Some Combinatorial and Analytical Identities

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Abstract

We give new proofs and explain the origin of several combinatorial identities of Fu and Lascoux, Dilcher, Prodinger, Uchimura, and Chen and Liu. We use the theory of basic hypergeometric functions, and generalize these identities. We also exploit the theory of polynomial expansions in the Wilson and Askey-Wilson bases to derive new identities which are not in the hierarchy of basic hypergeometric series. We demonstrate that a Lagrange interpolation formula always leads to very-well-poised basic hypergeometric series. As applications we prove that the Watson transformation of a balanced ${}_4\phi_3$ to a very-well-poised ${}_8\phi_7$ is equivalent to the Rodrigues-type formula for the Askey-Wilson polynomials. By applying the Leibniz formula for the Askey-Wilson operator we also establish the ${}_8\phi_7$ summation theorem.

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1 Introduction

The interest in combinatorial identities goes back a long way but the interest in the combinatorial q-identities is of a more recent vintage. Riordan's book [26] crystalized the interest in combinatorial identities but it appeared before the combinatorial community realized the importance of q-series outside the theory of partitions. Since the 1970's it was realized that combinatorial identities and special function identities enrich and complement each other. Some of the early combinatorial proofs of q-series identities are in [2] and [15].

This work two main goals. Our first is to give new proofs and generalizations of the identities in §2, and to identify their origin within the theory of basic hypergeometric functions. For example we show that not only the left and right sides of (2.1) are equal, but also that each sum can be found explicitly (Theorem 2.1). We show that (2.3) follows from the ${}_2\phi_1$ to ${}_3\phi_1$ transformation, [14, (III.8)], and we generalize (2.2), (2.4), (2.8) in Theorems 2.2, 2.3, and 2.4. That such basic hypergeometric proofs exist is not surprising, moreover they indicate the depth of the identity.

Our second goal is to shed some new light on q-series by showing how they follow from polynomial expansions. We obtain new identities which do not follow from the basic hypergeometric framework. The Watson transformation expresses a terminating very-well-poised ${}_8\phi_7$ as a terminating balanced ${}_4\phi_3$.

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In §5 we show that the Watson transformation is simply the Rodrigues type formula for the Askey-Wilson polynomials, a surprising result. Similarly the Whipple transformation is the Rodrigues type formula for the Wilson polynomials.

We follow the standard notation for q-series as in [5] and [14].

2 The combinatorial identities

In this section we prove (2.1), (2.3), and (2.5) and generalize (2.2), (2.4), (2.8). These are the combinatorial identities of Chen and Liu, Dilcher, Fu and Lascoux, Prodinger, and Uchimura. We give quick proofs of these results and and state the generalizations in Theorems 2.2, 2.3, and 2.4. Theorem 2.1 sharpens (2.1).

Fu and Lascoux [11] used the Newton and Lagrange interpolation to prove several combinatorial identities. One such identity is the following identity of Uchimura [27]

$$\sum_{j=1}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(-1)^{j-1} q^{\binom{j+1}{2}}}{1-q^{j+m}} = \sum_{j=1}^{n} \frac{q^j}{1-q^j} \bigg/ \begin{bmatrix} m+j \\ j \end{bmatrix}_q . \tag{2.1}$$

This generalized the case $m = 0$ which was proved in [28]. Fu and Lascoux also treated the Dilcher identity [10]

$$\sum_{j=1}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(-1)^{j-1} q^{\binom{j}{2}+mj}}{(1-q^j)^m} = \sum_{1 \le j_1 \le \cdots \le j_m \le n} \frac{q^{j_1}}{1-q^{j_1}} \cdots \frac{q^{j_m}}{1-q^{j_m}}. \tag{2.2}$$

In another paper Fu and Lascoux [12] proved the identities

$$\frac{(z;q)_{n+1}}{(q;q)_n} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(-1)^j x^j (-1/x;q)_j}{1-zq^j} q^j = \sum_{j=0}^{n} (-1)^j \frac{(z;q)_j}{(q;q)_j} x^j q^j, \tag{2.3}$$

$$\begin{aligned} &\sum_{j=1}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(-x)^j (-1/x;q)_j}{(1-q^j)^m} q^{jm} \\ &= \sum_{j=1}^{n} \frac{(-1)^j [x^j - (-1)^j]}{1-q^j} q^j \sum_{j \le j_2 \le \cdots \le j_m \le n} \frac{q^{\sum_{k=2}^m j_k}}{\prod_{k=2}^m (1-q^{j_k})}. \end{aligned} \tag{2.4}$$

Prodinger [24] established the following q-analogue of an earlier result of Kirchenhofer [23]

$$\sum_{j=0, j \ne M} \begin{bmatrix} n \\ j \end{bmatrix}_q (-1)^{j-1} \frac{q^{\binom{j+1}{2}}}{1-q^{j-M}} = (-1)^M q^{\binom{M+1}{2}} \begin{bmatrix} n \\ M \end{bmatrix}_q \sum_{j=0, j \ne M} \frac{q^{j-M}}{1-q^{j-M}}. \tag{2.5}$$

We show that (2.5) is an immediate consequence of our version of (2.1).

We establish (2.1) by establishing a stronger statement, that each side is summable.

Theorem 2.1. *Each side of* (2.1) *is equal to*

$$\frac{1}{1-q^m} - \frac{(q;q)_n}{(q^m;q)_{n+1}}.$$

Proof. A special case of the terminating ${}_2\phi_1$ evaluation is [14, (II.7)]

$$\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(-1)^j q^{\binom{j+1}{2}}}{1-zq^j} = \frac{1}{1-z} {}_2\phi_1\left(\begin{matrix} q^{-n}, z \\ zq \end{matrix} \middle| q; q^{n+1} \right) = \frac{(q;q)_n}{(z;q)_{n+1}}. \tag{2.6}$$

Put $z = q^m$ to evaluate the left side of (2.1).

