SOME PROBLEMS

DENNIS STANTON

Contents

Some of these problems do not originate with me.

Date: May 4, 2019.

1. Alternating sign matrices

An $n \times n$ *alternating sign matrix* A is an $n \times n$ matrix, with entries $0, \pm 1$, whose row and column sums are 1, and non-zero entries in each row and column alternate in sign, see [13].

The number of $n \times n$ alternating sign matrices is known [48] to be

$$ASM(n) = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}.$$

This sequence starts $1, 2, 7, 42, 429, 7436, \cdots$.

It is known that $ASM(n)$ is equal to two other numbers

(1) the number of totally symmetric self-complementary plane partitions inside a $2n \times 2n \times 2n$ box, $TSSCPP(n)$,
(2) the number of descending plane partitions whose largest part is at most n, $DPP(n)$.

A *descending plane partition (DPP)* [3] is a column strict tableau of shifted shape, decreasing along rows, and strictly decreasing down columns, such that the lead element of a row is greater than the number of elements in a that row. Here are the 7 descending plane partitions counted by $DPP(3)$

$$\emptyset, \quad 2, \quad 3, \quad 3\ 2, \quad 3\ 1, \quad 3\ 3, \quad \begin{matrix} 3 & 3 \\ & 2 \end{matrix}.$$

Open Problem 1.1. *Find a bijection between the elements counted by $ASM(n)$ and those counted by $TSSCPP(n)$ or $DPP(n)$.*

It is not even known how to do this via the involution principle [22].

It is known [42] that the values of n for which $ASM(n)$ is odd are

$$n = \sum_{t/2 \geq k \geq 0} 2^{t-2k} + \{1 \text{ if } t \text{ is odd}\}.$$

The first few values are $1, 3, 5, 11, 21, 43, 85, \cdots$.

Open Problem 1.2. *Find a Franklin type involution which proves that $ASM(n)$ is even when n avoids the above sequence.*

Open Problem 1.3. *Find a statistic on a subset of permutations, $T_n \subset S_n$, $stat(w)$, such that*

$$ASM(n) = \sum_{w \in T_n} 2^{stat(w)}.$$

Andrews conjectured [3], and Mills, Robbins, and Rumsey proved [28], that the generating function for $DPP(n)$ is

$$DPP(n, q) = \prod_{k=0}^{n-1} \frac{(3k+1)!_q}{(n+k)!_q},$$

for example

$$\begin{aligned} DPP(3, q) = \frac{7!_q * 4!_q}{3!_q * 4!_q * 5!_q} = \frac{7_q * 6_q}{2_q * 3_q} &= (1 - q + q^2)(1 + q + q^2 + q^3 + q^4 + q^5 + q^6) \\ &= 1 + q^2 + q^3 + q^4 + q^5 + q^6 + q^8. \end{aligned}$$

Open Problem 1.4. *Find a statistic on $ASM(n)$ whose generating function is $DPP(n, q)$.*

It is easy to see that

$$\begin{aligned} DPP(\infty, q) = \lim_{n\to\infty} DPP(n, q) =& \frac{1}{(1-q^2)(1-q^3)(1-q^4)(1-q^5)^2(1-q^6)^2\cdots} \\ =& \prod_{k=1}^{\infty} (1-q^{3k-1})^{-k}(1-q^{3k})^{-k}(1-q^{3k+1})^{-k} \\ =& \prod_{k=2}^{\infty} (1-q^k)^{-\lfloor (k+1)/3 \rfloor}. \end{aligned}$$

This infinite product may be rewritten using Cauchy's formula for Schur functions as

$$DPP(\infty, q) = \sum_{\lambda} s_\lambda(q^2, q^3, q^4, \cdots) s_\lambda(1, q^3, q^6, \cdots). \tag{1}$$

Open Problem 1.5. *Find a weight preserving bijection between $DPP(\infty)$ and pairs of column strict tableaux of the same shape which proves* (1)*.*

Open Problem 1.6. *Find a weight preserving bijection between $ASM(\infty)$ and pairs of column strict tableaux of the same shape which proves* (1)*. Restrict this bijection to find a bijection between $DPP(n)$ and $ASM(n)$, also a q-statistic for $ASM(n)$.*

Let G be the cyclic group of order 4 which acts by rotations on the set of $n \times n$ alternating sign matrices. It is known [36] that $(ASM(n), DPP(q), G)$ is an example of the cyclic sieving phenomenon. Thus $DPP(i)$ is the number of $n \times n$ alternating sign matrices fixed under a 90 degree rotation.

Open Problem 1.7. *Find an insightful (non computational) proof that*

$$(ASM(n), DPP(n, q), G)$$

is an example of the CSP.

Tom Sundquist [43] defined, for positive integers n and p,

$$\begin{aligned} A(n, p; q) =& \prod_{k=0}^{n-1} \frac{(np+k)!_q k!_q}{((p+1)k+p)!_q((p+1)k)!_q} \\ =& q^{-P} \frac{s_{(p\delta_n)'}(1, q, \cdots, q^{np-1})}{s_{p\delta_n}(1, q, \cdots, q^{n-1})}. \end{aligned}$$

where

$$\delta_n = (n-1, n-2, \cdots, 0), \quad P = \binom{p}{2} \sum_{i=1}^{n-1} i^2.$$

Sundquist proved this was always a polynomial in q with integer coeffcients, but did not prove positivity.

Open Problem 1.8. *Prove $A(n, p; q) \in N[q]$ if n and p are positive integers and what does $A(n, p; q)$ count?*

Note that the DPP and Catalan are both special cases, so positivity is known.

$$\begin{aligned} A(n,2;q) =& DPP(n,q) \\ A(2,p;q) =& \frac{1}{[p+1]_q}\begin{bmatrix} 2p \\ p \end{bmatrix} = Cat_p(q) \end{aligned}$$

Sundquist also gives a combinatorial interpretation for $A(\infty, p; q)$. For $p = 2$, Jessica Striker has noted that this result should be the following.

