use derivatives to reduce risk in the market from potential future market movements.
- Speculators - trade derivatives to bet on the future direction of a market variable.
- Arbitrageurs - take offsetting positions in two or more instruments to lock in a profit. (usually short-lived)

Arbitrageurs

Third important group of traders in derivatives. Involves locking in riskless profit by simultaneously entering into transactions in two or more markets.

Mostly possible when future prices become out of line with spot prices.

- Transaction costs may gobble up most of the profit for small investments, but large financial institutions could profit. Furthermore, arbitrage opportunities are quickly lost.

- Useful for determining the monetary value of certain derivatives.

Wiener Processes and Itô's Lemma

More sophisticated approach to modeling the behavior of assets underlying derivatives - view motion as a stochastic process.

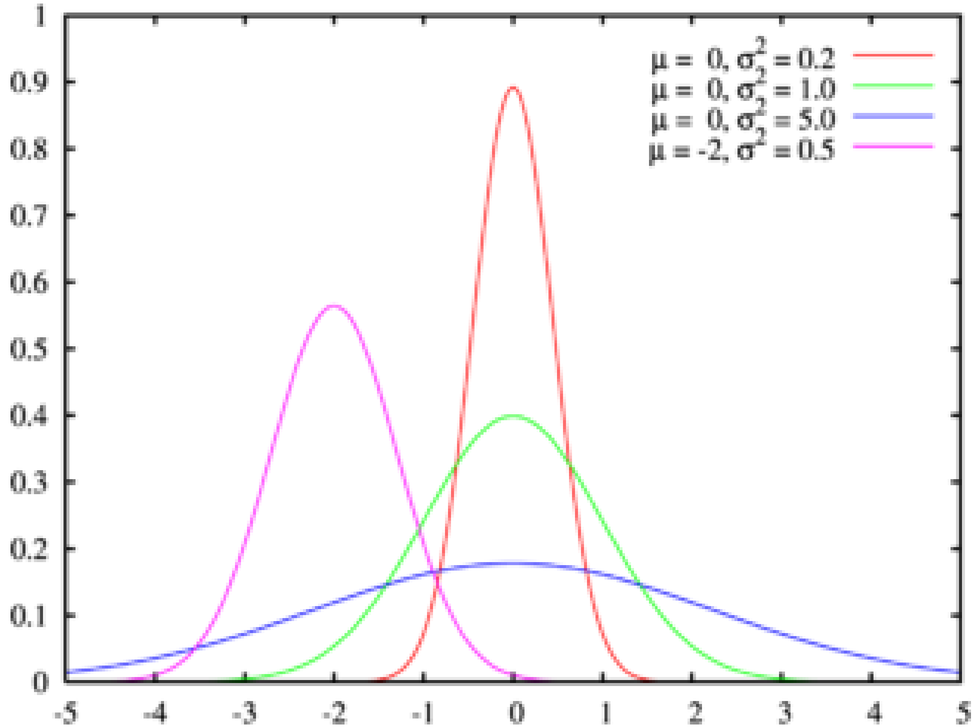
A stochastic process is a process where future evolution is described by probability distributions.

Two types: discrete-time stochastic process changes values at discrete time steps. A continuous-time stochastic process changes value at any time.

Stochastic process can be continuous variable or discrete variable. A continuous-variable process can take any value within a certain range. (motion of a particle in fluid). A discrete-variable process can take only certain prescribed values. (coin flips)

Markov Process is a stochastic process where only present value of a variable is relevant for predicting the future. Coin flips are Markovian. If we flip the coin 30 times and comes up heads 30 straight times, next flip still 50/50 chance.

Normal Distribution

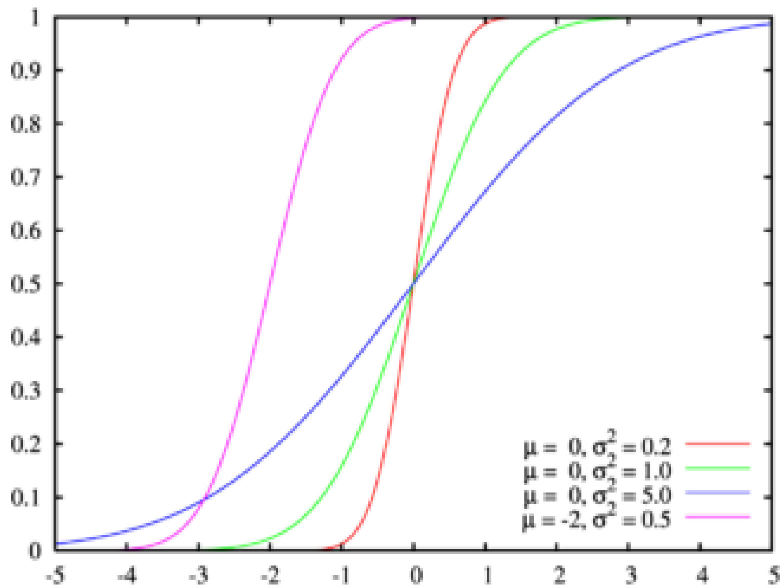


Let $\phi(\mu, \sigma)(x)$ denote the normal distribution. Then ϕ satisfies

$$\phi(\mu, \sigma)(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Probability of sum of

Normal Distribution



$$N(\mu, \sigma) = \int_{-\infty}^{x} \phi(\mu, \sigma)(s)ds \qquad N(\mu, \sigma)(+\infty) = 1$$

Consider a random number $x \in (0, 1)$ then most likely within the middle of the curve, if we undo it.