One can easily verify that the left side L of (2.1) is the partial fraction expansion in $z = q^m$ for the right side R. It also follows from a ${}_3\phi_2$ transformation [14, (III.12)]

$$\begin{aligned} R =& \frac{q}{1-zq} \lim_{\epsilon\to 0^+} {}_3\phi_2\left(\begin{matrix} q^{1-n}, q^{1+\epsilon}, q \\ q^{1-n+\epsilon}, zq^2 \end{matrix} \middle| q, q \right) = \frac{q^n}{1-zq^n} \lim_{\epsilon\to 0^+} {}_3\phi_2\left(\begin{matrix} q^{1-n}, q^{-n}, q \\ q^{1-n+\epsilon}, q^{1-n}/z \end{matrix} \middle| q, q^\epsilon/z \right) \\ =& \frac{1}{1-zq^n} \left[q^n {}_2\phi_1\left(\begin{matrix} q, q^{-n} \\ q^{-n+1}/z \end{matrix} \middle| q, 1/z \right) - \frac{(q;q)_n (q^{-n};q)_n}{(q;q)_n (q^{1-n}/z;q)_n} q^n z^{-n} \right]. \end{aligned}$$

The ${}_2\phi_1$ can be evaluated as a product, again by [14, (II.7)], and the proof is complete. ∎

Our version (2.6) of (2.1) immediately proves (2.5).

Proof of (2.5). Fix an integer M between 0 and n, and write (2.6) as

$$\sum_{j=0, j\neq M}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(-1)^j q^{\binom{j+1}{2}}}{1-zq^j} = \frac{(q;q)_n}{1-zq^M} \left[\frac{1}{(z;q)_M (zq^{M+1};q)_{n-M}} - \frac{(-1)^M q^{\binom{M+1}{2}}}{(q;q)_M (q;q)_{n-M}} \right].$$

If $f(z) = \{(z;q)_M (zq^{M+1};q)_{n-M}\}^{-1}$, then the quantity in the square bracket is $f(z) - f(q^{-M})$. Taking the limit as $z \to q^{-M}$ of the resulting identity yields (2.5). ∎

Proof of (2.3). Recall the basic hypergeometric transformation [14, (III.8)]

$${}_2\phi_1\left(\begin{matrix} q^{-n}, B \\ C \end{matrix} \middle| q, Z \right) = \frac{(C/B;q)_n}{(C;q)_n} B^n {}_3\phi_1\left(\begin{matrix} q^{-n}, B, q/Z \\ Bq^{1-n}/C \end{matrix} \middle| q, \frac{Z}{C} \right).$$

The right side of (2.3) is a limit of a ${}_2\phi_1$. Upon applying the above transformation

$$\lim_{\epsilon\to 0^+} {}_2\phi_1\left(\begin{matrix} q^{-n}, z \\ q^{\epsilon-n} \end{matrix} \middle| q, -qx \right) = \frac{(q^{-n}/z;q)_n}{(q^{-n};q)_n} z^n {}_3\phi_1\left(\begin{matrix} q^{-n}, z, -1/x \\ qz \end{matrix} \middle| q, -q^{n+1}x \right),$$

we obtain the left side of (2.3). ∎

For the proof of (2.2) we prove a more general identity.

Theorem 2.2. *For any positive integer m,*

$$\begin{aligned} &\sum_{j=1}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q (-1)^{j-1} q^{\binom{j}{2}+jm} \frac{1-q^j}{(1-zq^j)^{m+1}} \\ &= \frac{(q;q)_n}{(zq;q)_n} \sum_{j_1+j_2+\cdots+j_n=m} \frac{q^{j_1}}{(1-zq)^{j_1}} \frac{q^{2j_2}}{(1-zq^2)^{j_2}} \cdots \frac{q^{nj_n}}{(1-zq^n)^{j_n}}. \end{aligned}$$

It is evident that (2.2) is the special case $z = 1$ of Theorem 2.2.

Proof. Rewrite (2.6) as

$$\sum_{j=1}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q (-1)^{j-1} q^{\binom{j}{2}} \left(1 - \frac{1-q^j}{1-zq^j} \right) = 1 - \frac{(q;q)_n}{(zq;q)_n}.$$

Now differentiate the above identity m times with respect to z to obtain Theorem 2.2. ∎

The final variation of (2.6) generalizes (2.4).

Theorem 2.3. *For any positive integer m*

$$\sum_{j=1}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(-x)^j(-1/x;q)_j}{(1-zq^j)^m} q^{jm}$$
$$= (q;q)_n \sum_{j=1}^{n} \frac{(-1)^j[x^j-(-1)^j]}{(q;q)_j(zq^j;q)_{n+1-j}} q^j \sum_{j\le j_2\le\cdots\le j_m\le n} \frac{q^{\sum_{k=2}^m j_k}}{\prod_{k=2}^m (1-zq^{j_k})}.$$

Proof. Observe that the Chu-Vandermonde sum [14, (II.6)] implies

$$\sum_{j=0}^{n} \frac{(z;q)_j}{(q;q)_j} q^j = \lim_{\epsilon\to 0^+} {}_2\phi_1\left(\begin{matrix} q^{-n}, z \\ q^{-n-\epsilon} \end{matrix} \,\middle|\, q, q \right) = \frac{(q^{-n}/z;q)_n}{(q^{-n};q)_n} z^n = \frac{(qz;q)_n}{(q;q)_n}. \tag{2.7}$$

Now subtract the right-hand side of (2.7) from the left-hand side of (2.3) and the left-hand side of (2.7) from the right-hand side of (2.3) to establish

$$\frac{(z;q)_{n+1}}{(q;q)_n} \sum_{j=1}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(-x)^j(-1/x;q)_j}{1-zq^j} q^j = \sum_{j=1}^{n} (-1)^j \frac{(z;q)_j}{(q;q)_j} (x^j-(-1)^j) q^j.$$

Dividing the above identity by $(z;q)_{n+1}/(q;q)_n$ and differentiating $m-1$ times with respect to z proves Theorem 2.3. □

In a recent paper Chen and Liu [7] generalized earlier work of Alladi [1] involving weighted partition theorems. Chen and Liu were interested in Franklin type involutions, and as an application of their technique, they proved the identity

$$\begin{aligned} &\sum_{n=0}^{\infty} q^{2mn}(q^{2mn+2m};q^{2m})_\infty (aq^{2mn+1};q^2)_\infty \\ &\quad = 1 + \sum_{k=1}^{\infty} (-a)^k q^{k^2} \prod_{j=1}^{k} \left[1+q^{2j}+q^{4j}+\cdots+q^{2(m-1)j}\right]. \end{aligned} \tag{2.8}$$

The case $a=-1$ is due to Andrews, see [3, p. 157]. We generalize (2.8) in Theorem 2.4, and give its combinatorial interpretation in Theorem 2.5. We need only the q-binomial theorem.

