BINOMIAL ANDREWS-GORDON-BRESSOUD IDENTITIES

DENNIS STANTON

Abstract. Binomial versions of the Andrews-Gordon-Bressoud identities are given.

1. Introduction

The Rogers-Ramanujan identities

$$\sum_{s=0}^{\infty} \frac{q^{s^2}}{(q;q)_s} = \frac{1}{(q,q^4;q^5)_\infty}, \qquad \sum_{s=0}^{\infty} \frac{q^{s^2+s}}{(q;q)_s} = \frac{1}{(q^2,q^3;q^5)_\infty}$$

where

$$(A;q)_n = \prod_{i=0}^{n-1}(1-Aq^i), \qquad (A,B;q)_n = (A;q)_n(B;q)_n,$$

were generalized to odd moduli at least five by the Andrews [1]. These identities are called the Andrews-Gordon identities

$$\begin{aligned} \sum_{s_1\ge s_2\ge\cdots\ge s_k\ge 0} &\frac{q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}} \\ &= \frac{(q^{k+1-r},q^{k+2+r},q^{2k+3};q^{2k+3})_\infty}{(q;q)_\infty}, \qquad 0\le r\le k. \end{aligned} \tag{1}$$

(If the base q is understood, we sometimes abbreviate $(A;q)_n$ as $(A)_n$. We also assume that $0<q<1$.)

Bressoud [7],[9] gave a version of these identities for even moduli

$$\begin{aligned} \sum_{s_1\ge s_2\ge\cdots\ge s_k\ge 0} &\frac{q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} \\ &= \frac{(q^{k+1-r},q^{k+1+r},q^{2k+2};q^{2k+2})_\infty}{(q;q)_\infty}, \qquad 0\le r\le k. \end{aligned} \tag{2}$$

Bressoud's beautiful and efficient proof [8] established both sets of identities when $r=0$. Moreover he had other closely related identities, for example, [9, (3.3), p. 15]

$$\begin{aligned} \sum_{s_1\ge s_2\ge\cdots\ge s_k\ge 0} &\frac{q^{s_1^2+\cdots+s_k^2-(s_1+\cdots+s_j)}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}} \\ &= \sum_{s=0}^{j} \frac{(q^{k+1+j-2s},q^{k+2-j+2s},q^{2k+3};q^{2k+3})_\infty}{(q;q)_\infty}, \qquad 0\le j\le k. \end{aligned} \tag{3}$$

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The purpose of this paper is to examine Bressoud's proof, and develop new variations and generalizations of these Andrews-Gordon-Bressoud identities. The new results are given §3, §4, and §5.

2. The motivating question

In [4], which was presented by George Andrews at the May 2015 UCF meeting in honor of Mourad Ismail, Andrews reconsidered Bressoud's elementary proof [8]. He asked a specific question (see Question 2.6) about Bressoud's proof that we answer in this section.

We shall need a few of the relevant definitions and facts in a recapitulation Bressoud's simple proof [8]. We shall also use these facts in later sections. Bressoud's key idea was to use the following Laurent polynomials, which have arbitrary quadratic exponents.

Definition 2.1. Let

$$H_{2n}(z,a|q)=\sum_{s=-n}^{n}\begin{bmatrix}2n\\ n-s\end{bmatrix}_q q^{as^2}z^s.$$

Bressoud's main lemma [8, Lemma 2], which allowed the quadratic exponent to change, is next.

Lemma 2.2.

$$\frac{H_{2n}(z,a|q)}{(q;q)_{2n}}=\sum_{s=0}^{n}\frac{q^{s^2}}{(q;q)_{n-s}}\frac{H_{2s}(z,a-1|q)}{(q;q)_{2s}}.$$

This lemma may be iterated.