Sums of two normal distributions mean zero is a normal distribution with mean zero and variance that's the sum of the two variances. Proof: next time.

Continuous-Time Stochastic Processes

- Consider a Markov stochastic process. Suppose that the current value is 10 and the change in its value during 1 year is $\phi(0, 1)$.
- After two years? The change in two years is a some of two one year Markov stochastic process with mean zero and standard deviation 1.
- Therefore, the sum is a normal distribution with mean zero and variance $1 + 1 = 2$. Thus the standard deviation is $\sqrt{2}$.
- Consider now the change in the variable during 6 months. The variance of the change in the value of the variable during 1 year equals the variance of the change during the first 6 months plus the variance of the change during the second 6 months.
- We assume they are the same. Then variance of change during a 6-month period must be 0.5. Thus, the standard deviation of the change is $\sqrt{0.5}$. Thus 6-month distribution is $\phi(0, \sqrt{0.5})$.
- Consider a small time step $\Delta t = \frac{1}{N}$, during which each period is an independent normal distribution. Then sum of the variances are equal to 1, so each variance should be Δt.
- The standard deviation then is $\sqrt{\Delta t}$.

Uncertainty is proportional to square root of time.

Wiener Processes

We continue letting $\Delta t \to 0$ carefully! This is called the Wiener process or Brownian motion. It is a Markov stochastic process with mean zero and variance 1.0 per year. Therefore, it has

1. Change Δz during a small period of time Δt is

$$\Delta z = \epsilon \sqrt{\Delta t}$$

   where ϵ has a standardized normal distribution $\phi(0, 1)$. Therefore, Δz has a normal distribution with

   - mean of $\Delta z = 0$
   - standard deviation of $\Delta z = \sqrt{\Delta t}$
   - variance of $\Delta z = \Delta t$.

2. Values of Δz for any two different short intervals of time Δt are independent. Therefore, z follows a Markov process.

Wiener Process cont.

Measure the value of $z(T) - z(0)$ over a long period of time T.

View as a sum of N small changes over small time changes Δt, i.e.

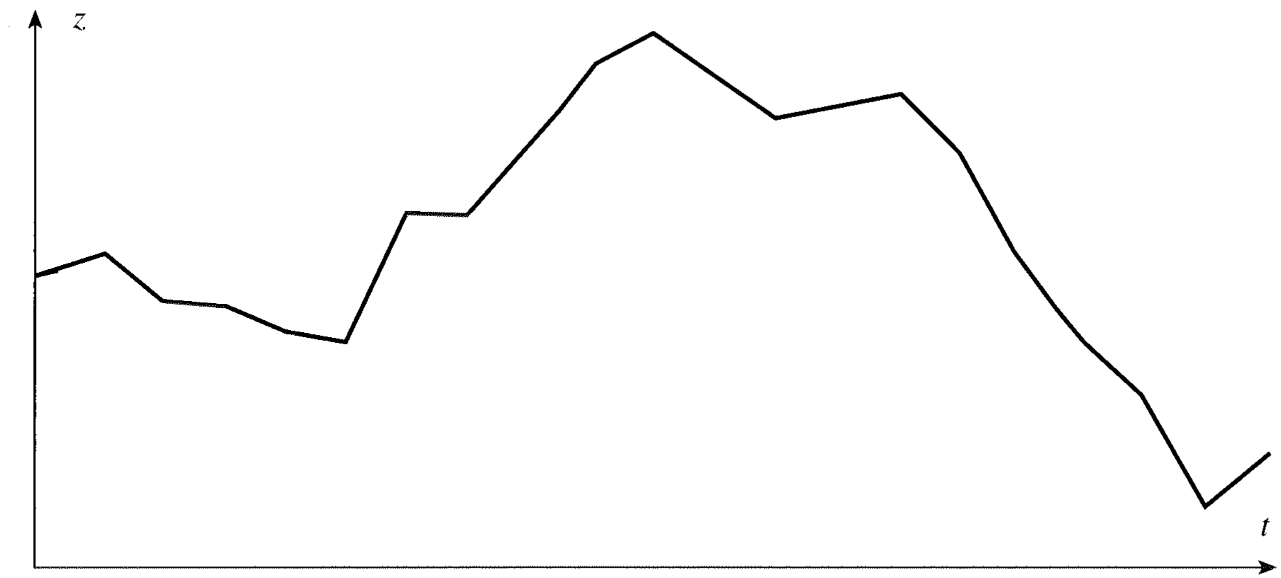
$$N = \frac{T}{\Delta t} \Longrightarrow z(T) - z(0) = \sum_{i=1}^{N} \epsilon_i \sqrt{\Delta t}$$

where ϵ_i for $i \in \{1, \ldots, N\}$ are distributed $\phi(0, 1)$. The ϵ_i's are independent of each other.