Theorem 2.4. *For any positive integer m,*

$$\sum_{n=0}^{\infty} q^{2mn}(q^{2mn+2m};q^{2m})_\infty \frac{(aq^{2mn+1};q^2)_\infty}{(abq^{2mn+1};q^2)_\infty}$$
$$= 1 + \sum_{k=1}^{\infty} (abq)^k (1/b;q^2)_k \prod_{j=1}^{k} \left[1+q^{2j}+q^{4j}+\cdots+q^{2(m-1)j}\right].$$

It is clear that (2.8) is the special case $b\to 0$ of Theorem 2.4.

Proof. Using the q-binomial theorem

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z|<1,$$

we write the left side of Theorem 2.4 as

$$\sum_{n=0}^{\infty} q^{2mn}(q^{2mn+2m};q^{2m})_\infty \sum_{k=0}^{\infty} \frac{(1/b;q^2)_k}{(q^2;q^2)_k}(abq)^k q^{2mnk}$$
$$= (q^{2m};q^{2m})_\infty \sum_{k=0}^{\infty} \frac{(1/b;q^2)_k}{(q^2;q^2)_k}(abq)^k \sum_{n=0}^{\infty} \frac{q^{2mn(k+1)}}{(q^{2m};q^{2m})_n}$$
$$= \sum_{k=0}^{\infty} \frac{(1/b;q^2)_k}{(q^2;q^2)_k}(abq)^k \frac{(q^{2m};q^{2m})_\infty}{(q^{2m+2mk};q^{2m})_\infty} = \sum_{k=0}^{\infty}(1/b;q^2)_k(abq)^k \frac{(q^{2m};q^{2m})_k}{(q^2;q^2)_k},$$

and the proof is complete. □

Next we give an integer partition interpretation of Theorem 2.4 which generalizes Theorem 7.2 in [7]. We give two sets of integer partitions whose generating functions are the respective sides of Theorem 2.4. For the left side, let $A_{k,s,m}(n)$ be the set of integer partitions λ of n such that

1. the even parts of λ are distinct non-negative integers,
2. each even part of λ is a multiple of $2m$,
3. the smallest part of λ is even,
4. the odd parts of λ form an overpartition with k total parts and $k-s$ barred parts.

An example is $\lambda = (13, \overline{13}, 12, 9, 7, 5, 5, \overline{5}, 4) \in A_{7,5,2}(73)$.

We see that if $2mj$ is the smallest part of λ, the generating function of $|A_{k,s,m}(n)|$ is

$$\sum_{k,s,n\geq 0} |A_{k,s,m}(n)|a^k b^s q^n = \sum_{j=0}^{\infty} q^{2mj}(-q^{2mj+2m};q^{2m})_\infty \frac{(-aq^{2mj+1};q^2)_\infty}{(abq^{2mj+1};q^2)_\infty},$$

which is nearly the left side of Theorem 2.4, with $a \to -a$, $b \to -b$. If the even parts greater than $2mj$ are weighted by -1, we do obtain the left side of Theorem 2.4.

The right side of Theorem 2.4, with $a \to -a$, $b \to -b$ is

$$1 + \sum_{k=1}^{\infty} a^k \prod_{p=1}^{k}(bq + q^{2p-1}) \prod_{j=1}^{k} \left[1 + q^{2j} + q^{4j} + \cdots + q^{2(m-1)j}\right].$$

For the combinatorial interpretation of the coefficient of $a^k b^s$, we choose s parts of size 1, and $k-s$ distinct odd parts, $2k-1 \geq \theta_1 > \theta_2 > \cdots > \theta_{k-s}$ each at most $2k-1$. Then $\mu_1 = (\theta_1 - (2(k-s)-1), \theta_2 - (2(k-s)-3), \cdots, \theta_{k-s} - 1)$ is a partition with even parts lying inside a $(k-s) \times 2s$ rectangle. The s 1's, and the $k-s$ odd parts can be concatenated to obtain a partition γ_1 with k parts, $\gamma_1 = (2(k-s)-1, 2(k-s)-3, \cdots, 1, 1, 1, \cdots 1)$.

The product $\prod_{j=1}^{k}\left[1 + q^{2j} + q^{4j} + \cdots + q^{2(m-1)j}\right]$ is the generating function for partitions γ_2 with exactly k parts, $0's$ allowed, each part even, with difference of consecutive parts at most $2(m-1)$. Define $\mu_2 = \gamma_1 + \gamma_2$, the partition obtained by adding the respective parts.

We can now define a set $B_{k,s,m}(n)$ for the right side of Theorem 2.4. Let $B_{k,s,m}(n)$ be the set of pairs of integer partitions (μ_1, μ_2), $|\mu_1| + |\mu_2| = n$, such that

1. μ_1 is a partition with even parts which lies inside a $(k-s) \times 2s$ rectangle,
2. $\mu_2 = (m_1, \cdots, m_k)$ has exactly k parts, all of which are odd, the first $k-s$ of which are distinct,

3. $m_i - m_{i+1} \leq 2m$ for $1 \leq i \leq k-s-1$,
4. $m_i - m_{i+1} \leq 2(m-1)$ for $k-s \leq i \leq k-1$,
5. $m_k \leq 2m-1$.

The right side of Theorem 2.4 is

$$\sum_{k,s,n\geq 0} |B_{k,s,m}(n)| a^k b^s q^n.$$

Theorem 7.2 in [7] is the $s=0$ case of the next Theorem.