Proposition 1.9. *$DPP(\infty, q)$ is the generating function for all plane partitions T of the following type. For any i, the elements of the i^{th} column are $1, 2, \cdots, i-1$ or $i+1, i+4, i+7, \cdots$.*

Open Problem 1.10. *What restriction on the plane partitions T in Proposition 1.9 allows the generating function $DPP(n,q)$? Find a bijection between this class and any of the three known $ASM(n)$-equivalent objects.*

2. Eigenvalues of graphs

Let G be a finite simple graph with n vertices $\{v_1, \cdots, v_n\}$. The Laplacian matrix $L(G)$ is an $n \times n$ matrix whose entries are

$$L(G)_{ij} = \begin{cases} deg(v_i) \text{ if } i = j, \\ -1 \text{ if } \quad i \neq j \text{ and } v_i - v_j \text{ is an edge}, \\ 0 \text{ if } \quad i \neq j \text{ and } v_i - v_j \text{ is not an edge} \end{cases}$$

It is known that $L(G)$ is singular, diagonalizable, and positive semidefinite. So one eigenvalue of $L(G)$ is 0, and let $\lambda_1, \cdots, \lambda_{n-1}$ be the remaining non-negative eigenvalues. It is known that

$$\lambda_1 * \cdots * \lambda_{n-1} = e_{n-1}(\lambda_1, \cdots, \lambda_{n-1})$$

is the number of rooted spanning trees of G. Moreover the combinatorial interpretation of the coefficients of the characteristic polynomial of $L(G)$ shows that

$$e_{n-k}(\lambda_1, \cdots, \lambda_{n-1})$$

is the number of spanning forests of G consisting of k rooted trees.

Open Problem 2.1. *What is the combinatorial interpretation of the Schur function*

$$s_\mu(\lambda_1, \cdots, \lambda_{n-1})?$$

This is a non-negative integer, because of the Jacobi-Trudi identity and the non-negativity of the eigenvalues.

3. Ranks and Cranks

The *Dyson rank* [15] of an integer partition $\lambda = (\lambda_1, \lambda_2, \cdots)$

$$rank(\lambda) = \lambda_1 - \lambda_1'$$

(largest part- number of parts) proves the Ramanujan congruences

$$p(5n+4) \equiv 0 \mod 5, \quad p(7n+5) \equiv 0 \mod 7$$

by considering the rank modulo 5 and 7. No one knows bijections for these rank classes.

The generating function for the rank polynomial is known to be

$$\sum_{n=0}^{\infty} rank_n(z)q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq;q)_n(q/z;q)_n}$$

The rank generating function $rank_{5n+4}(z)$ for partitions of $5n+4$ does have an explicit factor of 5, but not positively. For example

$$\begin{aligned} rank_4(z) =& 1 + z^{-3} + z^{-1} + z^3 + z^1 = (1+z+z^2+z^3+z^4) * (1-z+z^2)/z^3, \\ rank_{14}(z) =& (1+z+z^2+z^3+z^4) * p(z)/z^{13} \end{aligned}$$

where $p(z)$ is an irreducible polynomial of degree 22 which has negative coeffcients. For an explicit 5-cycle which would be a rank class bijection, one would expect the factor $1+z+z^2+z^3+z^4$ times a positive Laurent polynomial in z. Here is a conjectured modification that does this.

Definition 3.1. *For $n \geq 2$ let*

$$Mrank_n(z) = rank_n(z) + (z^{n-2} - z^{n-1} + z^{2-n} - z^{1-n}).$$

Conjecture 3.2. *For $n \geq 0$,*

$$Mrank_{5n+4}(z)/(1+z+z^2+z^3+z^4)$$

is a non-negative Laurent polynomial in z. Also

$$Mrank_{7n+5}(z)/(1+z+z^2+z^3+z^4+z^5+z^6)$$

is a non-negative Laurent polynomial in z.

This conjecture says that the rank definition only needs to be changed for $\lambda = n, 1^n$ to have the "correct" symmetry. I do not know a modification which will also work modulo 11. Frank Garvan has verified Conjecture 3.2 for $5n+4 \leq 1000$ and $7n+5 \leq 1000$.

The Andrew-Garvan [5] *crank* of a partition λ is

$$AGcrank(\lambda) = \begin{cases} \lambda_1 \text{ if } \lambda \text{ has no 1's} \\ \mu(\lambda) - (\#1's \text{ in } \lambda) \text{ if } \lambda \text{ has at least one 1}, \end{cases}$$

where $\mu(\lambda)$ is the number of parts of λ which are greater than the number of 1's of λ. For example

$$\begin{aligned} AGcrank(1111) =& 0-4, \quad AGcrank(211) &= 0-2, \quad & AGcrank(22) = 2-0 \\ AGcrank(31) =& 1-1, \quad AGcrank(4) &= 4-0 \quad & . \end{aligned}$$

The generating function of the AGcrank over all partitions of n is $AGcrank_n(z)$. For example

$$AGcrank_4(z) = z^{-4} + z^{-2} + z^2 + z^0 + z^4.$$

The generating function for the AGcrank polynomial is known to be (after modifying $AGcrank_1(z)$)

$$\sum_{n=0}^{\infty} AGcrank_n(z)q^n = \frac{(q;q)_\infty}{(zq;q)_\infty(q/z;q)_\infty}$$

Open Problem 3.3. *Show*

$$AGcrank_{5n+4}(z) = (1+z^2+z^4+z^6+z^8) * (\textit{a positive Laurent polynomial in } z).$$

Frank Garvan has verified Open Problem 3.3 for $5n+4 \leq 1000$.