Then $z(T) - z(0)$ is normally distributed with

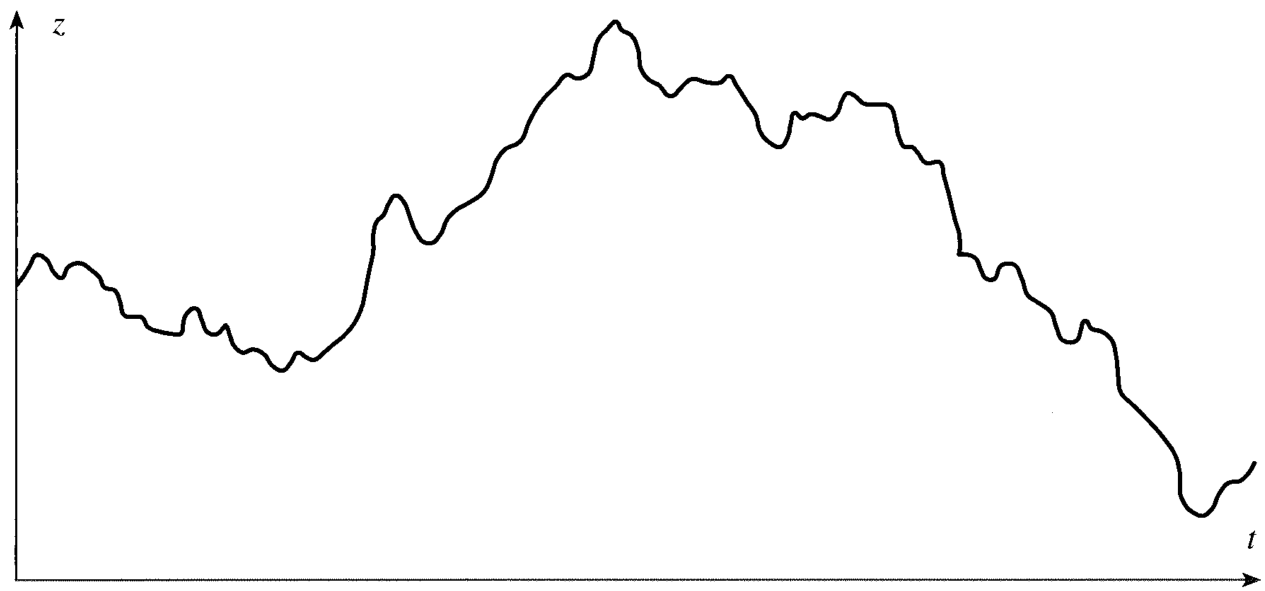
- mean of $[z(T) - z(0)] = 0$
- variance of $[z(T) - z(0)] = N\Delta t = T$
- standard deviation of $[z(T) - z(0)] = \sqrt{\Delta T}$.