Theorem 2.5. *For any positive integer m,*

$$|B_{k,s,m}(n)| = \sum_{\lambda \in A_{k,s,m}(n)} (-1)^{(\#even\ parts\ of\ \lambda)-1}.$$

For example if $n=10$, $k=4$, $s=3$, and $m=2$,

$$B_{4,3,2}(10) = \{(\emptyset, 5311), (\emptyset, 3331), (2, 3311), (4, 3111), (6, 1111)\},$$

$$A_{4,3,2}(10) = \{711\overline{1}0, \overline{7}1110, \overline{5}3110, 5\overline{3}110, 531\overline{1}0, 33\overline{3}10, 333\overline{1}0, 4\overline{3}1110, 4311\overline{1}0\}.$$

3 Polynomial Expansions

In the previous section we applied the beginning results in basic hypergeometric series to derive recent combinatorial identities. In this section we will derive new identities, which are not special cases of the basic hypergeometric literature. We use polynomial expansions which are extensions of Taylor's theorem. Along the way we give very short proofs of Jackson's summation theorem and Watson's transformation for very well-poised ${}_8\phi_7$'s.

Let

$$x = (z+1/z)/2, \quad z = e^{i\theta}, \quad x = \cos\theta. \tag{3.1}$$

The Askey-Wilson basis for the vector space of polynomials in x is

$$\phi_n(x;a) = (ae^{i\theta}, ae^{-i\theta}; q)_n = \prod_{i=0}^{n-1}(1 - 2axq^i + a^2q^{2i}), \quad n = 0, 1, \cdots. \tag{3.2}$$

The question of finding the coefficients when an arbitrary polynomial is expanded in this basis is answered using the Askey-Wilson operators [18], which generalize the derivative.

Put

$$f(x) = f((z+1/z)/2) = \breve{f}(z).$$

The Askey-Wilson operator $\mathcal{D}_q$ is defined by

$$(\mathcal{D}_q f)(x) = \frac{\breve{f}(q^{1/2}z) - \breve{f}(q^{-1/2}z)}{(q^{1/2} - q^{-1/2})[(z-1/z)/2]}.$$

It is easy to see that

$$\mathcal{D}_q \phi_n(x;a) = -\frac{2a(1-q^n)}{1-q}\phi_{n-1}(x; aq^{1/2}). \tag{3.3}$$

$$\mathcal{D}_q \frac{1}{\phi_n(x;a)} = \frac{2aq(1-q^n)}{1-q}\frac{1}{\phi_{n+1}(x; aq^{-1/2})}. \tag{3.4}$$

The expansions we shall use are embodied in the following theorem.

Theorem 3.1. *Let*

$$x_n \quad = \quad [aq^{n/2} + q^{-n/2}/a]/2, \quad 0 < q < 1, \ 0 < a < 1,$$

and assume that $f(x)$ is a polynomial. Then

$$f(x) = \sum_{k=0}^{\infty} f_{k,\phi} \ \phi_k(x;a),$$

with

$$f_{k,\phi} = \frac{(q-1)^k}{(2a)^k (q;q)_k} q^{-k(k-1)/4} \ (\mathcal{D}_q^k f)(x_k).$$

Theorem 3.1 for the ϕ_n basis is in [17].

Theorem 3.2. *The action of $\mathcal{D}_q^n$ is given by*

$$\begin{aligned}\mathcal{D}_q^n f(x) = &\frac{2^n q^{n(1-n)/4}}{(q^{1/2} - q^{-1/2})^n} \\ &\times \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k(n-k)} z^{2k-n} \breve{f}(q^{(n-2k)/2} z)}{(q^{1+n-2k} z^2; q)_k (q^{2k-n+1} z^{-2}; q)_{n-k}},\end{aligned}$$

where $x = \cos\theta, z = e^{i\theta}$.

Theorem 3.2 is due to S. Cooper [8].

It is clear that Theorem 3.2 is a summation result whenever $\mathcal{D}_q^n f(x)$ is explicitly known. For example, from (3.3) we see that

$$f(x) = \phi_m(x;A), \quad \mathcal{D}_q^n f(x) = (q^{m-n+1};q)_n q^{\frac{1}{2}\binom{n}{2}} \frac{(2a)^n}{(q-1)^n} \phi_{m-n}(x; Aq^{n/2}).$$

In this case Theorem 3.2 becomes the terminating very well-poised ${}_6\phi_5$ evaluation [14, II.20].

By taking a product of two ϕ_k's we obtain a terminating very-well poised ${}_8\phi_7$ from Theorem 3.2. It only remains to choose an appropriate product for a summation result.

Proposition 3.3. *For any positive integer j,*

$$\begin{aligned}\mathcal{D}_q^n \left(\frac{\phi_{n+j-1}(x;A)}{\phi_j(x;B)} \right) = &\frac{(2B)^n q^{-\frac{1}{2}\binom{n}{2}}}{(1-q)^n} \\ &\times \frac{\phi_{j-1}(x; Aq^{n/2})}{\phi_{n+j}(x; Bq^{-n/2})} \frac{(q;q)_{n+j-1}}{(q;q)_{j-1}} (ABq^{j-1};q)_n (A/B;q)_n.\end{aligned}$$

Proof. The Leibniz rule [17, (1.22)] for the Askey-Wilson operator is

$$\mathcal{D}_q^n \left(f(x)g(x) \right) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)/2} (\eta^k (\mathcal{D}_q^{n-k} f))(\eta^{k-n} (\mathcal{D}_q^k g)), \tag{3.5}$$

where η^k is the map

$$(\eta^k f)(x) = \breve{f}(q^{k/2} z).$$

When (3.5), and (3.3)- (3.4) are applied in Proposition 3.3, the sum which results is a 1-balanced ${}_3\phi_2$, which is evaluable [14, II.12]. ☐

Observe the unusual fact that the right-hand side of the formula in Proposition 3.3 is a constant multiple of

$$\mathcal{D}_q^n \phi_{n+j-1}(x;A) \ \times \mathcal{D}_q^n \left(\frac{1}{\phi_j(x;B)} \right),$$

a very curious fact.