Ramanujan factored the first 21 AGcrank polynomials, $\lambda_n = AGcrank_n(a)$, see the paper of Berndt, Chan, Chan and Liaw [9, p. 12]. Ramanujan found the factor $\rho_5 = z^4 + z^{-4} + z^2 + z^{-2} + 1$ for $n = 4, 9, 14, 19$ but the other factors did not always have positive coefficients. For example Ramanujan had

$$AGcrank_{14}(z) = (z^4 + z^2 + 1 + z^{-2} + z^{-4}) * \rho_9 * (a_5 - a_3 + a_1 + 1),$$

where

$$\begin{aligned}\rho_9 * (a_5 - a_3 + a_1 + 1) =& (z^2 + z^{-2} + 1)(z^3 + z^{-3} + 1) * (z^5 + z^{-5} - z^3 - z^{-3} + z + z^{-1} + 1) \\ =& 3 + 1/z^{10} + 1/z^7 + 1/z^6 + 1/z^5 + 2/z^4 + 2/z^3 + 2/z^2 + 2/z \\ &+ 2z + 2z^2 + 2z^3 + 2z^4 + z^5 + z^6 + z^7 + z^{10}.\end{aligned}$$

A modified version of the AGcrank works for modulo 5, 7, and 11, with only the values at partitions $n, 1^n$ changed.

Definition 3.4. *For $n \geq 2$ let*

$$MAGcrank_{n,a}(z) = AGcrank_n(z) + (z^{n-a} - z^n + z^{a-n} - z^{-n}).$$

Conjecture 3.5. *The following are non-negative Laurent polynomials in z*

$MAGcrank_{5n+4,5}(z)/(1 + z + z^2 + z^3 + z^4)$,

$MAGcrank_{7n+5,7}(z)/(1 + z + z^2 + z^3 + z^4 + z^5 + z^6)$,

$MAGcrank_{11n+6,11}(z)/(1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7 + z^8 + z^9 + z^{10})$.

Frank Garvan has verified Conjecture 3.5 for $tn + r \leq 1000$.

The 5corecrank (see [23, 1990]) may be defined from the integer parameters $(a_0, a_1, a_2, a_3, a_4)$ involved in the 5-core of a partition λ. Its generating function for partitions of $5n+4$ is

$$\sum_{n=0}^{\infty} q^{n+1} \sum_{\lambda \vdash 5n+4} z^{5corecrank(\lambda)} = \frac{1}{(q;q)_\infty^5} \sum_{\substack{\vec{a}\cdot\vec{1}=1 \\ \vec{a}\in\mathbb{Z}^5}} q^{Q(a)} z^{\sum_{i=0}^4 i a_i}$$

where

$$Q(a) = \sum_{i=0}^{4} a_i^2 - \sum_{i=0}^{4} a_i a_{i+1}, \quad a_5 = a_0.$$

Frank Garvan also noted the following version of the previous conjectures holds for the 5corecrank for $n \leq 100$, and $n \leq 8$, see [7]. Ken Ono [33], in work with Bringmann and Rolen, has established the first statement.

Conjecture 3.6. *The following are non-negative Laurent polynomial in z*

$5corecrank_{5n+4}(z)/(1 + z + z^2 + z^3 + z^4)$,

$5corecrank_{5n+4,j}(z)/(1 + z + z^2 + z^3 + z^4)$ *when restricted to $BGcrank = j$.*

4. The Borwein and Bressoud conjectures

The Borwein conjecture ([1], proven by Chen Wang [44] in 2019, see also [10]) is the following positivity conjecture.

Let

$$(q;q^3)_n(q^2;q^3)_n = A_n(q^3) - qB_n(q^3) - q^2C_n(q^3)$$

for polynomials $A_n(q), B_n(q), C_n(q)$. Then $A_n(q), B_n(q), C_n(q)$ have non-negative coefficients.

There are explicit alternating forms for these polynomials

$$A_n(q) = \sum_k \begin{bmatrix} 2n \\ n-3k \end{bmatrix}_q (-1)^k q^{(9k^2-k)/2}$$
$$B_n(q) = \sum_k \begin{bmatrix} 2n \\ n-3k-1 \end{bmatrix}_q (-1)^k q^{(9k^2+5k)/2}$$
$$C_n(q) = \sum_k \begin{bmatrix} 2n \\ n-3k+1 \end{bmatrix}_q (-1)^k q^{(9k^2-7k)/2}$$

Note that $A_n(1) = 2 * 3^{n-1}$, $B_n(1) = 3^{n-1} = C_n(1)$.

The $n = \infty$ case follows from the Jacobi triple product

$$\begin{aligned} A_\infty(q) &= \frac{(q^4, q^5, q^9; q^9)_\infty}{(q;q)_\infty}, \\ B_\infty(q) &= \frac{(q^7, q^2, q^9; q^9)_\infty}{(q;q)_\infty}, \\ C_\infty(q) &= \frac{(q^8, q^1, q^9; q^9)_\infty}{(q;q)_\infty}. \end{aligned} \tag{2}$$

Open Problem 4.1. *Identify finite subsets of partitions whose parts are restricted modulo 9 via* (2) *whose generating functions are* $A_n(q), B_n(q), C_n(q)$.

It is known that the *hook difference polynomials* do have positive coefficients and count certain partitions which lie inside a rectangle (see [4]).

Let $N, M, i, K, \alpha, \beta$ be postive integers such that

$$\alpha + \beta < K, \quad -i + \beta \le N - M \le K - i - \alpha.$$

Then the hook difference polynomials are

$$\begin{aligned} D_{K,i}(N, M, \alpha, \beta) = &\sum_\lambda q^{\lambda(K\lambda+i)(\alpha+\beta)-K\beta\lambda} \begin{bmatrix} N+M \\ N-K\lambda \end{bmatrix}_q \\ &- \sum_\lambda q^{\lambda(K\lambda-i)(\alpha+\beta)-K\beta\lambda+\beta i} \begin{bmatrix} N+M \\ N-K\lambda+i \end{bmatrix}_q \end{aligned}$$

A special case is

$$D_{2k,k}(N, N, \alpha, \beta) = \sum_s (-1)^s \begin{bmatrix} 2N \\ N-ks \end{bmatrix}_q q^{ks(s+1)(\alpha+\beta)/2-\beta ks} \tag{3}$$

for

$$\alpha + \beta < 2k, \quad -k + \beta \le 0 \le k - \alpha.$$

The Borwein polynomial $A_n(q) = D_{6,3}(N, N, 4/3, 5/3)$.