Proposition 2.3.

$$\frac{H_{2n}(z,a+k+1|q)}{(q;q)_{2n}}=\sum_{n\geq s_1\geq s_2\geq\cdots\geq s_{k+1}\geq 0}\frac{q^{s_1^2+\cdots s_{k+1}^2}H_{2s_{k+1}}(z,a|q)}{(q)_{n-s_1}(q)_{s_1-s_2}\cdots(q)_{s_k-s_{k+1}}(q)_{2s_{k+1}}}$$

The value of $a=1/2$ is nice because the polynomials $H_{2n}(z,1/2|q)$ factor by the q-binomial theorem.

$$H_{2n}(-zq^{1/2},1/2|q)=(qz,1/z;q)_n. \tag{4}$$

So we have

Theorem 2.4. *[Bressoud* [8, (14)]*]*

$$\begin{aligned}&\frac{H_{2n}(-zq^{1/2},k+3/2|\ q)}{(q;q)_{2n}}\\ &=\sum_{n\geq s_1\geq s_2\geq\cdots\geq s_{k+1}\geq 0}\frac{q^{s_1^2+\cdots s_{k+1}^2}(qz,1/z;q)_{s_{k+1}}}{(q)_{n-s_1}(q)_{s_1-s_2}\cdots(q)_{s_k-s_{k+1}}(q)_{2s_{k+1}}}\end{aligned} \tag{5}$$

We now take the $n\to\infty$ limit of Theorem 2.4. The right side has a clear limit. For the left side we show that Definition 2.1 has a limit as an infinite product when $n\to\infty$.

If $0<q<1$ and $-n\leq s\leq n$, we have

$$\begin{bmatrix}2n\\ n-s\end{bmatrix}_q\leq\frac{1}{(q;q)_\infty},$$

because the q-binomial coefficient $\begin{bmatrix} 2n \\ n-s \end{bmatrix}_q$ is the generating function for partitions inside an $(n-s)\times(n+s)$ rectangle, and right side is the generating function for all partitions. For any fixed integer s this also shows that

$$\lim_{n\to\infty} \begin{bmatrix} 2n \\ n-s \end{bmatrix}_q = \frac{1}{(q;q)_\infty}.$$

So the $n\to\infty$ limit in (2.1) converges uniformly

$$\begin{aligned} \lim_{n\to\infty} H_{2n}(-z,a|q) =& \frac{1}{(q;q)_\infty} \lim_{n\to\infty} \sum_{s=-n}^{n} q^{as^2}(-z)^s \\ =& \frac{(q^{2a}, zq^a, q^a/z; q^{2a})_\infty}{(q)_\infty}, \end{aligned} \tag{6}$$

using the Jacobi Triple Product identity.

We obtain

Corollary 2.5. *For a non-negative integer k,*

$$\frac{(q^{2k+3}, zq^{k+2}, q^{k+1}/z; q^{2k+3})_\infty}{(q;q)_\infty} = \sum_{s_1\geq s_2\geq\cdots\geq s_{k+1}\geq 0} \frac{q^{s_1^2+\cdots s_{k+1}^2}(qz,1/z;q)_{s_{k+1}}}{(q)_{s_1-s_2}\cdots(q)_{s_k-s_{k+1}}(q)_{2s_{k+1}}}.$$

Note that Corollary 2.5 immediately gives the Andrews-Gordon identities (1) for $r=0$. If $z=1$, this choice of z forces $s_{k+1}=0$. The choice of $z=q^r$ does give the right side of the Andrews-Gordon identities (1), but not the left side. There is an extra sum over s_{k+1}, and the power of q does not match.

Andrews' Question 2.6. Is there a simple way to understand why the choice of $z=q^r$ eliminates the s_{k+1} sum and replaces it with a power of q?

We now answer Andrews' question, and we will use this answer in subsequent sections. The ingredient we need appeared in a paper of Garrett, Ismail and Stanton, [10].

Proposition 2.7. *For any c,*

$$H_{2n}(-q^c, c|q) = q^n H_{2n}(-q^{c-1}, c|q).$$

To answer Andrews' question, start with $H_{2n}(-q^{r+1/2}, k+3/2|q)$, apply Lemma 2.2 $k-r+1$ times to obtain $H_{2s_{k-r+1}}(-q^{r+1/2}, r+1/2|q)$. Next apply Proposition 2.7 once to obtain $q^{s_{k-r+1}}H_{2s_{k-r+1}}(-q^{r-1/2}, r+1/2|q)$. This is the linear exponent in q we need. The remaining exponents arise from again applying Lemma 2.2 followed by Proposition 2.7. The final sum on s_{k+1} is now eliminated, because the final term becomes $H_{2s_{k+1}}(-q^{1/2}, 1/2|q) = (q,1;q)_{s_{k+1}}$, which forces $s_{k+1}=0$.