Applying Proposition 3.3 to Theorem 3.2 does yield Jackson's terminating very well-poised ${}_8\phi_7$ evaluation [14, II.22], the parameters are $a = z^2q^{-n}$, $b = Azq^{n/2+j-1}$, $c = zq^{-n/2+1}/A$, $d = Bzq^{-n/2}$, and $e = zq^{-n/2-j+1}/B$.

If we choose $f(x)$ to be a product of two arbitrary ϕ's, $f(x) = \phi_m(x;A)\phi_j(x;B)$, then the right side of Theorem 3.2 becomes a very-well poised ${}_8\phi_7$. The Leibniz rule (3.5) yields a 1-balanced ${}_4\phi_3$ for the left side of Theorem 3.2. The resulting equality is Watson's transformation [14, (III.18)]. If a general product of $p+1$ ϕ's is chosen, the resulting equality is Andrews' transformation [4, Theorem 4] of a very-well poised ${}_{2p+6}\phi_{2p+5}$ to a p-fold sum.

We combine Theorems 3.1 and 3.2 in the next result. Theorem 3.4 will be applied throughout the rest of this section.

Theorem 3.4. *For polynomials f of degree at most n and with $x = \cos\theta$, we have the expansion*

$$\frac{(q, qa^2;q)_n}{(aqe^{i\theta}, aqe^{-i\theta};q)_n} \ f(x) = \sum_{k=0}^{n} \frac{1-a^2q^{2k}}{1-a^2} \frac{(q^{-n}, a^2, ae^{i\theta}, ae^{-i\theta};q)_k}{(q, a^2q^{n+1}, aqe^{i\theta}, aqe^{-i\theta};q)_k} q^{k(1+n)} \breve{f}(aq^k).$$

Proof. Combine Theorem 3.1 and Theorem 3.2 to find that

$$f(x) = \sum_{0\le j\le k\le n} \frac{q^{-(k-j)^2+k}\breve{f}(aq^{k-j})(ae^{i\theta}, ae^{-i\theta};q)_k}{a^{2k-2j}(q, a^2q^{1+2k-2j};q)_j(q, q^{2j-2k+1}/a^2;q)_{k-j}},$$

holds for polynomial f. Replace k by $k+j$ and the new j sum becomes

$$\lim_{\epsilon\to 0^+} {}_3\phi_2 \left(\begin{matrix} q^{k-n}, aq^ke^{i\theta}, aq^ke^{-i\theta} \\ q^{k-n+\epsilon}, a^2q^{2k+1-\epsilon} \end{matrix} \ \middle| \ q, q \right) = \frac{(aq^{k+1}e^{i\theta}, aq^{k+1}e^{-i\theta};q)_{n-k}}{(q, a^2q^{2k+1};q)_{n-k}},$$

by the q-Pfaff-Saalschütz sum, [14, (II.12)]. Theorem 3.4 now follows from simple manipulations. □

Theorem 3.4 can be considered as another polynomial expansion. Upon multiplying both sides by $\phi_n(x;aq)$ note that

$$\frac{\phi_k(x;a)\phi_n(x;aq)}{\phi_k(x;aq)} = \frac{\phi_{n+1}(x;a)}{1-2xaq^k+a^2q^{2k}}.$$

so that the k^{th} term of the right side is a polynomial in x. Theorems 3.1 and 3.4 lead to distinct results. Theorem 3.4 is related to very-well-poised sums, while Theorem 3.1 is related to balanced sums. From the classical theory of interpolation, Theorem 3.1 is an expansion using divided difference operators while Theorem 3.4 is the Lagrange interpolation, [9], [16].

The first application of Theorem 3.4 chooses $f(x) = (2x+b)^n$.

Corollary 3.5. *For non-negative integers n we have*

$$\frac{a^n(q, qa^2;q)_n}{(aqe^{i\theta}, aqe^{-i\theta};q)_n} \ (2x+b)^n = \sum_{k=0}^{n} \frac{1-a^2q^{2k}}{1-a^2} \frac{(q^{-n}, a^2, ae^{i\theta}, ae^{-i\theta};q)_k}{(q, a^2q^{n+1}, aqe^{i\theta}, aqe^{-i\theta};q)_k} q^k[1+abq^k+a^2q^{2k}]^n.$$

By replacing n by $n+s$ in Corollary 3.5 then equating coefficients of b^s we establish the identity

$$\begin{aligned}(3.6)\qquad &\frac{(q, qa^2; q)_{n+s}}{(aqe^{i\theta}, aqe^{-i\theta}; q)_{n+s}}\,(2a\cos\theta)^n \\ &= \sum_{k=0}^{n+s} \frac{1-a^2q^{2k}}{1-a^2}\,\frac{(q^{-n-s}, a^2, ae^{i\theta}, ae^{-i\theta}; q)_k}{(q, a^2q^{n+s+1}, aqe^{i\theta}, aqe^{-i\theta}; q)_k}\, q^{k(1+s)}[1+a^2q^{2k}]^n.\end{aligned}$$

The special case $b = 0$ of Corollary 3.5, or equivalently the case $s = 0$ of (3.6), is the attractive identity

$$(3.7)\qquad \sum_{k=0}^{n} \frac{(a^2, ac, a/c, q^{-n}; q)_k}{(q, aq/c, aqc, a^2q^{n+1}; q)_k}\,\frac{1-a^2q^{2k}}{1-a^2} q^k(1+a^2q^{2k})^n = \frac{(qa^2, q; q)_n}{(aq/c, aqc; q)_n} a^n(c+1/c)^n.$$

The above identity is a partial fraction expansion in c and can be proved by computing residues.

The limiting case $s \to \infty$ of (3.6) is the sum

$$\begin{aligned}(3.8)\qquad &\frac{(2a\cos\theta)^n(q, qa^2; q)_\infty}{(aqe^{i\theta}, aqe^{-i\theta}; q)_\infty} \\ &= \sum_{k=0}^{\infty} \frac{1-a^2q^{2k}}{1-a^2}\,\frac{(a^2, ae^{i\theta}, ae^{-i\theta}; q)_k}{(q, aqe^{i\theta}, aqe^{-i\theta}; q)_k}(-1)^k q^{-nk+\binom{k+1}{2}}[1+a^2q^{2k}]^n.\end{aligned}$$

Our next result is a bibasic sum.