Approximating Wiener Process



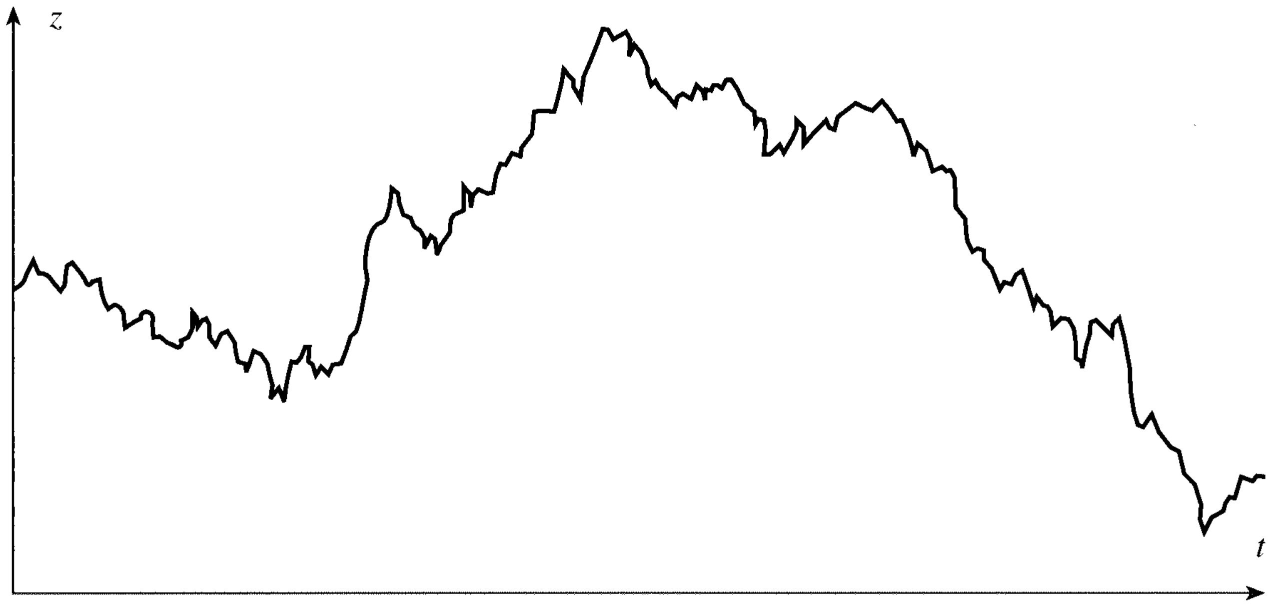
Relatively large value of Δt

Approximating Wiener Process



Smaller value of Δt

Approximating Wiener Process

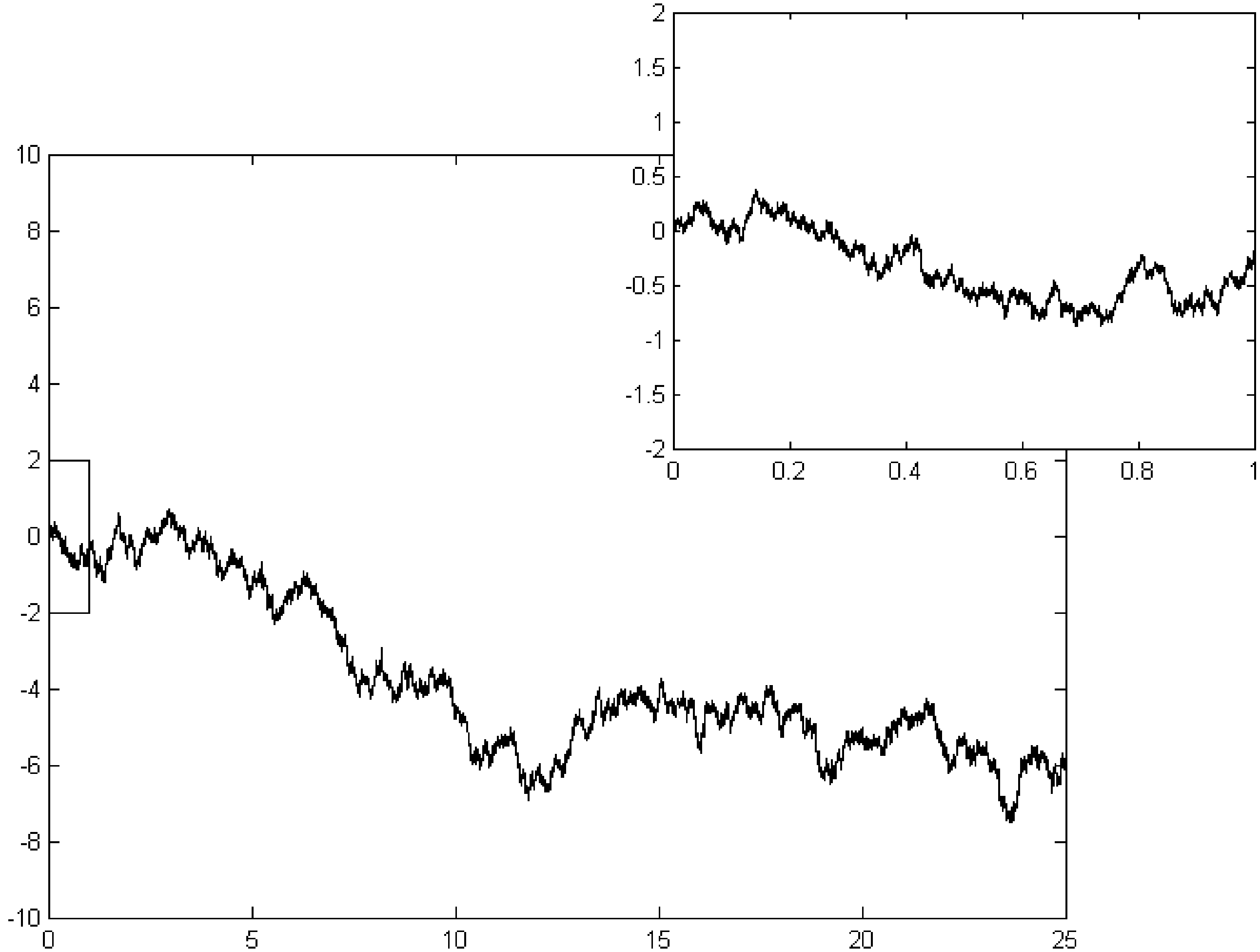


The true process obtained as $\Delta t \to 0$

1. Expected length of the path followed by z in any time interval is infinite!

2. Expected number of times z equals any particular value in any time interval is infinite!

Wiener Process



Self-similar structure. Lies in $C^{0,\frac{1}{2}}$.

Generalized Wiener Process

The mean change per unit time for a stochastic process is known as the drift rate.

The variance per unit time for a stochastic process is known as the variance rate.

A generalized Wiener process for a variable x can be defined in terms of dz as

$$dx = adt + bdz$$

where adt is the expected drift rate of a per unit time.

Holds since $dx = adt \implies \frac{dx}{dt} = a$. Therefore,

$$x = x_0 + at$$

After time T the variable x travels T units.

Generalized Wiener Process

The term $b\,dz$ regarded as noise added to the system, which is b times a Wiener process.

In a small time interval Δt, the change Δx in the variable of x is given by

$$\Delta x = a\Delta t + b\epsilon\sqrt{\Delta t}$$

where ε has a standard normal distribution.

