ORTHOGONAL POLYNOMIALS AND SMITH NORMAL FORM

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ABSTRACT. Smith normal form evaluations found by Bessenrodt and Stanley for some Hankel matrices of q-Catalan numbers are proven in two ways. One argument generalizes the Bessenrodt–Stanley results for the Smith normal form of a certain multivariate matrix that refines one studied by Berlekamp, Carlitz, Roselle, and Scoville. The second argument, which uses orthogonal polynomials, generalizes to a number of other Hankel matrices, Toeplitz matrices, and Gram matrices. It gives new results for q-Catalan numbers, q-Motzkin numbers, q-Schröder numbers, q-Stirling numbers, q-matching numbers, q-factorials, q-double factorials, as well as generating functions for permutations with eight statistics.

1. Introduction

In [3] Bessenrodt and Stanley gave a Smith normal form evaluation for a certain matrix that generalizes one studied by Berlekamp [1, 2], and Carlitz, Roselle, and Scoville [4]. They specialized this result to give a Smith normal form result on Hankel matrices of q-Catalan numbers. These evaluations use induction and elementary row and column operations. In §5 we give a short direct combinatorial argument which generalizes the results in [3]. But the main purpose of the present paper is to put the Hankel results into the combinatorial framework of orthogonal polynomials. This combinatorial theory developed over the last 30 years immediately implies Bessenrodt and Stanley's two Hankel evaluations as well as many new ones, see §4.

The main new results in this paper are

(1) Theorem 1 for the Smith normal form of Hankel matrices of moments of orthogonal polynomials,
(2) Theorem 5 for the Smith normal form of Toeplitz matrices of moments of biorthogonal polynomials,
(3) Theorem 6 for the Smith normal form of a rank matrix of a lattice.

2. Definitions

Let A be an m-by-n matrix with entries in a commutative ring R.

2.1. We say that A has *Smith normal form* (or *SNF* for short) D over R if

(a) $PAQ = D$ for some $P \in \mathrm{GL}(m, R)$ and $Q \in \mathrm{GL}(n, R)$,
(b) D is a diagonal ($m \times n$) matrix in the sense that $D_{ij} = 0$ for $i \neq j$,
(c) d_{ii} is a multiple of d_{jj} whenever $i \geq j$.

Most of the time R will be a unique factorization domain such as $\mathbf{Z}[q]$ so that the SNF of A is unique up to units if it exists [19, Prop. 8.1]. Existence is guaranteed for $R = \mathbf{Z}$ or any other principal ideal domain, but not for other types of unique

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factorization domains. For example if $R = \mathbf{Z}[q]$, then a diagonal matrix of the form $\mathrm{diag}(q + a_1, q + a_2, \ldots, q + a_n)$ $(a_i \in \mathbf{Z})$ admits a Smith normal form if and only if the a's are chosen from a set of two consecutive integers [19, Prop. 8.9].

2.2. If A is square-shaped with Smith normal form D, then $\det A$ equals $d_1 d_2 \ldots d_n$ up to a unit factor in R. For example if $R = \mathbf{Z}[q]$, then $\det A = \pm d_1 d_2 \ldots d_n$. Call D a *special Smith normal form* (*SSNF*) of A over R if in addition to (a)–(c) it holds that

(a′) $PAQ = D$ for some $P \in \mathrm{SL}(m, R)$ and $Q \in \mathrm{SL}(n, R)$.

Proposition 1. *A has SSNF over $R \Leftrightarrow A$ has SNF over R. If A has $n \times n$ SSNF D, then*

$$\det A = \det D = d_1 d_2 \ldots d_n. \tag{1}$$

Proof. If A has Smith normal form D and P, Q satisfy (a), then scaling the first row of D by $\det P^{-1} \det Q^{-1}$ gives SSNF D' of A. The other implications are clear. □