Corollary 3.6. *For non-negative integers n we have*

$$\begin{aligned}&\frac{(q, qa^2; q)_n}{(aqe^{i\theta}, aqe^{-i\theta}; q)_n}\,(be^{i\theta}, be^{-i\theta}; p)_n \\ &= \sum_{k=0}^{n} \frac{1-a^2q^{2k}}{1-a^2}\,\frac{(q^{-n}, a^2, ae^{i\theta}, ae^{-i\theta}; q)_k}{(q, a^2q^{n+1}, aqe^{i\theta}, aqe^{-i\theta}; q)_k} q^{k(1+n)}(abq^k, bq^{-k}/a; p)_n.\end{aligned}$$

It must be noted that the known bibasic results are proved using telescopy [14] but Corollary 3.6 does not seem to be suitable for a proof by telescopy. The limiting case $n \to \infty$ is our earlier result, [20, (5.3)],

$$\begin{aligned}(3.9)\qquad &\frac{(q, qa^2; q)_\infty}{(aqe^{i\theta}, aqe^{-i\theta}; q)_\infty}\,(be^{i\theta}, be^{-i\theta}; p)_\infty \\ &= \sum_{k=0}^{\infty} \frac{1-a^2q^{2k}}{1-a^2}\,\frac{(a^2, ae^{i\theta}, ae^{-i\theta}; q)_k}{(q, aqe^{i\theta}, aqe^{-i\theta}; q)_k}\,(-1)^k q^{\binom{k+1}{2}}(abq^k, bq^{-k}/a; p)_\infty,\end{aligned}$$

which is valid for $0 < p < q < 1$, or $0 < p = q < 1$ and $|b| < |a|$. The case $p = 1$ of Corollary 3.6 is Corollary 3.5.

The following examples use the very-well-poised functions W. The W notation, due to W. N. Bailey, is defined by

$$\begin{aligned}(3.10)\qquad &{}_{3+m}W_{2+m}(a; b_1, \cdots, b_m; q, z) = \\ &{}_{3+m}\phi_{2+m}\left(\left.\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b_1, \cdots, b_m \\ \sqrt{a}, -\sqrt{a}, qa/b_1, \cdots, qa/b_m \end{matrix}\right| q; z\right).\end{aligned}$$

The very special case $p = q$ of Corollary 3.6 is the following corollary.

Corollary 3.7. *The summation theorem*

$$\frac{(q, qa^2; q)_n}{(aqe^{i\theta}, aqe^{-i\theta}; q)_n} \frac{(be^{i\theta}, be^{-i\theta}; q)_n}{(b/a, ab; q)_n}$$
$$= {}_8W_7(a^2, ae^{i\theta}, ae^{-i\theta}, abq^n, qa/b, q^{-n}; q, q),$$

holds.

Corollary 3.7 is a special case of Jackson's q-analogue of Dougall's ${}_7F_6$ sum [14, (II.22)]. The general theorem of Jackson contains one more parameter.

This is surprising because if we calculate the coefficients $f_{k,\phi}$ for $f(\cos\theta) = (be^{i\theta}, be^{-i\theta}; q)_n$ then apply the expansion as in Theorem 3.1 we will get the q-analogue of the Pfaff-Saalschütz theorem, as noted in [17]. This indicates that the Pfaff-Saalschütz theorem is equivalent to Corollary 3.7.

Corollary 3.8. *We have the following ${}_{10}W_9$ summation theorem*

$$\frac{(q, qa^2; q)_n}{(aqe^{i\theta}, aqe^{-i\theta}; q)_n} \frac{(be^{i\theta}, be^{-i\theta}; q)_s (ce^{i\theta}, ce^{-i\theta}; q)_{n-s}}{(b/a, ab; q)_s (ac, c/a; q)_{n-s}}$$
$$= {}_{10}W_9(a^2, ae^{i\theta}, ae^{-i\theta}, abq^s, acq^{n-s}, qa/b, qa/c, q^{-n}; q, q).$$

The proof consists of taking $f(\cos\theta) = (be^{i\theta}, be^{-i\theta}; q)_s (ce^{i\theta}, ce^{-i\theta}; q)_{n-s}$ in Theorem 3.4.

Similarly we have the following theorem.

Corollary 3.9. *For $s_1 + s_2 + \cdots + s_m = n$ we have the summation theorem*

$$\frac{(q, qa^2; q)_n}{(aqe^{i\theta}, aqe^{-i\theta}; q)_n} \prod_{j=1}^{m} \frac{(b_j e^{i\theta}, b_j e^{-i\theta}; q)_{s_j}}{(ab_j, b_j/a; q)_{s_j}}$$
$$= {}_{6+2m}W_{5+2m}(a^2, ae^{i\theta}, ae^{-i\theta}, q^{-n}, qa/b_1, ab_1 q^{s_1}, \cdots, qa/b_m, ab_m q^{s_m}; q, q).$$

Corollary 3.9 gives the sum of a terminating W function. To find a nonterminating version we first replace m by $m+1$, then let $n, s_{m+1} \to +\infty$ while $N = n - s_{m+1}$ remains constant. Let k be the summation index in the W series. The terms involving n or s_{m+1} are

$$\frac{(q^{-n}, ab_{m+1} q^{s_{m+1}}; q)_k}{(a^2 q^{n+1}, aq^{1-s_{m+1}}/b_{m+1}; q)_k} \to q^{-k(1+N)} \left(\frac{b_{m+1}}{a}\right)^k,$$

as $n \to \infty$. Thus a limiting case of Corollary 3.9 is

$$\frac{(q, qa^2, b_{m+1}e^{i\theta}, b_{m+1}e^{-i\theta}; q)_\infty}{(aqe^{i\theta}, aqe^{-i\theta}, ab_{m+1}, b_{m+1}/a; q)_\infty} \prod_{j=1}^{m} \frac{(b_j e^{i\theta}, b_j e^{-i\theta}; q)_{s_j}}{(ab_j, b_j/a; q)_{s_j}} \tag{3.11}$$
$$= {}_{6+2m}W_{5+2m}(a^2, ae^{i\theta}, ae^{-i\theta}, qa/b_{m+1}, qa/b_1, ab_1 q^{s_1}, \cdots, qa/b_m, ab_m q^{s_m}; q, q^{-N} \frac{b_{m+1}}{a}),$$

where $N = s_1 + \cdots + s_m$ and $|b_{m+1}| < |a| q^N$ if the W series does not terminate. The summation theorem (3.11) is due to George Gasper [13]. In fact Corollary 3.9 is equivalent to (3.11) because we may choose $b_{m+1} = aq^{N+1}$ in (3.11) and recover Theorem 3.9. Note that (3.8) is the limiting case $b_{m+1} \to 0$ of (3.11) when $b_j = \sqrt{-1}$, $m = n, s_1 = s_2 = \cdots = s_m = 1$.