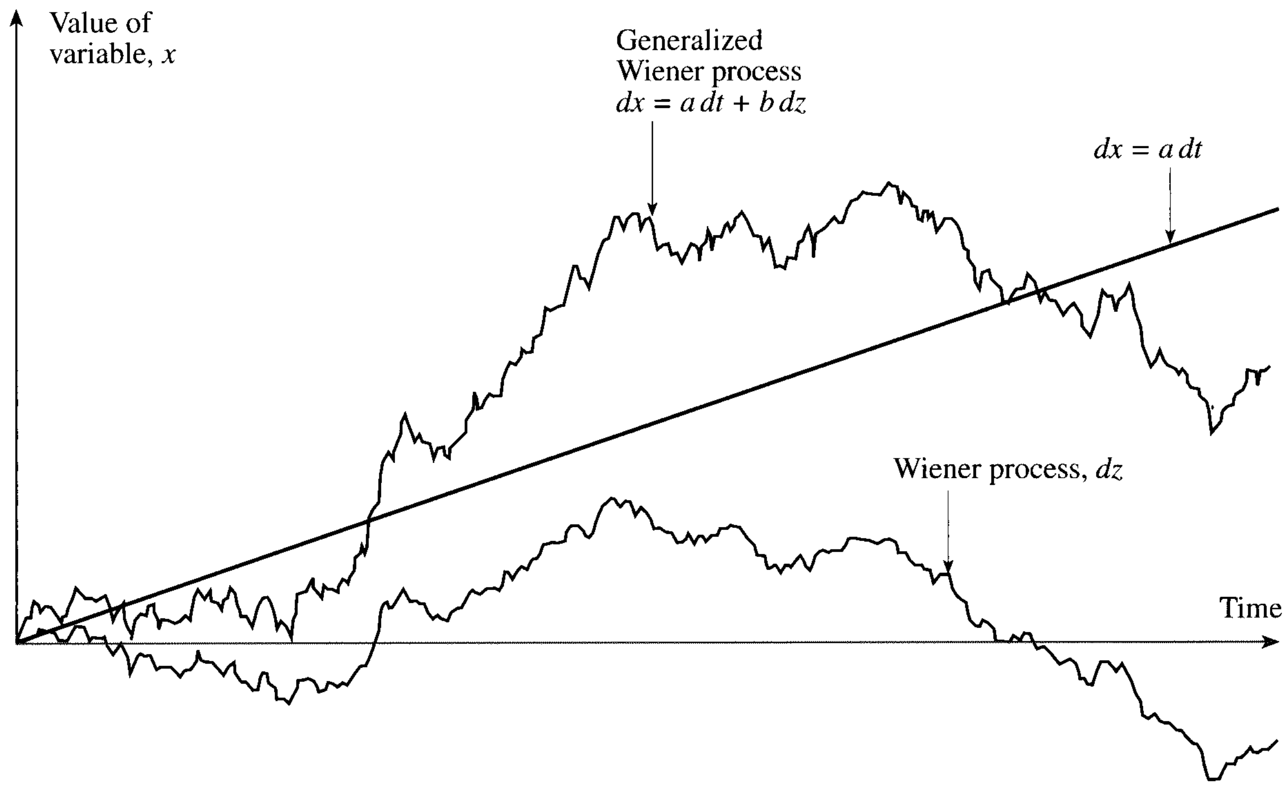
- mean of $\Delta x = a\Delta t$
- standard deviation of $\Delta x = b\sqrt{\Delta t}$
- variance of $\Delta x = b^2\Delta t$.

The same argument show that the change in the value of x in any time interval T is normally distributed with

- mean of $x = aT$
- standard deviation of $x = b\sqrt{T}$
- variance of $x = b^2 T$.

Generalized Wiener Process

Figure 12.2 Generalized Wiener process with $a = 0.3$ and $b = 1.5$.



Itô Process

A generalized Wiener process in which the parameters a and b are functions of the value of the underlying variable x and t. An Itô process can be written as

$$dx = a(x,t)dt + b(x,t)dz$$

Both the expected drift rate and the variance rate of an Itô process are liable to change over time. In a small time interval between t and $t + \Delta t$, the variable changes from x to $x + \Delta x$, where

$$\Delta x = a(x,t)\Delta t + b(x,t)\epsilon\sqrt{\Delta t}$$

Thus b^2 is the variance and a is the mean during the interval between t and $t + \Delta t$.

Process for a stock price

Discuss the process that models stock movements for a nondividend paying stock:

- Expected return $= \frac{\text{Expected drift}}{\text{Stock price}}$ is constant
- If S is the stock price a time t, then the expected drift rate in S should be assumed to be μS for some constant parameter μ.
- So in short period of time Δt the expected increase in S should be $\mu S \Delta t$.

If volatility of the stock is zero then model implies

$$\Delta S = \mu S \Delta t$$

In the limit $\Delta t \to 0$,

$$dS = \mu S dt$$

or

$$\frac{dS}{S} = \mu dt$$

Then

$$S_T = S_0 e^{\mu T}$$

Process for a stock price

Including volatility then expect: variability of the percentage return in a short period of time Δt is the same regardless of the stock price. This suggests that the standard deviation of the change in a short period of time Δt should be proportional to the stock price and leads to

$$dS = \mu S dt + \sigma S dz$$

or

$$\frac{dS}{S} = \mu dt + \sigma dz \tag{1}$$

We use (1) to price stocks. Here σ is the volatility and μ is the expected return rate.