2.3. A number of well-studied determinant evaluations in combinatorics can be sharpened into interesting Smith normal form evaluations over the rings $\mathbf{Z}[q]$ and $\mathbf{Z}[x_1, x_2, \ldots, x_n]$. But there seems to be no generic explanation why certain matrices admit a Smith normal form. Each one uses a different trick. Bessenrodt and Stanley gave two recent examples (Corollary 1 below). They refine

$$\det (C_{i+j})_{0 \le i,j \le n} = 1 \quad \text{and} \quad \det (C_{i+j+1})_{0 \le i,j \le n} = 1$$

by first replacing the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ with the q-Catalan numbers $C_n(q)$ below in (8), and then giving Smith normal form evaluations over $\mathbf{Z}[q]$ for the Hankel matrices $(C_{i+j}(q))$ and $(C_{i+j+1}(q))$. The determinants of these two q-Hankel matrices are not new. They are well known in the combinatorial study of orthogonal polynomials, and Theorem 1 tells us that Bessenrodt and Stanley's Smith normal form evaluations are completely elucidated by the combinatorics of orthogonal polynomials as well.

3. SNF of Hankel matrices of moments of orthogonal polynomials

Take two sequences $b = (b_0, b_1, \ldots)$ and $\lambda = (\lambda_1, \lambda_2, \ldots)$ in the commutative ring R. Define $p_0(x), p_1(x), \ldots$ in $R[x]$ by the classical three-term recurrence relation

$$p_{n+1}(x) = (x - b_n) p_n(x) - \lambda_n p_{n-1}(x), \quad p_{-1}(x) = 0, p_0(x) = 1. \tag{2}$$

The p_n's are orthogonal in that $\mathcal{L}(p_n(x) p_m(x)) = 0$ whenever $n \neq m$ for some unique linear functional $\mathcal{L} : R[x] \to R$ with $\mathcal{L}(1) = 1$. The moments $\mathcal{L}(x^n)$ are called *the moments of* $\{p_n(x)\}_{n \ge 0}$ and they are described by Motzkin paths.

3.1. A *Motzkin path of length n* is a map $\omega : \{1, 2, \ldots, n+1\} \to \mathbf{N}$ such that $|\omega'| \le 1$ for $\omega' : \{1, 2, \ldots, n\} \to \mathbf{Z}$ defined by $\omega'(i) = \omega(i+1) - \omega(i)$. Put

$$\mathrm{wt}(\omega) = \prod b_{\omega(i)} \lambda_{\omega(j)} \tag{3}$$

over i and j such that $\omega'(i) = 0$ and $\omega'(j) = -1$. Denote by $\mathcal{L} : R[x] \to R$ the linear functional whose n-th moment $\mathcal{L}(x^n)$ is the weighted generating function

$$\mu_n = \mathcal{L}(x^n) = \sum_{\omega} \mathrm{wt}(\omega) \tag{4}$$

($n = 0, 1, \ldots$) over all Motzkin paths ω of length n such that $\omega(1) = \omega(n+1) = 0$. Then a sign-reversing involution [24] tells us that

$$\mathcal{L}(p_i(x)p_j(x)) = \lambda_1\lambda_2\ldots\lambda_i\delta_{ij}. \tag{5}$$

The moments μ_n of $\mathcal{L}$ are therefore the moments of $\{p_n(x)\}_{n\geq 0}$.

3.2. Our main theorem is the observation that the Hankel matrix $H = (\mu_{i+j})_{0\leq i,j\leq n}$ has Smith normal form over $\mathbf{Z}[b,\lambda] = \mathbf{Z}[b_0, b_1, \ldots, \lambda_1, \lambda_2, \ldots]$.

Theorem 1. *$(\mu_{i+j})_{0\leq i,j\leq n}$ has SSNF* $\mathrm{diag}(1, \lambda_1, \lambda_1\lambda_2, \ldots, \lambda_1\lambda_2\ldots\lambda_n)$ *over* $\mathbf{Z}[b,\lambda]$.