An interesting limiting case of Corollary 3.9 is to let $z = e^{i\theta} \to +\infty$. The result is

$$\frac{(q, qa^2; q)_n}{a^n q^{n(n+1)/2}} \prod_{j=1}^{m} \frac{b_j^{s_j} q^{s_j(s_j-1)/2}}{(ab_j, b_j/a; q)_{s_j}} \tag{3.12}$$
$$= {}_{4+2m}W_{3+2m}(a^2, q^{-n}, qa/b_1, ab_1 q^{s_1}, \cdots, qa/b_m, ab_m q^{s_m}; q, 1).$$

For another instance where a basic hypergeometric series is evaluated at 1, see [14, §1.9].

We now consider a bivariate version of Theorem 3.4.

Theorem 3.10. *Let $f(x,y)$ be a polynomial of degree at most n in x and in y. With $x=\cos\theta, y=\cos\phi$, the following expansion holds*

$$\frac{(q,q,qa^2,qb^2;q)_n}{(aqe^{i\theta},aqe^{-i\theta},be^{i\phi},be^{-i\phi};q)_n}\, f(x,y)=\sum_{j,k=0}^{n}\frac{(1-a^2q^{2j})(1-b^2q^{2k})}{(1-a^2)(1-b^2)}\, q^{(j+k)(1+n)}$$
$$\times\frac{(q^{-n},a^2,ae^{i\theta},ae^{-i\theta};q)_j(q^{-n},b^2,be^{i\phi},be^{-i\phi};q)_k}{(q,a^2q^{n+1},aqe^{i\theta},aqe^{-i\theta};q)_j(q,b^2q^{n+1},bqe^{i\phi},bqe^{-i\phi};q)_k}\breve{f}(aq^j,bq^k).$$

Applying Theorem 3.10 to

$$f(x,y)=(\alpha e^{i(\theta+\phi)},\alpha e^{i(\phi-\theta)},\alpha e^{i(\theta-\phi)},\alpha e^{-i(\theta+\phi)};q)_n,$$

a polynomial in x and y of degree $2n$, leads to the next result.

Corollary 3.11. *The following summation theorem holds.*

$$\frac{(q,q,qa^2,qb^2;q)_{2n}\ (\alpha e^{i(\theta+\phi)},\alpha e^{i(\phi-\theta)},\alpha e^{i(\theta-\phi)},\alpha e^{-i(\theta+\phi)};q)_n}{(\alpha ab,\alpha/ab;q)_n(aqe^{i\theta},aqe^{-i\theta},bqe^{i\phi},bqe^{-i\phi};q)_{2n}}$$
$$=\sum_{j,k=0}^{2n}\frac{(\alpha abq^n,qab/\alpha;q)_{j+k}}{(\alpha ab,abq^{1-n}/\alpha;q)_{j+k}}\frac{1-a^2q^{2j}}{1-a^2}\frac{(q^{-2n},a^2,ae^{i\theta},ae^{-i\theta};q)_j}{(q,a^2q^{2n+1},aqe^{i\theta},aqe^{-i\theta};q)_j}$$
$$\times\frac{1-b^2q^{2k}}{1-b^2}\frac{(q^{-2n},b^2,be^{i\phi},be^{-i\phi};q)_k}{(q,b^2q^{2n+1},bqe^{i\phi},bqe^{-i\phi};q)_k}q^{(j+k)(n+1)}$$
$$\times\frac{(\alpha b/a;q)_{n+k-j}(\alpha a/b;q)_{n+j-k}(\alpha aq^{j-k}/b;q)_k(\alpha bq^{k-j}/a;q)_j}{(\alpha b/a;q)_k(\alpha a/b;q)_j}.$$

A bivariate version of Corollary 3.5 is

$$\frac{a^n(q,qa^2;q)_n}{(aqe^{i\theta},aqe^{-i\theta};q)_n}\frac{b^n(q,qb^2;q)_n}{(bqe^{i\phi},bqe^{-i\phi};q)_n}\ (2cos\theta+2\cos\phi+c)^n$$
$$=\sum_{j,k=0}^{n}\frac{1-b^2q^{2j}}{1-b^2}\frac{1-a^2q^{2k}}{1-a^2}\frac{(q^{-n},b^2,be^{i\phi},be^{-i\phi};q)_j}{(q,b^2q^{n+1},bqe^{i\phi},bqe^{-i\phi};q)_j}q^{j+k} \tag{3.13}$$
$$\times\frac{(q^{-n},a^2,ae^{i\theta},ae^{-i\theta};q)_k}{(q,a^2q^{n+1},aqe^{i\theta},aqe^{-i\theta};q)_k}[aq^k+bq^j+abcq^{k+j}+ab^2q^{2j+k}+a^2bq^{2k+j}]^n.$$