Open Problem 4.2. *What is the combinatorial meaning of the rational parameters* $\alpha = 4/3, \beta = 5/3$?

Bressoud [11] investigated this question and formulated a more general conjecture for rational parameters (his Conjecture 6).

Conjecture 4.3. *Let α and β be positive rational numbers, and let $k>1$ be an integer such that αk and βk are integers. If*

$$\begin{aligned} 1 \le \alpha+\beta \le 2k-1, &\quad (\text{ with strict inequalities for } k=2) \\ -k+\beta \le n-m \le k-\alpha & \end{aligned}$$

then $D_{2k,k}(n,m,\alpha,\beta)$ has non-negative coefficients.

There is also a corresponding Borwein type conjecture for special cases of these polynomials (see [11, Conjecture 5]). If k is odd, $1 \le a < k/2$, let

$$(q^a;q^k)_m(q^{k-a};q^k)_n = \sum_{\nu=(1-k)/2}^{(k-1)/2} (-1)^\nu q^{k(\nu^2+\nu)/2-a\nu} F_\nu(q^k)$$

then

$$F_\nu(q) = G_{2k,k}(n,m,\nu+(k+1)/2-a/k, -\nu+(k-1)/2+a/k).$$

Conjecture 4.4. *If a is relatively prime to k and $m=n$, then the coefficients of $F_\nu(q)$ are non-negative.*

The Borwein conjecture is the case $k=3$, $a=1$.

Conjecture 4.4 says that the coefficients of $q^p, p \equiv a\nu \mod k$ in

$$(q^a;q^k)_n(q^{k-a};q^k)_n$$

all have sign $(-1)^\nu$.

The refined Borwein conjecture [1, (1.5)] for the coeffcients of z^t in

$$(q,q^2;q^3)_m(zq,zq^2;q^3)_n$$

has been proven false in general by Yee, see [46].

If $q=1$ there is polynomial version [26] which replaces the sign $(-1)^s$ in (3) by a Chebychev polynomial.

Theorem 4.5. *If $|N-M| \le k$, then*

$$\sum_s \binom{N+M}{M-ks} \cos(sx)$$

is a positive polynomial in $1+\cos(x)$, and thus is positive for a real value of x.

Open Problem 4.6. *Find a q-version of this result which contains Bressoud's conjecture. See* [26].

5. Assorted q-binomial questions

In [39, Theorem 1] it was proven that

$$\frac{1}{n!_q} = \frac{(1-q)^n}{(q;q)_n} \tag{4}$$

has alternating power series coefficients.

Open Problem 5.1. *What is the algebraic meaning in terms of Koszul duality of this result?*

A generalization [39, Theorem 2] was given for the alternating behavior of

$$(1-q)^n \begin{bmatrix} n+k \\ k \end{bmatrix}_q \tag{5}$$

where nk is even.

Open Problem 5.2. *What is the algebraic meaning in terms of Koszul duality of this result?*

Open Problem 5.3. *Find sign-reversing involutions which prove* (4) *and* (5) *have alternating coefficients.*

In [20] a polynomial expansion for the q-binomial coefficient is given

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\omega \in \Omega_{n,k}} q^{s(\omega)} (1+q)^{t(\omega)}.$$

The set $\Omega_{n,k}$ is a subset of the set of all words with k 1's and $n-k$ 0's.

Open Problem 5.4. *Is there an apriori definition of $\Omega_{n,k}, s(\omega), t(\omega)$ using coset representatives or root systems?*

Franklin [2, Ex. 13-14, p. 50] had a generalization of the q-binomial coefficient $\begin{bmatrix} m+k \\ k \end{bmatrix}_q$ being the generating function for partitions with at most k parts, largest part at most m.

Theorem 5.5. *(Franklin) Let $1 \le j \le k$. The generating function for all partitions λ with at most k parts such that $\lambda_1 - \lambda_{j+1} \le m$ is*

$$\frac{(1-q^{m+1}) \cdots (1-q^{m+j})}{(1-q) \cdots (1-q^k)}.$$

This has an inductive proof by a sign-reversing involution.

Open Problem 5.6. *What is the analogue of Franklin's Theorem 5.5 for the MacMahon box theorem? Is there a result for each symmetry class of plane partitions?*

The KOH identity [25, 47] for the q-binomial coefficient

$$\begin{bmatrix} n+k \\ k \end{bmatrix}_q = \sum_{\lambda \vdash k} q^{2n(\lambda)} \prod_{i=0}^{k-1} \begin{bmatrix} (k-i)n - 2i + d_{k-i} + \sum_{j=0}^{i-1} 2(i-j) d_{k-j} \\ d_{k-i} \end{bmatrix}_q \tag{6}$$

where $\lambda = 1^{d_1} 2^{d_2} \cdots$ is a partition of k, proves unimodality of the q-binomial coefficient as a polynomial in q. Stanley [38] proved a stronger theorem for Weyl groups, which implies that $(-q;q)_n$ is unimodal polynomial in q.

Open Problem 5.7. *Find a KOH-type identity for $(-q;q)_n = \prod_{i=1}^n (1+q^i)$.*

The KOH identity was combinatorially proved under the assumption that

$$\begin{bmatrix} N \\ s \end{bmatrix}_q = 0$$

if $s \geq 0$ and $N < 0$. Macdonald [32] proved a version, called MACKOH, which assumes the definition

$$\begin{bmatrix} N \\ s \end{bmatrix}_q = \frac{(1-q^N)(1-q^{N-1})\cdots(1-q^{N-s+1})}{(1-q)(1-q^2)\cdots(1-q^s)}$$

for all $s \geq 0$, and all N. In this version, both sides of (6) are polynomials in $x = q^n$ of degree k, true for infinitely many x. Thus (KOH) implies (MACKOH).