Proof. Write P_{ik} for the coefficient of x^k in $p_i(x)$. Let $P = (P_{ik})_{0\leq i,k\leq n}$. Then by (5)

$$PHP^t = \mathrm{diag}(1, \lambda_1, \lambda_1\lambda_2, \ldots, \lambda_1\lambda_2\ldots\lambda_n), \quad H = (\mu_{i+j})_{0\leq i,j\leq n}. \tag{6}$$

Since $p_m(x)$ is a polynomial over $\mathbf{Z}[b,\lambda]$ which is monic of degree m, P is a matrix over $\mathbf{Z}[b,\lambda]$ which is lower triangular with 1's on the diagonal. In other words P is a lower unitriangular matrix over $\mathbf{Z}[b,\lambda]$. □

3.2.1. For example if $b_n = 0$ and $\lambda_n = 1$, then by (4) the n-th moment μ_n equals the number of length-n Dyck paths (Motzkin paths where $|\omega'| = 1$). Hence

$$\mu_n = \begin{cases} C_{n/2} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \tag{7}$$

where C_n is the n-th Catalan number given by $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$, $C_0 = 1$. In this case $\mu_n = \binom{n}{\lfloor n/2\rfloor} - \binom{n}{\lfloor (n-1)/2\rfloor}$ and Theorem 1 says that the Hankel matrix $(\mu_{i+j})_{0\leq i,j\leq n}$ has special Smith normal form $\mathrm{diag}(1, 1, \ldots, 1)$ over $\mathbf{Z}$ so that $\det(\mu_{i+j})_{0\leq i,j\leq n} = 1$.

3.2.2. We know of only two previous results about the Smith normal form of a Hankel matrix of q-Catalan numbers over a polynomial ring. They are the two mentioned above that Bessenrodt–Stanley found [3, pp. 81–82] for the q-Catalan numbers

$$C_{n+1}(q) = \sum_{k=0}^{n} q^k C_k(q) C_{n-k}(q), \quad C_0(q) = 1. \tag{8}$$

We record them here in parts (a) and (b) of Corollary 1. They are elucidated in §4.8 by Theorem 1 applied to the natural q-analogue of our first example from §3.2.1.

Corollary 1. (a) *The matrix* $(C_{i+j}(q))_{0\leq i,j\leq n}$ *has SSNF* $\mathrm{diag}(q^{\binom{0}{2}}, q^{\binom{2}{2}}, q^{\binom{4}{2}}, \ldots, q^{\binom{2n}{2}})$ *over* $\mathbf{Z}[q]$.

(b) *The matrix* $(C_{i+j+1}(q))_{0\leq i,j\leq n}$ *has SSNF* $\mathrm{diag}(q^{\binom{1}{2}}, q^{\binom{3}{2}}, q^{\binom{5}{2}}, \ldots, q^{\binom{2n+1}{2}})$ *over* $\mathbf{Z}[q]$.

4. Examples

Theorem 1 also gives new results for q-Catalan numbers, q-Motzkin numbers, q-Stirling numbers, q-Matching numbers, q-factorials, q-double factorials, as well as more striking generating functions such as Simion and Stanton's octabasic Laguerre moments which count permutations according to eight different statistics. There are many interesting moment sequences and this is just a sampling.

4.1. q-Catalan. If $b_n = 0$ and $\lambda_n = q^{n-1}$, then μ_n counts length-n Dyck paths according to area between the path and the zig-zag one of height 1 (Fig. 1) so that

$$\mu_n = C_n^*(q) = \begin{cases} C_{n/2}(q) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{9}$$

Corollary 2. *$(C_{i+j}^*(q))_{0\le i,j\le n}$ has SSNF* $\mathrm{diag}(1, q^{\binom{1}{2}}, q^{\binom{2}{2}}, \ldots, q^{\binom{n}{2}})$ *over* $\mathbf{Z}[q]$. □

In §4.8 below we use a general result (Theorem 2) to read off the two Bessenrodt–Stanley results directly from this first and most basic example of ours.

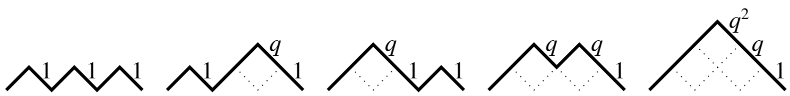

FIGURE 1. $C_3^*(q) = 1 + 2q + q^2 + q^3$.

4.2. q-Motzkin. The q-Motzkin number given by $\mathrm{Motz}_n(q) = \sum_{k=0}^{n/2} \binom{n}{2k} C_k(q)$ is the n-th moment μ_n when $b_n = 1$ and $\lambda_n = q^{n-1}$.