Again replace n by $n+s$ then equate the coefficient of c^s. The result is

$$\frac{a^n(q,qa^2;q)_{n+s}}{(aqe^{i\theta},aqe^{-i\theta};q)_{n+s}}\frac{b^n(q,qb^2;q)_{n+s}}{(bqe^{i\phi},bqe^{-i\phi};q)_{n+s}}\ (2cos\theta+2\cos\phi)^n$$
$$=\sum_{j,k=0}^{n+s}\frac{1-b^2q^{2j}}{1-b^2}\frac{1-a^2q^{2k}}{1-a^2}\frac{(q^{-n-s},b^2,be^{i\phi},be^{-i\phi};q)_j}{(q,b^2q^{n+s+1},bqe^{i\phi},bqe^{-i\phi};q)_j}q^{(j+k)(1+s)} \tag{3.14}$$
$$\times\frac{(q^{-n-s},a^2,ae^{i\theta},ae^{-i\theta};q)_k}{(q,a^2q^{n+s+1},aqe^{i\theta},aqe^{-i\theta};q)_k}(aq^k+bq^j)^n(1+abq^{j+k})^n.$$

4 The Wilson basis

All of the results of §3 can be given for the corresponding Wilson basis

$$w_n(x;a)=(a+i\sqrt{x},a-i\sqrt{x})_n,$$

and Wilson operator $\mathcal{W}$

$$(\mathcal{W}f)(x) = \frac{\tilde{f}(z+i/2) - \tilde{f}(z-i/2)}{2i\sqrt{x}},$$

where

$$f(x) = \tilde{f}(z), \quad z = \sqrt{x},$$

The analogues of Theorems 3.1, 3.2, and 3.4 are given here. Theorem 4.2 is due to Cooper.

Theorem 4.1. *Let*

$$y_k = -(a+k/2)^2,$$

and assume that $f(x)$ is a polynomial. Then

$$f(x) = \sum_{k=0}^{\infty} f_{k,w} w_k(x;a), \quad f_{k,w} = \frac{1}{k!}(\mathcal{W}^k f)(y_k).$$

Theorem 4.2. *The action of $\mathcal{W}^n$ is given by*

$$\mathcal{W}^n f(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k \tilde{f}(z+i(n-2k)/2))}{(-2iz+1+n-2k)_k (2iz+1+2k-n)_{n-k}}, \; z = \sqrt{x},$$

Theorem 4.3. *For polynomials f of degree at most n we have the following expansion*

$$\begin{aligned} &\frac{n!(2a+1)_n}{(a+1+i\sqrt{x})_n (a+1-i\sqrt{x})_n} f(x) \\ &= \sum_{k=0}^{n} \frac{a+k}{a} \frac{(-n)_k (2a)_k (a+i\sqrt{x})_k (a-i\sqrt{x})_k}{k!(2a+n+1)_k (a+1+i\sqrt{x})_k (a+1-i\sqrt{x})_k} \tilde{f}(i(a+k)). \end{aligned}$$

It is clear that Theorem 4.3 is closely related to very-well-poised hypergeometric series.

5 The Watson Transformation

In this section we reverse the use of Theorem 3.2, namely use it to write very-well-poised series as the n^{th} iterate of an Askey-Wilson operator. When this is realized for the Askey-Wilson polynomials, we see that the Rodrigues formula is equivalent to Watson's transformation.

The Askey-Wilson polynomials have the basic hypergeometric representation

$$p_n(x;\mathbf{t}\,|\,q) = t_1^{-n}\,(t_1t_2, t_1t_3, t_1t_4;q)_n\, {}_4\phi_3\left(\left.\begin{matrix} q^{-n}, t_1t_2t_3t_4q^{n-1}, t_1e^{i\theta}, t_1e^{-i\theta} \\ t_1t_2,\; t_1t_3,\; t_1t_4 \end{matrix}\right| q, q\right), \tag{5.1}$$

where $\mathbf{t}$ stands for the ordered quadruple (t_1, t_2, t_3, t_4), [6], [18]. Their weight function is

$$w(x,\mathbf{t}|q) = \frac{(e^{2i\theta}, e^{-2i\theta};q)_\infty}{\prod_{j=1}^4 (t_je^{i\theta}, t_je^{-i\theta};q)_\infty} \frac{1}{\sqrt{1-x^2}} = \frac{2ie^{-i\theta}(e^{2i\theta}, qe^{-2i\theta};q)_\infty}{\prod_{j=1}^4 (t_je^{i\theta}, t_je^{-i\theta};q)_\infty}, \tag{5.2}$$

with $x = \cos\theta \in (-1,1)$. They have the Rodrigues type formula [6], [14], [18]

$$p_n(x;\mathbf{t}) = \left(\frac{q-1}{2}\right)^n \frac{q^{n(n-1)/4}}{w(\cos\theta;\mathbf{t})} \mathcal{D}_q^n [w(x;q^{n/2}\mathbf{t})]. \tag{5.3}$$

From Theorem 3.2 it follows that the right-hand of (5.3), with $z = e^{i\theta}$, is

$$
\begin{aligned}
&\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1 - z^2 q^{n-2k}}{1 - z^2} \frac{q^{k(1+n-k)} z^{2k-n}}{(q/z^2;q)_k (qz^2;q)_{n-k}} \prod_{j=1}^{4} (t_j z;q)_{n-k} (t_j/z;q)_k \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1 - z^2 q^{-n} q^{2k}}{1 - z^2} \frac{q^{(n-k)(1+k)} z^{n-2k}}{(q/z^2;q)_{n-k} (qz^2;q)_k} \prod_{j=1}^{4} (t_j z;q)_k (t_j/z;q)_{n-k}.
\end{aligned} \tag{5.4}
$$

After routine manipulations we arrive at the representation

$$
\begin{aligned}
p_n(\cos\theta;\mathbf{t}) = \frac{z^n \prod_{j=1}^{4} (t_j/z;q)_n}{(1/z^2;q)_n} \\
\times {}_8W_7(q^{-n} z^2; q^{-n}, t_1 z, t_2 z, t_3 z, t_4 z; q, q^{2-n}/t_1 t_2 t_3 t_4).
\end{aligned} \tag{5.5}
$$

The equality of the right sides of (5.5) and (5.1) is the Watson transformation [14, (III.18)] after the application of the iterated Sears transformation [14, (II.15)]. Note that Ismail [17] proved the Sears transformation using q-Taylor series in the Askey-Wilson operator. This is reproduced in Chapter 12 of [18].

Rodrigues formulas for other very-well-poised series, including ${}_{10}W_9$'s, may be found in this manner, see [22].

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