Open Problem 5.8. *Find an involution which proves that the (MACKOH) implies the (KOH).*

For example, if $k = 4$, then the KOH identity is

$$\begin{bmatrix} n+4 \\ 4 \end{bmatrix}_q = \begin{bmatrix} 4n+1 \\ 1 \end{bmatrix}_q + q^2 \begin{bmatrix} 3n-1 \\ 1 \end{bmatrix}_q \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q + q^4 \begin{bmatrix} 2n-2 \\ 2 \end{bmatrix}_q + q^6 \begin{bmatrix} 2n-3 \\ 1 \end{bmatrix}_q \begin{bmatrix} n-2 \\ 2 \end{bmatrix}_q + q^{12} \begin{bmatrix} n-2 \\ 4 \end{bmatrix}_q .$$

What is the involution, assuming (MACKOH), which shows that terms with negative parameters may be dropped?

This appears to be related to the $M = N$ conjecture in quantum integrable systems [17, Conj. 2.8].

Gessel [24] defined a collection of Super Catalan numbers which are integers

$$T(m,n) = \frac{(2n)!(2m)!}{n!m!(m+n)!}.$$

Combinatorial interpretations have been given for m small or $n-m$ small, and also signed versions for all m, n [6].

Sundquist generalized this result by considering

$$T(a_1, a_2, \cdots, a_k; q) = \frac{\prod_{j=1}^{k} (2^{k-1} a_j)!_q}{\prod_{S \subset [k]} a_S!_q},$$

where

$$a_S = \sum_{s \in S} a_s$$

and the product in the denominator does not include $S = [k]$. The Super Catalan numbers are the case $k = 2$, $a_1 = m$, $a_2 = n$. He proved $T(a_1, a_2, \cdots, a_k; q)$ is a polynomial in q. The positivity for the q-super Catalan numbers was established in [45, Prop. 2].

Open Problem 5.9. *Prove $T(a_1, a_2, \cdots, a_k; q) \in N[q]$ and find a combinatorial interpretation for this generating function. Is there a result for other posets?*

6. Finite fields

If the cyclic group of order n acts on the set of k-element subsets of $[n]$, the number of orbits is of size e is

$$O(n,k,e) = \frac{1}{e} \sum_{d|s|GCD(k,n)} \mu\left(\frac{s}{d}\right) \binom{n/s}{k/s}.$$

A q-version of this result was given by Drudge [14], [35, Prop. 9.2]. The number of orbits of the Singer cycle c in $GL_n(\mathbb{F}_q)$ on the k-dimensional subspaces of size $[e]_{q^d}$, $n = de$ is

$$O(n,k,e;q) = \frac{1}{[e]_{q^d}} \sum_{d|s|GCD(k,n)} \mu\left(\frac{s}{d}\right) \begin{bmatrix} n/s \\ k/s \end{bmatrix}_{q^s}.$$

The special case $e = n$ is attractive

$$O(n,k,n;q) = \frac{1}{[n]_q} \sum_{s|GCD(k,n)} \mu(s) \begin{bmatrix} n/s \\ k/s \end{bmatrix}_{q^s}.$$

In [35, Conj. 10.3] the polynomiality in q of this number is proven, but not positivity.

Open Problem 6.1. *Prove $O(n,k,n;q) \in N[q]$ and find a combinatorial interpretation for this generating function.*

When $GCD(n,k) = 1$ this is

$$O(n,k,n;q) = \frac{1}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q$$

and this has a combinatorial interpretation.

In [30, Theorem 1.1], using the complex irreducible characters of $GL_n(\mathbb{F}_q)$, the number of factorizations of the Singer cycle c into n reflections was found to be

$$(q^n - 1)^{n-1}.$$

No simple proof is known. It is the q-version of the number of factorizations of an n-cycle into transpositions being n^{n-2}.

Open Problem 6.2. *Find a direct bijection for this result.*

More generally [30, Theorem 1.2], the generating function for the number of factorizations, $t(c,\ell)$, of the Singer cycle c into ℓ reflections is

$$\sum_{\ell=n}^{\infty} t(c,\ell)x^\ell = (q^n-1)^{n-1} \frac{x^n}{1+x[n]_q} \prod_{k=0}^{n-1} (1 + x[n]_q(1+q^k-q^{k+1}))^{-1}.$$

Open Problem 6.3. *Find an apriori reason for the rationality of this generating function. Do the zeros in the denominator have geometric meaning?*

Let $S = \mathbb{F}_q[x_1, \cdots, x_n]$ be the polynomial ring on which $GL_n(\mathbb{F}_q)$ naturally acts. Let

$$Q_m = S/ < x_1^{q^m}, \cdots, x_n^{q^m} > .$$

on which $GL_n(\mathbb{F}_q)$ also acts. Then [31, Conj. 1.2]

Conjecture 6.4. *The Hilbert series for the $GL_n(\mathbb{F}_q)$ fixed subalgebra of Q is*

$$\sum_{k=0}^{min(m,n)} t^{(n-k)(q^m-q^k)} \begin{bmatrix} m \\ k \end{bmatrix}_{q,t}.$$

In [34] and [40] some results are given for a theory of partitions and plane partitions whose parts sizes are $[n]_q$ instead of integers n.