Corollary 3. *$(\mathrm{Motz}_{i+j}(q))_{0\le i,j\le n}$ has SSNF* $\mathrm{diag}(1, q^{\binom{1}{2}}, q^{\binom{2}{2}}, \ldots, q^{\binom{n}{2}})$ *over* $\mathbf{Z}[q]$. □

4.3. q-Stirling. The Charlier polynomials $C_n^a(x)$ have moments $\mu_n = \sum_{k=0}^n S(n,k)a^k$ where $S(n,k)$ is the Stirling number of the second kind which counts partitions of $[n] = \{1, 2, \ldots, n\}$ into k blocks. Médicis–Stanton–White [18] defined q-Charlier polynomials

$$C_{n+1}^a(x; q) = (x - aq^n - [n]_q)C_n^a(x; q) - aq^{n-1}[n]_q C_{n-1}^a(x; q) \tag{10}$$

and showed that the moments are the q-analogues $\mu_n = B_n(a, q)$ given by q-Stirling numbers

$$B_n(a, q) = \sum_{k=0}^n S_q(n, k)a^k, \tag{11}$$

$$S_q(n, k) = S_q(n-1, k-1) + [k]_q S_q(n-1, k), \quad S_q(0, k) = \delta_{0,k} \tag{12}$$

where $[n]_q = 1 + q + \ldots + q^{n-1}$. The combinatorial interpretation of these moments in terms of set partitions π uses the number of blocks, block(π), and another statistic $rs(\pi)$. If Π_n is the set of all set partitions of $[n]$,

$$\mu_n = B_n(a, q) = \sum_{\pi\in\Pi_n} a^{\mathrm{blocks}(\pi)} q^{rs(\pi)}. \tag{13}$$

Corollary 4. *$(B_{i+j}(a, q))_{0\le i,j\le n}$ has SSNF* $\mathrm{diag}(1, a^1 q^{\binom{1}{2}}[1]!_q, a^2 q^{\binom{2}{2}}[2]!_q, \ldots, a^n q^{\binom{n}{2}}[n]!_q)$ *over* $\mathbf{Z}[a, q]$. □

4.4. Kim–Stanton–Zeng [12] defined another sequence of q-Charlier polynomials

$$C_{n+1}(x,a;q) = (x - a - [n]_q)C_n(x,a;q) - a[n]_qC_{n-1}(x,a;q). \tag{14}$$

They showed that the moments are the generating functions $\mu_n = \tilde{B}_n(a,q)$ given by

$$\tilde{B}_n(a,q) = \sum_{\pi\in\Pi_n} a^{\mathrm{block}(\pi)}q^{\mathrm{crossing}(\pi)}. \tag{15}$$

Here crossing(π) is the number of crossings in the diagram that has $1, 2, \ldots, n$ written out along a horizontal line and an upper arc $i \to j$ for each pair $i < j$ such that j is the next largest element in the block containing i. See Figure 2.

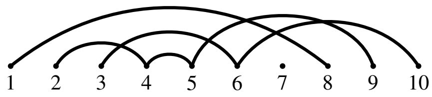

FIGURE 2. The partition $\pi = \{\{1,8\},\{2,4,5,9\},\{3,6,10\},\{7\}\}$ drawn above has block(π) equal to 4 and crossing(π) equal to 5.

Corollary 5. $(\tilde{B}_{i+j}(a,q))_{0\le i,j\le n}$ *has SSNF* $\mathrm{diag}(1, a^1[1]!_q, a^2[2]!_q, \ldots, a^n[n]!_q)$ *over* $\mathbf{Z}[a,q]$. □

4.5. q-Matchings. Ismail–Stanton–Viennot [9] tell us that the polynomials given by

$$h_{n+1}(x) = (x-1)h_n(x) - q^{n-1}[n]_qh_{n-1}(x) \tag{16}$$

have moments the matching polynomials $\mu_n = \mathrm{Match}_n(q)$ given by

$$\mathrm{Match}_n(q) = \sum_m q^{\mathrm{crossing}(m)+2\mathrm{nest}(m)} = \sum_{k=0}^{n/2}\binom{n}{2k}[1]_q[3]_q\ldots[2k-1]_q. \tag{17}$$

The first sum is over all matchings m of $[n]$ (partitions of $[n]$ into blocks of size at most 2) and nest(m) is the number of pairs $\{i,j\},\{k,l\} \in m$ such that $i < k < l < j$.