Limiting case of the random walk we saw with binomial trees.

Discrete-Time Model

Discrete-time version of the model is

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t} \tag{2}$$

so the change in the stock value over a short period of time is

$$\Delta S = \mu S \Delta t + \sigma \epsilon S \sqrt{\Delta t}$$

- Variable ΔS is the change in the stock price S over a small interval of time Δt and ϵ has a standard normal distribution (normal distribution with $\sigma = 1$ and $\mu = 0$).
- μ is the expected rate of return by the stock in a short period of time Δt.
- σ is the volatility of the stock price.

Discrete-Time Model

Left-hand-side of (2) is the return provided by the stock in a short period of time.

- Term $\mu\Delta t$ is the expected value of the return
- Term $\sigma\epsilon\sqrt{\Delta t}$ is the stochastic component of the return. Variance is $\sigma^2\Delta t$ (consistent with the definition of volatility defined earlier).

Then $\Delta S/S$ is normally distributed with mean $\mu\Delta t$ and standard deviation $\sigma\sqrt{\Delta t}$, so

$$\frac{\Delta S}{S} \sim \phi(\mu\Delta t, \sigma\sqrt{\Delta t})$$

Parameters

The development of the pricing model depends on μ and σ so far.

For derivatives that depend on the stock, not important to have μ. However, very important to have σ. We saw this with binomial tree pricing.

The standard deviation of the proportional change in the stock price in a small interval of time Δt is $\sigma\sqrt{\Delta t}$. The standard deviation of the proportional change in the stock price over a relatively long period of time T is $\sigma\sqrt{T}$.

Itô's Lemma

An Itô process is one in which the drift and the volatility depend on both x and t. Suppose x is an Itô's process then

$$dx = a(x,t)dt + b(x,t)dz$$

where dz is a Wiener process and a, b are functions of x and t. Then x has a variance b^2.

Itô's Lemma states that a function G of x and t follows the following process:

$$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b dz$$

In particular G is an Itô process with drift rate

$$\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2$$

and variance

$$\left(\frac{\partial G}{\partial x} \right)^2 b^2$$

Itô's Lemma - formal argument

Assume G is a function of two variables x and t then we can formally take a power series expansion

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\left(\Delta x\right)^2 + \frac{\partial^2 G}{\partial x \partial t}\left(\Delta x \Delta t\right) + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}\left(\Delta t\right)^2 + O(\Delta^3)$$

An Itô process satisfies

$$dx = a(x,t)dt + b(x,t)dz$$

where dz is a Wiener process. Then approximately (at the discrete level) we have

$$\Delta x = a(x,t)\Delta t + b(x,t)\epsilon\sqrt{\Delta t}.$$

Then returning to ΔG we have

$$\begin{aligned}
\Delta G &= \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\left(\Delta x\right)^2 + \frac{\partial^2 G}{\partial x \partial t}\left(\Delta x \Delta t\right) + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}\left(\Delta t\right)^2 + O(\Delta^3) \\
&= \frac{\partial G}{\partial x}\left[a\Delta t + b\epsilon\sqrt{\Delta t}\right] + \frac{\partial G}{\partial t}\Delta t \\
&\quad + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\left[a\Delta t + b\epsilon\sqrt{\Delta t}\right]^2 + \frac{\partial^2 G}{\partial x \partial t}\Delta t\left[a\Delta t + b\epsilon\sqrt{\Delta t}\right] + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}\left(\Delta t\right)^2 + O(\Delta^3)
\end{aligned}$$

Expand out:

Itô's Lemma - formal argument

$$
\begin{aligned}
\Delta G &= \frac{\partial G}{\partial x}\left[a\Delta t + b\epsilon\sqrt{\Delta t}\right] + \frac{\partial G}{\partial t}\Delta t \\
&\quad + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\left[a\Delta t + b\epsilon\sqrt{\Delta t}\right]^2 + \frac{\partial^2 G}{\partial x \partial t}\Delta t\left[a\Delta t + b\epsilon\sqrt{\Delta t}\right] + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}(\Delta t)^2 + O(\Delta^3) \\
&= \sqrt{\Delta t}\, b\epsilon \frac{\partial G}{\partial x} \\
&\quad + \Delta t\left[\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + b^2\epsilon^2\frac{1}{2}\frac{\partial^2 G}{\partial x^2}\right] \\
&\quad + (\Delta t)^{\frac{3}{2}}\left[2ab\epsilon\frac{1}{2}\frac{\partial^2 G}{\partial x^2}\right] \\
&\quad + (\Delta t)^2\left[a^2\frac{1}{2}\frac{\partial^2 G}{\partial x^2} + a\frac{\partial^2 G}{\partial x \partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}\right] + O((\Delta t)^3) \\
&= \left[\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + b^2\epsilon^2\frac{1}{2}\frac{\partial^2 G}{\partial x^2}\right]\Delta t + \frac{\partial G}{\partial x}b\epsilon\sqrt{\Delta t} + O((\Delta t)^{\frac{3}{2}})
\end{aligned}
$$

Itô's Lemma - formal argument

Therefore,

$$\Delta G = \left[\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + b^2 \epsilon^2 \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \right] \Delta t + \frac{\partial G}{\partial x} b \epsilon \sqrt{\Delta t} + O((\Delta t)^{\frac{3}{2}})$$

Since is a normal distribution, then the variance ϵ^2 is 1. Thus $1 = E(\epsilon^2) - (E(\epsilon))^2 = E(\epsilon^2)$.