3. New Andrews-Gordon identities

In this section we prove two new Andrews-Gordon identities for odd moduli. The first has binomial factors.

Theorem 3.1. *For $0 \le j, r \le k$, and $j + r \le k$,*

$$\sum_{s_1\ge s_2\ge\cdots\ge s_k\ge 0} \frac{q^{-s_1-\cdots-s_j}(1+q^{s_1+s_2})(1+q^{s_2+s_3})\cdots(1+q^{s_{j-1}+s_j})}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}}$$
$$\times q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k}$$
$$=\sum_{s=0}^{j}\binom{j}{s}\frac{(q^{k+1-r+j-2s},q^{k+2+r-j+2s},q^{2k+3};q^{2k+3})_\infty}{(q;q)_\infty}.$$

Moreover, the j factors of q^{-s_1} and $q^{-s_i}(1+q^{s_{i-1}+s_i})$, $2 \le i \le j$ may be replaced by any j-element subset of $\{q^{-s_1}\} \cup \{q^{-s_i}(1+q^{s_{i-1}+s_i}) : 2 \le i \le k-r\}$.

For example, the binomial factors could occur as the last j of the first $k-r$ summation indices instead of the first j indices, namely

$$\prod_{t=0}^{j-1} q^{-s_{k-r-t}}(1+q^{s_{k-r-1-t}+s_{k-r-t}}).$$

A corollary of Theorem 3.1 is an identity which contains the Andrews-Gordon identities (1) when $j = 0$ and Bressoud's identities (3) when $r = 0$.

Theorem 3.2. *For $0 \le j, r \le k$, and $j + r \le k$,*

$$\sum_{s_1\ge s_2\ge\cdots\ge s_k\ge 0} \frac{q^{s_1^2+\cdots+s_k^2-(s_1+\cdots+s_j)+(s_{k-r+1}+\cdots+s_k)}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}}$$
$$=\sum_{s=0}^{j}\frac{(q^{k+1-r+j-2s},q^{k+2+r-j+2s},q^{2k+3};q^{2k+3})_\infty}{(q;q)_\infty}.$$

We need a new fact about the Laurent polynomials $H_{2n}(z, a|q)$.

Proposition 3.3. *For a non-negative integer n,*

$$\frac{H_{2n}(zq,a+1|q)+H_{2n}(q/z,a+1|q)}{(q;q)_{2n}}=\sum_{s=0}^{n}\frac{q^{s^2-s}(1+q^{n+s})}{(q;q)_{n-s}}\frac{H_{2s}(z,a|q)}{(q;q)_{2s}},$$

Proof. Note that the left side of Proposition 3.3 is invariant under $z \to 1/z$ so it does have an expansion in terms of $H_{2s}(z, a|q)$. If the coefficient of $z^k q^{ak^2}$ is computed for each side, we must show

$$\begin{bmatrix}2n\\ n-k\end{bmatrix}_q q^k q^{k^2}+\begin{bmatrix}2n\\ n-k\end{bmatrix}_q q^{-k}q^{k^2}=\sum_{s=k}^{n}\frac{q^{s^2-s}(1+q^{n+s})}{(q)_{2s}}\frac{(q)_{2n}}{(q)_{n-s}}\begin{bmatrix}2s\\ s-k\end{bmatrix}_q.$$

The s-sum for the term q^{n+s} is summable as a product by a limiting case of the q-Vandermonde sum, see [11, p. 354, (II.6)]. The s-sum for the term 1 is nearly summable, it is a sum of two products. Putting these terms together yields the two terms on the left side. The details are not given. □

We need some functions which generalize the $H_{2n}(z, a|q)$.

Definition 3.4. For a non-negative integer j, let

$$F_n^{(0)}(z,a)=H_{2n}(z,a|q),\qquad F_n^{(j+1)}(z,a)=F_n^{(j)}(zq,a)+F_n^{(j)}(q/z,a),\quad j\ge 0.$$

Proposition 3.3 can be rewritten using these new functions. The proof is by induction on j.

Proposition 3.5. *For non-negative integers n and j,*

$$\frac{F_n^{(j+1)}(z,a+1)}{(q;q)_{2n}} = \sum_{s=0}^{n} \frac{q^{s^2-s}(1+q^{n+s})}{(q;q)_{n-s}} \frac{F_s^{(j)}(z,a)}{(q;q)_{2s}},$$

Iterating Proposition 3.5 is the next Proposition.