Corollary 6. $(\mathrm{Match}_{i+j}(q))_{0\le i,j\le n}$ *has SSNF* $\mathrm{diag}(1, q^{\binom{1}{2}}[1]!_q, q^{\binom{2}{2}}[2]!_q, \ldots, q^{\binom{n}{2}}[n]!_q)$ *over* $\mathbf{Z}[q]$. □

4.6. q-Perfect matchings. (Ismail–Stanton–Viennot [9]) Replacing x by $x+1$ in the last example gives the discrete q-Hermite polynomials

$$\tilde{h}_{n+1}(x) = x\tilde{h}_n(x) - q^{n-1}[n]_q\tilde{h}_{n-1}(x) \tag{18}$$

whose moments $\mu_n = PM_n(q)$ count perfect matchings by crossings and nestings:

$$PM_n(q) = \sum_m q^{\mathrm{crossing}(m)+2\mathrm{nest}(m)} = \begin{cases}[1]_q[3]_q\ldots[n-1]_q & \text{if } n \text{ is even,}\\ 0 & \text{if } n \text{ is odd,}\end{cases} \tag{19}$$

where the sum is over all perfect matchings m of $[n]$ (all blocks of size 2).

Corollary 7. $(PM_{i+j}(q))_{0\le i,j\le n}$ *has SSNF* $\mathrm{diag}(1, q^{\binom{1}{2}}[1]!_q, q^{\binom{2}{2}}[2]!_q, \ldots, q^{\binom{n}{2}}[n]!_q)$ *over* $\mathbf{Z}[q]$. □

4.7. Odd–Even trick. In general if the b_n's are all 0, then the polynomials $p_n(x)$ are alternately even and odd so that there exist polynomials $e_n(x)$ and $o_n(x)$ that satisfy

$$p_{2n}(x) = e_n(x^2), \quad p_{2n+1}(x) = xo_n(x^2). \tag{20}$$

The *odd-even trick* is the following observation. The polynomials $\{e_n(x)\}_{n\geq 0}$ and $\{o_n(x)\}_{n\geq 0}$ are themselves orthogonal polynomials and their moments are related to the moments μ_n of the original polynomials $\{p_n(x)\}_{n\geq 0}$ in a simple way.

Proposition 2 ([5, p. 40]). *If $b = (0, 0, \ldots)$, then $\mu_{2n+1} = 0$ and*

(i) *μ_{2n} is the n-th moment of the sequence $\{e_n(x)\}_{n\geq 0}$ defined by*

$$e_{n+1}(x) = (x - \lambda_{2n} - \lambda_{2n+1})e_n(x) - \lambda_{2n-1}\lambda_{2n}e_{n-1}(x), \quad e_{-1}(x) = 0,\ e_0(x) = 1,$$

(ii) *μ_{2n+2} is λ_1 times the n-th moment of the sequence $\{o_n(x)\}_{n\geq 0}$ defined by*

$$o_{n+1}(x) = (x - \lambda_{2n+1} - \lambda_{2n+2})o_n(x) - \lambda_{2n}\lambda_{2n+1}o_{n-1}(x), \quad o_{-1}(x) = 0,\ o_0(x) = 1,$$

for $\lambda_0 = 0$. These polynomials $e_n(x)$ and $o_n(x)$ are the unique ones that satisfy (20).

Corollary 8. *If $b = (0, 0, \ldots)$ and μ''_n, μ'_n are the n-th moments of the polynomials $\{e_n(x)\}$, $\{o_n(x)\}$ defined by* (20)*, then over the ring $\mathbf{Z}[\lambda] = \mathbf{Z}[\lambda_1, \lambda_2, \ldots]$*

(i) *the matrix $(\mu''_{i+j})_{0\leq i,j\leq n}$ has SSNF* $\mathrm{diag}(1, \lambda_1\lambda_2, \lambda_1\lambda_2\lambda_3\lambda_4, \ldots, \lambda_1\lambda_2\ldots\lambda_{2n-1}\lambda_{2n})$*,*
(ii) *the matrix $(\mu'_{i+j})_{0\leq i,j\leq n}$ has SSNF* $\mathrm{diag}(1, \lambda_2\lambda_3, \lambda_2\lambda_3\lambda_4\lambda_5, \ldots, \lambda_2\lambda_3\ldots\lambda_{2n}\lambda_{2n+1})$.