Open Problem 6.5. *Can these results be extended to other classical partition results?*

7. Rogers-Ramanujan identities

There are known polynomial idenitites which generalize the Rogers-Ramanujan (RR) identities

$$\sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k} = \frac{1}{(q,q^4;q^5)_\infty}, \qquad \sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q;q)_k} = \frac{1}{(q^2,q^3;q^5)_\infty}.$$

Schur knew that

$$D_n(q) = \sum_{k=0}^{(n+1)/2} q^{k^2} \begin{bmatrix} n+1-k \\ k \end{bmatrix}_q, \qquad E_n(q) = \sum_{k=0}^{n/2} q^{k^2+k} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \tag{7}$$

had alternative representations

$$D_{n-1}(q) = \sum_j (-1)^j q^{j(5j+1)/2} \begin{bmatrix} n \\ [(n+5j+1)/2] \end{bmatrix}_q,$$
$$E_n(q) = \sum_j (-1)^j q^{j(5j+3)/2} \begin{bmatrix} n+1 \\ [(n+5j+3)/2] \end{bmatrix}_q.$$

(Note that both satisfy the q-Fibonacci recurrence $p_n = p_{n-1} + q^n p_{n-2}$.) Using the Jacobi-Triple-Product on $D_\infty(q)$, $E_\infty(q)$, one obtains the RR identities. Easily (7) shows that $D_n(q)$ is the generating function for all partitions λ whose difference of parts is at least 2, and $\lambda_1 \le n$.

Open Problem 7.1. *Which subset of partitions whose parts are restricted modulo 5 do the polynomials in* (7) *generate?*

Bressoud [12, (1.1), (1.3)], [37, Sec. 6] gave another polynomial identity

$$\begin{aligned} \sum_{j=0}^{n} q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix}_q &= \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2} \begin{bmatrix} 2n \\ n+2j \end{bmatrix}_q \\ \sum_{j=0}^{n} q^{j^2+j} \begin{bmatrix} n \\ j \end{bmatrix}_q &= \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+3)/2} \begin{bmatrix} 2n+1 \\ n+2j+1 \end{bmatrix}_q \end{aligned} \tag{8}$$

This time the left side generates difference two partitions λ with $rank(\lambda) \le n-1$.

Open Problem 7.2. *Which subset of partitions whose parts are restricted modulo 5 do the polynomials in* (8) *generate?*

Ekhad-Tre [16] also found

$$\sum_{j=0}^{n} q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{2n}} \sum_{k=-n}^{n} (-1)^k q^{(5k^2-k)/2} \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q$$

Sills [37, Sec. 6] also gives polynomial identities for

$$\sum_{j=0}^{n/2} q^{j^2} \begin{bmatrix} n \\ 2j \end{bmatrix}_q.$$

There is a quintic transformation [21, Th. 7.1] which proves the RR identities

$$\sum_{n=0}^{\infty}\frac{q^{n^2}(tq)^{2n}}{(q;q)_n} = \frac{(t^4q^9,t^2q^5,t^4q^6;q^5)_\infty}{(t^2q^3;q)_\infty}\ {}_3\phi_2\left(\begin{matrix} t^2q^2, & t^2q^3, & t^2q^5\\ & t^4q^9, & t^4q^6\end{matrix}\,\middle|\, q^5;t^2q^5\right),$$
$$(9)\qquad = \frac{(t^4q^8,t^2q^6,t^4q^6;q^5)_\infty}{(t^2q^3;q)_\infty}\ {}_3\phi_2\left(\begin{matrix} t^2q, & t^2q^3, & t^2q^4\\ & t^4q^8, & t^4q^6\end{matrix}\,\middle|\, q^5;t^2q^6\right).$$

The only known proof of (9) uses orthogonal polynomials.

Open Problem 7.3. *Find another proof of* (9). *What does it mean combinatorially, or for representations of* $A_1^{(1)}$. *Is there a symmetric function version?*

One may show that (9) implies a finite rational function identity which is nearly positive

(10)

$$\sum_{k=0}^{n}\frac{q^{k^2}}{(q;q)_k} =$$
$$\sum_{\substack{a,b,c,k,s\\ a+b+c+k+s\le n}} \frac{q^{5ab+a(5k+1)+3b}}{(q^5;q^5)_a(q^5;q^5)_b}\frac{q^{5c(n-(a+b+c+k+s))+c(5k+2)}}{(q^5;q^5)_c(q^5;q^5)_{n-(a+b+c+k+s)}}\begin{bmatrix}k\\ s\end{bmatrix}_{q^5}\frac{q^{5\binom{s}{2}-s}(-1)^s q^{4k}}{(q^5;q^5)_k}.$$

No direct bijection for the Rogers-Ramanujan identities is known, although the involution principle of Garsia and Milne [22] was created to give an indirect bijection.

Open Problem 7.4. *Can this identity be mutated to one with only positive terms, and thereby lead to a direct RR bijection?*

Using the Cauchy identity, whose bijective version is RSK, one obtains

$$\sum_{\lambda} s_\lambda(xq,xq^2)s_\lambda(q^1,q^6,q^{11},\cdots,) = \frac{1}{(xq^2,xq^3;q^5)_\infty}.$$

Because the Schur function with 2 variables is zero unless the partition has at most two parts, one may rewrite this as

$$\frac{1}{(xq^2,xq^3;q^5)_\infty} = \sum_{N=0}^{\infty} x^N\frac{q^{2N}}{(q^5;q^5)_N}\sum_{b=0}^{[N/2]} q^b[N-2b+1]_q\left(\begin{bmatrix}N\\ b\end{bmatrix}_{q^5} - \begin{bmatrix}N\\ b-1\end{bmatrix}_{q^5}\right).$$

All terms here are positive. May this be extended to a proof of RR?

In [29] refinements of the Rogers-Ramanujan identities are given by marking parts. These are based upon some sporadic positive rational function identities.