Proposition 3.6. *For a non-negative integer n and a positive integer j,*

$$\frac{F_n^{(j)}(z,a)}{(q)_{2n}} = \sum_{n\geq s_1\geq s_2\geq\cdots\geq s_j\geq 0} \frac{q^{s_1^2-s_1}(1+q^{n+s_1})}{(q)_{n-s_1}} \prod_{t=2}^{j} \frac{q^{s_t^2-s_t}(1+q^{s_{t-1}+s_t})}{(q)_{s_{t-1}-s_t}} \frac{F_{2s_j}^{(0)}(z,a-j)}{(q)_{2s_j}}.$$

Finally the functions F also satisfy Lemma 2.2 because the H functions do.

Proposition 3.7. *For a non-negative integer n and a positive integer j,*

$$\frac{F_n^{(j)}(z,a)}{(q;q)_{2n}} = \sum_{s=0}^{n} \frac{q^{s^2}}{(q;q)_{n-s}} \frac{F_s^{(j)}(z,a-1|q)}{(q;q)_{2s}}.$$

Any of these functions may be written as a linear combination of $F_n^{(0)}(z,a) = H_{2n}(z,a|q)$.

Proposition 3.8. *For any non-negative integer j,*

$$F_n^{(j+1)}(z,a) = \sum_{s=0}^{j} \binom{j}{s} \left(F_n^{(0)}(zq^{j+1-2s},a) + F_n^{(0)}(q^{j+1-2s}/z,a)\right).$$

Proof. By induction on j we have

$$\begin{aligned} F_n^{(j+1)}(z,a) =& \sum_{s=0}^{j-1} \binom{j-1}{s} \left(F_n^{(0)}(zq^{j+1-2s},a) + F_n^{(0)}(q^{j-1-2s}/z,a)\right) \\ &+ \sum_{s=0}^{j-1} \binom{j-1}{s} \left(F_n^{(0)}(q^{j+1-2s}/z,a) + F_n^{(0)}(zq^{j-1-2s},a)\right) \\ =& \sum_{s=0}^{j} F_n^{(0)}(zq^{j+1-2s},a) \left(\binom{j-1}{s} + \binom{j-1}{s-1}\right) \\ &+ \sum_{s=0}^{j} F_n^{(0)}(q^{j+1-2s}/z,a) \left(\binom{j-1}{s-1} + \binom{j-1}{s}\right) \\ =& \sum_{s=0}^{j} \binom{j}{s} \left(F_n^{(0)}(zq^{j+1-2s},a) + F_n^{(0)}(q^{j+1-2s}/z,a)\right). \end{aligned}$$

□

We have two expressions for $F_n^{(j)}(z,a)$: Propositions 3.8 and 3.6. The proof of Theorem 3.1 uses these two expressions after taking a limit as $n \to \infty$. We record the appropriate $n \to \infty$ limit of Proposition 3.8.

Proposition 3.9. *If j is a non-negative integer,*

$$\lim_{n\to\infty} F_n^{(j+1)}(-z,a)=\frac{1}{(q)_\infty}\sum_{s=0}^{j+1}\binom{j+1}{s}(q^{2a},zq^{a+j+1-2s},q^{a-j-1+2s}/z;q^{2a})_\infty.$$

Proof. Applying (6) and Proposition 3.8 we have

$$\begin{aligned}\lim_{n\to\infty} F_n^{(j+1)}(-z,a)=&\frac{1}{(q)_\infty}\sum_{s=0}^{j}\binom{j}{s}\big((q^{2a},zq^{a+j+1-2s},q^{a-j-1+2s}/z;q^{2a})_\infty\\&+(q^{2a},q^{a+j+1-2s}/z,zq^{a-j-1+2s};q^{2a})_\infty\big)\\=&\frac{1}{(q)_\infty}\sum_{s=0}^{j+1}\left(\binom{j}{s}+\binom{j}{j+1-s}\right)(q^{2a},zq^{a+j+1-2s},q^{a-j-1+2s}/z;q^{2a})_\infty\\=&\frac{1}{(q)_\infty}\sum_{s=0}^{j+1}\binom{j+1}{s}(q^{2a},zq^{a+j+1-2s},q^{a-j-1+2s}/z;q^{2a})_\infty.\end{aligned}$$