Corollary 8 and Proposition 2 together refine the determinant identity (cf. [5, Ex. 8.8])

$$\det(\mu_{i+j})_{0\leq i,j\leq n} = \Big[\det(\mu_{2i+2j})_{0\leq i,j\leq \lfloor n/2\rfloor}\Big]\Big[\det(\mu_{2i+2j+2})_{0\leq i,j\leq \lfloor (n-1)/2\rfloor}\Big] \tag{21}$$

which holds for $b = (0, 0, \ldots)$. This is the next result.

Theorem 2. *Put $s_k = \lambda_1\lambda_2\ldots\lambda_k$ so that $s_0 = 1$. If $b = (0, 0, \ldots)$, then over $\mathbf{Z}[\lambda]$*

(i) *the matrix $(\mu_{i+j})_{0\leq i,j\leq n}$ has SSNF* $\mathrm{diag}(s_0, s_1, s_2, \ldots, s_n)$*,*
(ii) *the matrix $(\mu_{2i+2j})_{0\leq i,j\leq n}$ has SSNF* $\mathrm{diag}(s_0, s_2, s_4, \ldots, s_{2n})$*,*
(iii) *the matrix $(\mu_{2i+2j+2})_{0\leq i,j\leq n}$ has SSNF* $\mathrm{diag}(s_1, s_3, s_5, \ldots, s_{2n+1})$.

Proof. (i) restates Theorem 1. (ii) restates Corollary 8(i) by Proposition 2(i). For (iii) take Corollary 8(ii) which implies that the matrix $(\lambda_1\mu'_{i+j})_{0\leq i,j\leq n}$ has SSNF $\mathrm{diag}(s_1, s_3, \ldots, s_{2n+1})$ and then use Proposition 2(ii) to rewrite $\lambda_1\mu'_{i+j}$ as $\mu_{2i+2j+2}$. □

4.8. q-Catalan. Theorem 2 applies to our first and most basic example in §4.1 and gives the Bessenrodt–Stanley result in Corollary 1. In the case of the q-Chebyshev polynomials from §4.1 where

$$p_{n+1}(x) = xp_n(x) - q^{n-1}p_{n-1}(x) \tag{22}$$

Proposition 2 says that $C_0(q), C_1(q), C_2(q), \ldots$ is the moment sequence for

$$q_{n+1}(x) = (x - q^{2n} - q^{2n-1}1_{\{n>0\}})q_n(x) - q^{4n-3}q_{n-1}(x) \tag{23}$$

and $C_1(q), C_2(q), C_3(q), \ldots$ is the moment sequence for

$$q_{n+1}(x) = (x - q^{2n}(1 + q))q_n(x) - q^{4n-1}q_{n-1}(x). \tag{24}$$

4.9. q-Double factorials. Theorem 2 also applies to the q-Hermite example in §4.6 and gives Corollary 9 below. In this case Proposition 2 tells us that the q-double factorials $[2n-1]!!_q = [1]_q[3]_q \dots [2n-1]_q$ are the moments of the polynomials where $b_n = q^{2n-1}[2n]_q + q^{2n}[2n+1]_q$ and $\lambda_n = q^{4n-3}[2n-1]_q[2n]_q$.