Therefore, the expected value of $\epsilon^2 \Delta t$ is Δt (small fluctuations cancel out) and hence nonstochastic! Take the limit as $\Delta t \longrightarrow 0$ then get formally

$$dG = \left[\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + b^2 \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \right] dt + \frac{\partial G}{\partial x} b dz$$

Itô's Lemma: Modeling stock movements

We argued that a reasonable model of stock movements should be

$$dS = \mu S dt + \sigma S dz$$

with μ and σ constants.

From Itô's Lemma we can consider a process G that depends on t and S. Then

$$dG = \left(\frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S dz$$

so both S and G are affected by dz - the noise in the system.

Lognormal Property

Recall that our model requires

$$dS = \mu S dt + \sigma S dz$$

with μ and σ constants.

Define $G = \ln S$ then

$$\frac{\partial G}{\partial S} = \frac{1}{S} \qquad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2} \qquad \frac{\partial G}{\partial t} = 0$$

by Itô's Lemma we have

$$\begin{aligned} dG &= \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S}\sigma S dz \\ &= \left[\frac{1}{S}\mu S + 0 + \frac{1}{2}\frac{-1}{S^2}\sigma^2 S^2 \right] dt + \frac{1}{S}\sigma S dz \\ &= \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dz \end{aligned}$$

Therefore, G follows a generalized Wiener process with

- Drift $= \mu - \frac{\sigma^2}{2}$
- Variance $= \sigma^2$

Lognormal Property

Therefore, the change in $\ln S$ between 0 and a future time T is normally distributed with mean $(\mu - \frac{\sigma^2}{2})T$ and variance $\sigma^2 T$. Hence:

$$\ln S_T - \ln S_0 \approx \phi \left[\left(\mu - \frac{\sigma^2}{2} \right) T, \sigma \sqrt{T} \right]$$

or

$$\ln S_T \approx \phi \left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2} \right) T, \sigma \sqrt{T} \right]$$

This implies that $\log$ of the stock price is normally distributed:

A variable has a lognormal distribution if the natural $\log$ of the variable is normally distributed.

The standard deviation of the logarithm of the stock price is $\sigma\sqrt{T}$.

Derivation of Black-Scholes-Merton Differential Equation

We are now in position to derive the Black-Scholes or Black-Scholes-Merton differential equation. We build the model via a riskless portfolio, as we did for binomial trees. As for binomial trees, we carry some stock along with shorting the option. The amount of stock changes instantaneously. Special assumptions are required:

1. The stock price follows the process defined earlier for μ and σ:

$$\frac{dS}{S} = \mu dt + \sigma dz$$

2. Short selling of securities with full use of proceeds is permitted
3. There are no transactions costs or taxes. All securities are perfectly divisible
4. There are no dividends during the life of the derivative
5. There are no riskless arbitrage opportunities
6. Security trading is continuous
7. The risk-free rate of interest, r, is constant and the same for all maturities

Derivation of Black-Scholes-Merton Differential Equation

Recall our process for a continuous stock movement modeled on an Itô process with expected gain μ and volatility σ.

$$dS = \mu S dt + \sigma S dz$$

Let f be the price of a call option that depends on S. The variable f depends, then S and t. Then

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S}\sigma S dz$$

Recall the discrete-time analogues as

$$\Delta S = \mu S \Delta t + \sigma S \sqrt{\Delta t}$$

and so the discrete version of Itô's Lemma is:

$$\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right) \Delta t + \frac{\partial f}{\partial S}\sigma S \sqrt{\Delta t}$$

Derivation of Black-Scholes-Merton Differential Equation

We now build a portfolio that will eliminate the stochasticity of the process. The appropriate portfolio (as we will see) is

- -1 option
- $\frac{\partial f}{\partial S}$ shares ($\Delta = \frac{f_u - f_d}{S_0 u - S_0 d}$ is the Delta hedge found in binomial trees)

which changes continuously over time. Let Π be the value of the portfolio then

$$\Pi = -f + \frac{\partial f}{\partial S}$$

and $\Delta\Pi$ be the value of the portfolio in the time interval Δt then

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S$$

Derivation of Black-Scholes-Merton Differential Equation

Then

$$
\begin{aligned}
\Delta\Pi &= -\Delta f + \frac{\partial f}{\partial S}\Delta S \\
&= -\left[\left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t + \frac{\partial f}{\partial S}\sigma S \Delta z\right] \\
&\quad + \frac{\partial f}{\partial S}\left[\mu S \Delta t + \sigma S \Delta z\right] \\
&= -\left[\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right]\Delta t
\end{aligned}
$$