□

Proof of Theorem 3.1. We see from Proposition 3.9 that the right side of Theorem 3.1 is

$$\lim_{n\to\infty} F_n^{(j)}(-z,k+3/2),\qquad z=q^{-1/2-r}.$$

or at $z=q^{r+1/2}$ since all functions are symmetric under $z\to 1/z$. We apply Proposition 3.7 to obtain j sums and a factor of

$$\frac{F_{2s_j}^{(0)}(-z,k+3/2-j)}{(q)_{2s_j}}=\frac{H_{2s_j}(-z,k+3/2-j|q)}{(q)_{2s_j}}.$$

Now we are in the realm of the Andrews-Gordon proof in section 2. We finish the proof as before, by applying Lemma 2.2 $k-r$ times, and then inserting the linear factors r times.

Since the functions $F_n^{(j)}(z,a)$ also satisfy Proposition 3.7, we could apply Proposition 3.7 anytime before we use Proposition 3.5 in the first $k-r$ iterates. This gives the arbitrary choice of the binomials. □

Next we derive Theorem 3.2 from Theorem 3.1. The idea is take an appropriate linear combination of Theorem 3.1 to replace the binomial factors in Theorem 3.1 by a single term $q^{-s_1-s_2-\cdots-s_j}$. For example if $j=3$,

$$q^{-s_1-s_2-s_3}(1+q^{s_1+s_2})(1+q^{s_2+s_3})-q^{-s_3}(1+q^{s_2+s_3})-q^{-s_1}=q^{-s_1-s_2-s_3}\tag{7}$$

yields, for the right side of Theorem 3.1,

$$\begin{aligned}&\sum_{s=0}^{3}\binom{3}{s}\frac{(q^{k+1-r+3-2s},q^{k+2+r-3+2s},q^{2k+3};q^{2k+3})_\infty}{(q;q)_\infty}\\&-2\sum_{s=0}^{1}\binom{1}{s}\frac{(q^{k+1-r+1-2s},q^{k+2+r-1+2s},q^{2k+3};q^{2k+3})_\infty}{(q;q)_\infty}\\=&\sum_{s=0}^{3}\frac{(q^{k+1-r+1-2s},q^{k+2+r-1+2s},q^{2k+3};q^{2k+3})_\infty}{(q;q)_\infty}\end{aligned}$$

as predicted by Theorem 3.2.

The version of (7) we need for general j uses edges in a graph which is a path from 1 to j: $1-2-3-\cdots-j$. A pair of edges in this graph do not overlap if they do not share a vertex. For a set E of non-overlapping edges let

$$wt(E) = \prod_{i\notin E, i\geq 2} q^{-s_i}(1+q^{s_{i-1}+s_i}) \times \begin{cases} q^{-s_1} & \text{if } 1\notin E \\ 1 & \text{if } 1\in E. \end{cases}$$

Here are the three possible sets of non-overlapping edges E for $j=3$,

$$\begin{aligned} E =& \varnothing, \quad wt(E) = q^{-s_1-s_2-s_3}(1+q^{s_1+s_2})(1+q^{s_2+s_3}), \\ E =& 1-2, \quad wt(E) = q^{-s_3}(1+q^{s_2+s_3}), \\ E =& 2-3, \quad wt(E) = q^{-s_1}. \end{aligned}$$

These are the three terms in (7).

Lemma 3.10. *We have*

$$\sum_E (-1)^{|E|} wt(E) = q^{-s_1-s_2-\cdots-s_j},$$

where the sum is over all non-overlapping edge sets E of $1-2-3-\cdots-j$.