Corollary 9. $([2i{+}2j{-}1]!!_q)_{0\le i,j\le n}$ *has SSNF* $\operatorname{diag}(1, q^{\binom{2}{2}}[2]!_q, q^{\binom{4}{2}}[4]!_q, \dots, q^{\binom{2n}{2}}[2n]!_q)$ *over* $\mathbf{Z}[q]$. □

4.10. q-Factorials. Kasraoui–Stanton–Zeng [11] defined q-Laguerre polynomials

$$L_{n+1}(x;q) = (x - y[n+1]_q - [n]_q)L_n(x;q) - y[n]_q^2 L_{n-1}(x;q) \tag{25}$$

and showed that $\mu_n = W_n(y,q)$ counts permutations with respect to the number of weak excedances and crossings:

$$W_n(y,q) = \sum_{\sigma\in S_n} y^{\mathrm{wex}(\sigma)} q^{\mathrm{cr}(\sigma)}. \tag{26}$$

The number of *weak excedances of* σ is defined by

$$\mathrm{wex}(\sigma) = \#\{i \in [n] : i \le \sigma(i)\} \tag{27}$$

and the *number of crossings of* σ is defined by

$$\mathrm{cr}(\sigma) = \sum_{j=1}^{n} \#\{j : j < i \le \sigma(j) < \sigma(i)\} + \sum_{j=1}^{n} \#\{j : j > i > \sigma(j) > \sigma(i)\}. \tag{28}$$

This may be explained by the following diagram, see Figure 3. With 1 through n arranged in that order on a horizontal line, view σ graphically by taking each i and drawing an arc $i \to \sigma(i)$ above the line if $\sigma(i) > i$ and below the line if $\sigma(i) < i$. Then $\mathrm{wex}(\sigma)$ is the number of arcs above the line plus the number of isolated points, and $\mathrm{cr}(\sigma)$ is the number of proper crossings plus the number of points $1, 2, \dots, n$ at which two different upper arcs meet.

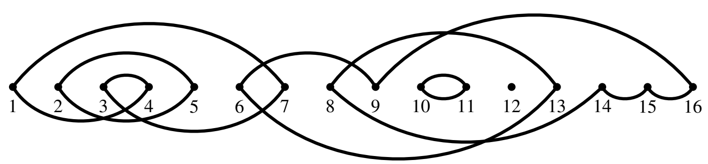

FIGURE 3. $\sigma = (1,7,3,4)(2,5)(6,9,16,15,14,8,13)(10,11)(12)$ is drawn above and has $\mathrm{wex}(\sigma)$ equal to 8 and $\mathrm{cr}(\sigma)$ equal to 9.

Corollary 10. $(W_{i+j}(y,q))_{0\le i,j\le n}$ *has SSNF* $\operatorname{diag}(1, y^1[1]!_q^2, y^2[2]!_q^2 \dots, y^n[n]!_q^2)$ *over* $\mathbf{Z}[y,q]$. □

4.11. Simion and Stanton's *octabasic Laguerre polynomials* with 8 independent q's are defined in terms of the 3-term recurrence relation (2) by setting

$$b_n = a[n+1]_{r,s} + b[n]_{t,u}, \quad \lambda_n = ab[n]_{p,q}[n]_{v,w}, \quad [n]_{r,s} = (r^n - s^n)/(r-s). \tag{29}$$

The moments are generating functions for permutations counted according to eight different statistics which specialize to many other combinatorial sets and related statistics [21]. In particular Simion–Stanton [20] gave specializations whose moments are basically $[n]!_q$. If we swap the a and b in their second specialization [20, Eq. 3.3], then we get the polynomials where μ_n is exactly $[n]!_q$:

$$p_{n+1}(x) = (x - q^n[n+1]_q - q^n[n]_q)p_n(x) - q^{2n-1}[n]_q[n]_q p_{n-1}(x). \tag{30}$$

Corollary 11. $([i+j]!_q)_{0\le i,j\le n}$ *has SSNF* $\mathrm{diag}(1, q^{1^2}[1]!_q^2, q^{2^2}[2]!_q^2, \ldots, q^{n^2}[n]!_q^2)$ *over* $\mathbf{Z}[q]$. □

5. Bessenrodt–Stanley: general results

Fix a Young diagram λ. View it in the top left part of a square-tiled fourth quadrant. Write (i, j) for the square in the i-th row and j-th column of the tiling. Let d be the length of the diagonal of λ, meaning $(d, d) \in \lambda$ and $(d+1, d+1) \notin \lambda$. Write

$$\lambda(i, j) = \{(u, v) \in \lambda : i \le u \text{ and } j \le v\}.$$

Associate to each square $s \in \lambda$ an indeterminate x_s and denote by A_{ij} the generating function for the skew shapes $\lambda(i, j) \setminus \mu$ so that

$$A_{ij} = \textstyle\sum_{\mu\subset\lambda(i,j)} \prod_{s\in\lambda(i,j)\setminus\mu} x_s. \tag{31}$$