Open Problem 7.5. *Can these identities be generated via computer algebra? Are they related to decompositions of polytopes, or Hilbert series in commutative algebra?*

8. Other questions

Type R_I and R_{II} orthogonal polynomials [27] satisfy the respective recurrence relations

$$P_n(x) = (x-b_n)P_{n-1}(x) - \lambda_n(x-a_n)P_{n-2}(z),$$

with the initial conditions

$$P_0(x) = 1, \quad P_1(x) = x - b_1.$$

and

$$Q_n(x) = (x - c_n)Q_{n-1}(x) - \lambda_n(x - a_n)(x - b_n)Q_{n-2}(z),$$

with the initial conditions

$$Q_0(x) = 1, \quad Q_1(x) = x - c_1.$$

There are linear functionals L_1 and L_2, defined on the appropriate vector space of rational functions, such that

$$L_1\left(x^j P_n(x)/\prod_{k=1}^{n}(x - a_{k+1})\right) = 0, \quad 0 \le j < n, \quad L_1(1) = \lambda_1$$

and

$$L_2\left(x^j Q_n(x)/\prod_{k=1}^{n}(x - a_{k+1})(x - b_{k+1})\right) = 0, \quad 0 \le j < n.$$

Open Problem 8.1. *Develop a Viennot theory for type R_I and R_{II} polynomials.*

(Note: September 12, 2019: Jang Soo Kim and I have done this for type R_I.)

Open Problem 8.2. *Does a $GL_n(\mathbb{F}_q)$ version of the cycle index generating function easily count involutions in $GL_n(\mathbb{F}_q)$ (see* [18]*) or explain the competing q-versions of the Poisson distribution* [19]*? Are there separate q-Central Limit Theorems for the discrete and continuous q-Hermite polynomials?*

A perfect Hamming 1-code is a subset S of the vertices of the n-dimensional cube $X_n = \{0, 1\}^n$ so that the balls of radius one about points of S are disjoint and cover X_n. Clearly for this to occur, $n + 1$ divides 2^n, so n must be one less than a power of two. Such perfect codes are known to exist.

The q-analogue of the Hamming scheme is a graph whose vertices are the maximal isotropic subspaces over $\mathbb{F}_q$, with edges if they overlap maximally. In types B_n and C_n it is known that there are $(1+q)(1+q^2) * \cdots * (1+q^n)$ such vertices, and the ball of radius 1 has size $(1-q^{n+1})/(1-q)$. Again the sphere packing condition implies that $n = 2^k - 1$ for some k. It is known that such perfect codes exist for $n = 3$, but it is unknown for $n \ge 7$.

Open Problem 8.3. *Settle the existence/non-existence question of perfect codes in the association schemes of dual polar spaces of types B_n and C_n for $n = 2^k - 1$, $k \ge 3$. See (*[41, §8]*.)*

The continuous q-Hermite polynomials $p_n(x)$ satisfy

$$p_{n+1}(x) = xp_n(x) - [n]_q p_{n-1}(x)$$

while the discrete q-Hermite polynomials $r_n(x)$ satisfy

$$r_{n+1}(x) = xr_n(x) - q^{n-1}[n]_q r_{n-1}(x).$$

There is another set of q-Hermite polynomials $s_n(x)$ which satisfy

$$s_{n+1}(x) = xs_n(x) - \frac{q^{-n} - q^n}{q^{-1} - q^1} s_{n-1}(x). \tag{11}$$

These polynomials remarkably have linearization formula

$$s_n s_m = \sum_{k=0}^{min(m,n)} c_{nm}^k s_k,$$

$$c_{nm}^k = \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} m \\ k \end{bmatrix}_q k!_q \frac{(-q^{m+n+1-2k};q)_k}{(1+q)^k} q^{-k/2(2m+2n-1-3k)},$$

where c_{nm}^k can be proven to be a non-negative polynomial in q. A combinatorial interpretation of the moments for the corresponding indeterminate moment problem is known.

Open Problem 8.4. *Find any of the following information about $s_n(x)$: an explicit formula, generating function, or measure. Is there an Askey scheme with these polynomials at the bottom?*

References

[1] G. Andrews, On a conjecture of Peter Borwein. J. Symbolic Comput. 20 (1995), no. 5-6, 487-501.

[2] G. Andrews, The Theory of Partitions. Addison-Wesley, 1976.

[3] G. Andrews, Plane partitions. III. The weak Macdonald conjecture. Invent. Math. 53 (1979), no. 3, 193-225.

[4] G. Andrews, R. Baxter, D. Bressoud, W. Burge, P. Forrester, and G. Viennot, Partitions with prescribed hook differences. European J. Combin. 8 (1987), no. 4, 341-350

[5] G. Andrews, and F. Garvan, Dyson's crank of a partition. Bull. Amer. Math. Soc. (N.S.) 18 (1988), no. 2, 167-171.

[6] E. Allen, and I. Gheorghiciuc, A weighted interpretation for the super Catalan numbers. J. Integer Seq. 17 (2014), no. 10,

[7] A. Berkovich and F. Garvan, The BG-rank of a partition and its applications. Adv. in Appl. Math. 40 (2008), no. 3, 377-400.

[8] A. Berkovich, and F. Garvan, On the Andrews-Stanley refinement of Ramanujan's partition congruence modulo 5 and generalizations. Trans. Amer. Math. Soc. 358 (2006), no. 2, 703-726.

[9] B. Berndt, H. H. Chan, S. H. Chan, and W.-C. Liaw, Cranks- really, the final problem. Ramanujan J. 23 (2010), no. 1-3, 3-15.

[10] G. Bhatnagar and M. Schlosser, A partial theta function Borwein conjecture. `https://arxiv.org/abs/1902.04447`

[11] D. Bressoud, The Borwein conjecture and partitions with prescribed hook differences. The Foata Festschrift. Electron. J. Combin. 3 (1996), no. 2, Research Paper 4

[12] D. Bressoud, Some identities for terminating q-series. Math. Proc. Cambridge Philos. Soc. 89 (1981), no. 2, 211-223.

[13] D. Bressoud, Proofs and confirmations. The story of the alternating sign matrix conjecture. MAA Spectrum. Mathematical Association of America, Washington, DC; Cambridge University Press, Cambridge, 1999.

[14] K. Drudge, On the orbits of Singer groups and their subgroups. Electron. J. Combin. 9 (2002), no. 1, Research Paper 15, 10 pp.