Proof. Again we do an induction on j. Suppose E is a set of non-overlapping edges for $1-2-3-\cdots-(j+1)$. If the last edge $j-(j+1)$ is in E, the remaining edges are non-overlapping for $j-1$, so by induction

$$\sum_{E, j-(j+1)\in E} (-1)^{|E|} wt(E) = -q^{-s_1-s_2-\cdots-s_{j-1}}.$$

If the last edge $j-(j+1)$ is not in E, the remaining edges are non-overlapping for j, so by induction

$$\sum_{E, j-(j+1)\notin E} (-1)^{|E|} wt(E) = q^{-s_{j+1}}(1+q^{s_j+s_{j+1}})q^{-s_1-s_2-\cdots-s_j}.$$

Because

$$-q^{-s_1-s_2-\cdots-s_{j-1}} + q^{-s_{j+1}}(1+q^{s_j+s_{j+1}})q^{-s_1-s_2-\cdots-s_j} = q^{-s_1-s_2-\cdots-s_{j+1}}$$

we are done. $\square$

Proof of Theorem 3.2. It remains to show that the linear combination given by Lemma 3.10 gives the correct constants for the infinite products on the right side of Theorem 3.2 (namely 1).

There are $\binom{j-t}{t}$ such non-overlapping E with t edges, where $2t\leq j$, and therefore $j-2t$ vertices not in E. So the right side becomes

$$\sum_{t=0}^{[j/2]} \binom{j-t}{t} (-1)^t \sum_{s=0}^{j-2t} \binom{j-2t}{s} \frac{(q^{k+1-r+j-2t-2s}, q^{k+2+r-j+2t+2s}, q^{2k+3}; q^{2k+3})_\infty}{(q;q)_\infty}$$

The coefficient of

$$\frac{(q^{k+1-r+j-2u}, q^{k+2+r-j+2u}, q^{2k+3}; q^{2k+3})_\infty}{(q;q)_\infty}$$

for $0\leq u\leq j$ is

$$\sum_{s,t,s+t=u} \binom{j-t}{t} (-1)^t \binom{j-2t}{s} = 1$$

by the Chu-Vandermonde evaluation, namely

$$\binom{j}{u}\ {}_2F_1\left(\begin{matrix}-u, & u-j\\ & -j\end{matrix}\,\middle|\, 1\right) = 1$$

for $0 \le u \le j$. $\square$

4. New Bressoud-Type Identities for Even Moduli

The Bressoud identities for even moduli can be proven the same way. The only change is to replace (4) by

$$\frac{H_{2s}(-1,1|q)}{(q)_{2s}} = \frac{1}{(q^2;q^2)_s}.$$

Here we state (without proof) the analogous binomial results for even moduli.

Theorem 4.1. *For $0 \le j, r \le k$, and $j + r \le k$,*

$$\sum_{s_1\ge s_2\ge\cdots\ge s_k\ge 0} \frac{q^{-s_1-\cdots-s_j}(1+q^{s_1+s_2})(1+q^{s_2+s_3})\cdots(1+q^{s_{j-1}+s_j})}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} \times q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k}$$
$$= \sum_{s=0}^{j}\binom{j}{s}\frac{(q^{k+1-r+j-2s}, q^{k+1+r-j+2s}, q^{2k+2}; q^{2k+2})_\infty}{(q;q)_\infty}.$$

Moreover, the j factors of q^{-s_1} and $q^{-s_i}(1+q^{s_{i-1}+s_i})$, $2 \le i \le j$ may be replaced by any j-element subset of $\{q^{-s_1}\} \cup \{q^{-s_i}(1+q^{s_{i-1}+s_i}) : 2 \le i \le k-r\}$.

Again using Lemma 3.10 we have the version without binomial coefficients.

Theorem 4.2. *For $0 \le j, r \le k$, and $j + r \le k$,*

$$\sum_{s_1\ge s_2\ge\cdots\ge s_k\ge 0} \frac{q^{s_1^2+\cdots+s_k^2-(s_1+\cdots+s_j)+(s_{k-r+1}+\cdots+s_k)}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}$$
$$= \sum_{s=0}^{j}\frac{(q^{k+1-r+j-2s}, q^{k+1+r-j+2s}, q^{2k+2}; q^{2k+2})_\infty}{(q;q)_\infty}.$$

The case $j = 0$ in Theorem 4.2 is given by Bressoud [9, (3.4), p. 15] while $r = 0$ is [9, (3.5), p. 16].

5. Overpartitions

Finally, for completeness, we give two analogous results for overpartitions, see [3]. Proposition 2.7, which is used to insert linear exponents, requires a special choice of z. But in the two results of this section we have a general z, so we cannot insert the linear factors as before.

The first result is a binomial version of Corollary 2.5.