Note that $\Delta\Pi$ does not depend on dz, therefore there is no risk during time Δt! Thus the portfolio must instantaneously earn the same rate of return as other short-term risk-free securities. If it earned more than this return, arbitrageurs could make a riskless profit by borrowing money to buy the portfolio; if it earned less they could make a riskless profit by shorting the portfolio and buying risk-free securities. Thus:

$$\Delta\Pi = r\Pi\Delta t$$

Derivation of Black-Scholes-Merton Differential Equation

where r is the risk-free rate. Then

$$-\left[\frac{\partial f}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}\right]\Delta t=\left[-f+\frac{\partial f}{\partial S}S\right]\Delta t$$

so

$$\frac{\partial f}{pt}+rS\frac{\partial f}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}=rf \tag{3}$$

Equation (3) is the Black-Scholes partial differential equation. Any solution corresponds to the price of a derivative overlying a particular stock.

In order to specify further what the derivative is, we use a boundary condition to constrain it.

Boundary conditions for European call options:

$$f = \max\{S - K, 0\}$$

when $t = T$. Boundary conditions for European put options:

$$f = \max\{K - S, 0\}$$

when $t = T$. The portfolio created is riskless only for infinitesimally short periods.

Tradeable Derivatives

Any function $f(S,t)$ that satisfies (3) is the theoretical price of a derivative that could be traded.

If a derivative with that price existed, then there would be no arbitrage opportunities

Conversely if a function $f(S,t)$ does not satisfy (3) then it cannot be the price of a derivative without creating arbitrage opportunities.

Example: Let $f(S,t) = e^S$ then $f_t = 0$, $f_S = e^S$, and $f_{SS} = e^S$ then $f_t + rSf_S + \frac{1}{2}\sigma^2 f_{SS} = rSe^S + \frac{1}{2}\sigma^2 e^S \neq re^S$. Thus this cannot be a derivative of a stock price.

On the other hand

$$f = \frac{e^{\left(\sigma^2 - 2r\right)(T-t)}}{S}$$

then

$$f_t = -\left(\sigma^2 - 2r\right) f$$

$$f_S = -\frac{e^{\left(\sigma^2-2r\right)(T-t)}}{S^2} \Longrightarrow rSf_S = -rf$$

$$f_{SS} = 2\frac{e^{\left(\sigma^2-2r\right)(T-t)}}{S^3} \Longrightarrow \frac{\sigma^2}{2}S^2 f_{SS} = \sigma^2 f$$

Tradeable Derivatives

so

$$f_t + rSf_S + \frac{\sigma^2}{2}S^2 f_{SS} = -\sigma^2 f + 2rf - rf + \sigma^2 f = rf$$

which is the Black-Scholes differential equation. This the price of a derivative that pays off $\frac{1}{S_T}$ at time T.

Black-Scholes Pricing Formulas

The Black-Scholes formulas for the price at time 0 of a European call option on a non-dividend-paying stock and for a European put option on a non-dividend paying stock are

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

and

$$p = KN(-d_2) - S_0 e^{-rT} N(-d_1)$$

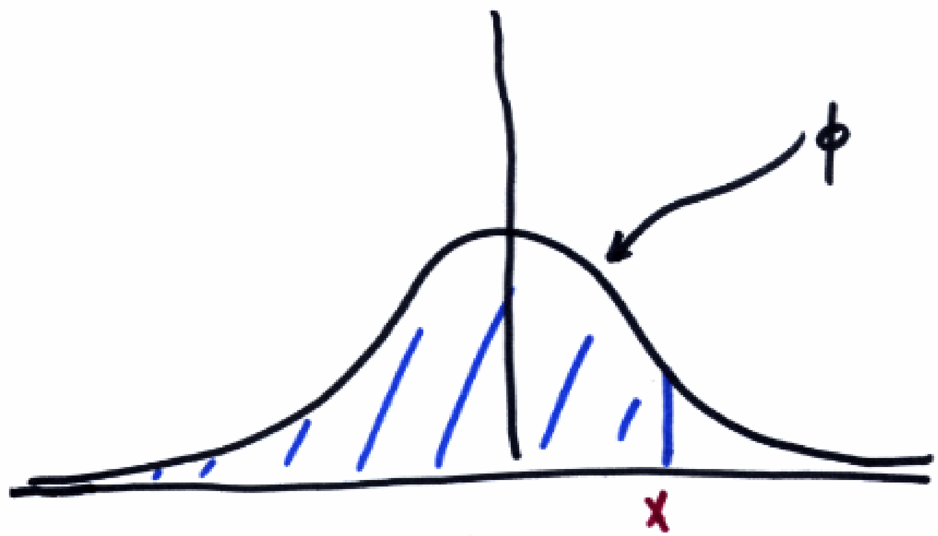
where

$$d_1 = \frac{\ln \frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}}$$

$$d_2 = \frac{\ln \frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T}$$

and $N(x)$ is the cumulative probability distribution function.

Black-Scholes Pricing Formulas



- $N(x) = \int_{-\infty}^{x} \phi(t)\,dt$
- $1 - N(x) = N(-x)$

The variables c and p are the European call and put prices, S_0 is the current stock price at time 0, K is the strike price, r is the continuously compounded risk-free rate, σ is the stock price volatility, and T is the time to maturity of the option. Why?