[15] F. J. Dyson, Some guesses in the theory of partitions, Eureka (Cambridge) 8 (1944), 10-15.

[16] S. B. Ekhad and S. Tre, A purely verification proof of the first Rogers-Ramanujan identity. J. Combin. Theory Ser. A 54 (1990), no. 2, 309-311.

[17] Ph. Di Francesco, and R. Kedem, Proof of the combinatorial Kirillov-Reshetikhin conjecture. Int. Math. Res. Not. IMRN 2008, no. 7, Art. ID rnn006, 57 pp.

[18] J Fulman R. Guralnick, and D. Stanton, Asymptotics of the number of involutions in finite classical groups. J. Group Theory 20 (2017), no. 5, 871-902.

[19] J. Fulman and D. Stanton, On the distribution of the number of fixed vectors for the finite classical groups. Ann. Comb. 20 (2016), no. 4, 755-773.

[20] S. Fu, V. Reiner, D. Stanton, and N. Thiem, The negative q-binomial. Electron. J. Combin. 19 (2012), no. 1, Paper 36, 24 pp.

[21] K. Garrett, M. E. H. Ismail, and D. Stanton, Variants of the Rogers-Ramanujan identities. Adv. in Appl. Math. 23 (1999), no. 3, 274-299

[22] A. Garsia and S. Milne, Method for constructing bijections for classical partition identities. Proc. Nat. Acad. Sci. U.S.A. 78 (1981), no. 4, part 1, 2026-2028.

[23] F. Garvan, D. Kim, and D. Stanton, Cranks and t-cores. Invent. Math. 101 (1990), no. 1, 1-17.

[24] I. M. Gessel, Super ballot numbers, J. Symbolic Comput. 14 (1992), no. 2-3, 179-194;

[25] F. Goodman and K. O'Hara, On the Gaussian polynomials. q-series and partitions, 57-66, IMA Vol. Math. Appl., 18, Springer, New York, 1989.

[26] M. E. H. Ismail, D. Kim, and D. Stanton, D. Lattice paths and positive trigonometric sums. Constr. Approx. 15 (1999), no. 1, 69-81.

[27] M. E. H. Ismail and D. Masson, Generalized orthogonality and continued fractions. J. Approx. Theory 83 (1995), no. 1, 1-40.

[28] W. Mills, D. Robbins, and H. Rumsey, Proof of the Macdonald conjecture. Invent. Math. 66 (1982), no. 1, 73-87.

[29] K. O'Hara and D. Stanton, Refinements of the Rogers-Ramanujan identities. Exp. Math. 24 (2015), no. 4, 410-418.

[30] J. B. Lewis, V. Reiner, and D. Stanton, Reflection factorizations of Singer cycles. J. Algebraic Combin. 40 (2014), no. 3, 663-691.

[31] J. Lewis, V. Reiner, and D. Stanton, Invariants of $GL_n(\mathbb{F}_q)$ in polynomials modulo Frobenius powers. Proc. Roy. Soc. Edinburgh Sect. A 147 (2017), no. 4, 831-873.

[32] I. G. Macdonald, An elementary proof of a q-binomial identity. q-series and partitions (Minneapolis, MN, 1988), 73-75, IMA Vol. Math. Appl., 18, Springer, New York, 1989.

[33] K. Ono, personal communication, July, 2019.

[34] V. Reiner and D. Stanton, (q,t)-analogues and $GL_n(\mathbb{F}_q)$. J. Algebraic Combin. 31 (2010), no. 3, 411-454.

[35] V. Reiner, D. Stanton, and D. White, The cyclic sieving phenomenon. J. Combin. Theory Ser. A 108 (2004), no. 1, 17-50.

[36] V. Reiner, D. Stanton, and D. White, What is ... cyclic sieving? Notices Amer. Math. Soc. 61 (2014), no. 2, 169-171.

[37] A. Sills, Finite Rogers-Ramanujan type identities. Electron. J. Combin. 10 (2003), Research Paper 13, 122 pp.

[38] R. Stanley, Weyl groups, the hard Lefschetz theorem, and the Sperner property. SIAM J. Algebraic Discrete Methods 1 (1980), no. 2, 168-184.

[39] D. Stanton and D. Zeilberger, The Odlyzko conjecture and O'Hara's unimodality proof. Proc. Amer. Math. Soc. 107 (1989), no. 1, 39-42.

[40] D. Stanton, q-analogues of Euler's odd = distinct theorem. Ramanujan J. 19 (2009), no. 1, 107-113.

[41] D. Stanton, Some q-Krawtchouk polynomials on Chevalley groups. Amer. J. Math. 102 (1980), no. 4, 625-662.

[42] X. Sun and V. Moll, The p-adic valuations of sequences counting alternating sign matrices J. Integer Sequences 12 (2009), Article 09.3.8

[43] T. Sundquist, Pfaffians, involutions, and Schur functions. Thesis (Ph.D.), University of Minnesota. 1992. 76 pp.

[44] C. Wang, An analytic proof of the Borwein conjecture, . `https://arxiv.org/pdf/1901.10886.pdf`

[45] S. Ole Warnaar and W. Zudilin, A q-rious positivity. Aequationes Math. 81 (2011), no. 1-2, 177-183.

[46] Ae Ja Yee, personal communication, 2014.

[47] D. Zeilberger, A one-line high school algebra proof of the unimodality of the Gaussian polynomials $\begin{bmatrix} n \\ k \end{bmatrix}$ for $k < 20$. q-series and partitions, 67-72, IMA Vol. Math. Appl., 18, Springer, New York, 1989.

[48] D. Zeilberger, Proof of the alternating sign matrix conjecture. The Foata Festschrift. Electron. J. Combin. 3 (1996), no. 2, Research Paper 13, approx. 84 pp.

School of Mathematics, University of Minnesota, Minneapolis, MN 55455
E-mail address: `stanton@math.umn.edu`