Theorem 5.1. *For $0 \le j \le k+1$,*

$$\sum_{s_1\ge s_2\ge\cdots\ge s_{k+1}\ge 0} \frac{q^{-s_1-\cdots-s_j}(1+q^{s_1+s_2})(1+q^{s_2+s_3})\cdots(1+q^{s_{j-1}+s_j})}{(q)_{s_1-s_2}\cdots(q)_{s_k-s_{k+1}}} \frac{(-z,-q/z;q)_{s_{k+1}}}{(q)_{2s_{k+1}}} \times q^{s_1^2+\cdots+s_{k+1}^2}$$
$$= \sum_{s=0}^{j} \binom{j}{s} \frac{(-zq^{k+1+j-2s}, -q^{k+2-j+2s}/z, q^{2k+3}; q^{2k+3})_\infty}{(q;q)_\infty}.$$

Moreover, the j factors of q^{-s_1} and $q^{-s_i}(1+q^{s_{i-1}+s_i})$, $2 \le i \le j$ may be replaced by any j-element subset of $\{q^{-s_1}\} \cup \{q^{-s_i}(1+q^{s_{i-1}+s_i}) : 2 \le i \le k+1\}$.

Theorem 5.2. *For $0 \le j \le k+1$,*

$$\sum_{s_1\ge s_2\ge\cdots\ge s_{k+1}\ge 0} \frac{q^{s_1^2-s_1+\cdots+s_j^2-s_j+s_{j+1}^2+\cdots+s_{k+1}^2}}{(q)_{s_1-s_2}\cdots(q)_{s_k-s_{k+1}}} \frac{(-z,-q/z;q)_{s_{k+1}}}{(q)_{2s_{k+1}}}$$
$$= \sum_{s=0}^{j} \frac{(-zq^{k+1+j-2s}, -q^{k+2-j+2s}/z, q^{2k+3}; q^{2k+3})_\infty}{(q;q)_\infty}.$$

We mention a different expansion for the infinite product in Theorem 5.2 when $j=0$. This final result comes from a version of the Laurent polynomials $H_{2n}(z,a|q)$ with an odd index. We do not develop the corresponding results here.

Theorem 5.3. *If k is a non-negative integer,*

$$\sum_{s_1\ge s_2\ge\cdots\ge s_{k+1}\ge 0} \frac{q^{s_1^2+\cdots+s_{k+1}^2+s_1+\cdots+s_{k+1}}}{(q)_{s_1-s_2}\cdots(q)_{s_k-s_{k+1}}} \frac{(-q^{k+1}/z)_{s_{k+1}+1}(-zq^{-k})_{s_{k+1}}}{(q)_{2s_{k+1}+1}}$$
$$= \frac{(-zq^{k+1}, -q^{k+2}/z, q^{2k+3}; q^{2k+3})_\infty}{(q;q)_\infty}.$$

If $k=0$ in Theorems 5.2 and 5.3, we have the curious result (see [3, (5.1)])

$$\text{(8)} \qquad \frac{(-zq,-q^2/z,q^3;q^3)_\infty}{(q)_\infty} = \sum_{s=0}^{\infty} \frac{(-z,-q/z;q)_s}{(q)_{2s}} q^{s^2} = \sum_{s=0}^{\infty} \frac{(-zq)_{s+1}(-1/z)_s}{(q)_{2s+1}} q^{s^2+s}.$$

6. Remarks

The Andrews-Gordon identities have combinatorial interpretations for integer partitions, three of which are (see [2]):

(1) those with modular conditions on parts,
(2) those with difference conditions on parts,
(3) those with conditions on iterated Durfee squares.

This paper offers no insightful versions of these results for the binomial versions given here.

Berkovich and Paule [5],[6] have versions of the Andrews-Gordon identities where the linear forms are also modified.

Griffin, Ono, and Warnaar [12] give new infinite families (e.g. [12, Theorem 1.1]) of Rogers-Ramanujan identities. See [12, (2.7)] for the Andrews-Gordon-Bressoud identities in their paper.

Seo and Yee [13] combinatorially study singular overpartitions, whose generating function is given by $j = 0$ in Theorem 5.1 with a special choice of z.

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School of Mathematics, University of Minnesota, Minneapolis, MN 55455
E-mail address: `stanton@math.umn.edu`