The full Bessenrodt–Stanley result [3, Theorem 1] is Corollary 12 below for a SSNF of the matrix

$$A(\lambda) = (A_{ij})_{1\le i,j\le d+1}. \tag{32}$$

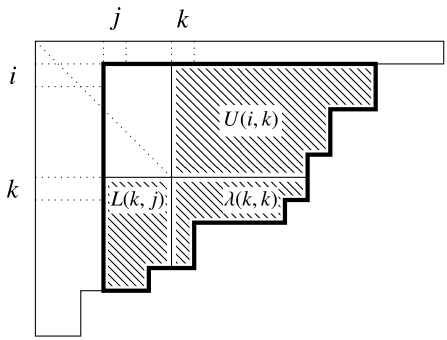

FIGURE 4. $\lambda(i, j)$ in bold.

Theorem 3. $A(\lambda) = UDL$ *where*

(i) $U = (U_{ik})_{1\le i,k\le d+1}$ *is the upper triangular matrix such that* U_{ik} *is the generating function for skew shapes of* $U(i, k) = \{(u, v) \in \lambda : i \le u < k \le v\}$ *when* $i \le k$,

(ii) $D = \operatorname{diag}(D_{11}, D_{22}, \ldots, D_{d+1,d+1})$ *is the diagonal matrix where* $D_{kk} = \prod_{s\in\lambda(k,k)} x_s$,
(iii) $L = (L_{kj})_{1\le k,j\le d+1}$ *is the lower triangular matrix such that* L_{kj} *is the generating function for skew shapes of* $L(k, j) = \{(u, v) \in \lambda : j \le v < k \le u\}$ *when* $j \le k$,

so that in particular L and U are lower and upper unitriangular.

Proof. Write $A_{ij} = \sum_{k=1}^{d+1} U_{ik} D_{kk} L_{kj}$ as illustrated by Figure 4. □

Corollary 12 ([3, Thm. 1])**.** *There are upper and lower unitriangular P and Q over* $\mathbf{Z}[x] = \mathbf{Z}[x_s : s \in \lambda]$ *such that*

$$PA(\lambda)Q = \operatorname{diag}(D_{11}, D_{22}, \ldots, D_{d+1,d+1}), \qquad D_{kk} = \textstyle\prod_{s\in\lambda(k,k)} x_s. \tag{33}$$

In particular, the matrix $A(\lambda)$ *has SSNF* $\operatorname{diag}(1, D_{dd}, D_{d-1,d-1}, \ldots, D_{11})$ *over* $\mathbf{Z}[x]$.

Proof. Theorem 3 implies (33) for $P = U^{-1}$ and $Q = L^{-1}$. But the inverse of an upper (resp. lower) unitriangular matrix is again upper (resp. lower) unitriangular. For the SSNF, let $D = \operatorname{diag}(D_{11}, D_{22}, \ldots, D_{d+1,d+1})$, $D' = \operatorname{diag}(1, D_{dd}, D_{d-1,d-1}, \ldots, D_{11})$, and let X be the permutation matrix such that $XDX^{-1} = D'$. If $\det X = -1$, then put $Y = \operatorname{diag}(-1, 1, 1, \ldots, 1)$ so that $\det(YX) = \det(X^{-1}Y) = 1$ and $YXDX^{-1}Y = D'$. □

Bessenrodt–Stanley's two q-Catalan results (Corollary 1 above) are the two cases of Corollary 12 where $x_s = q$ and $\lambda = (2n-1, 2n-2, \ldots, 1), (2n, 2n-1, \ldots, 1)$.

Remark 1. Bessenrodt–Stanley gave two more theorems in [3]. Their second theorem is essentially an inclusion-exclusion lemma used to recursively construct the P and Q in Corollary 12. But the nature of the factorization $PAQ = D$ implies $P = U^{-1}$ and $Q = L^{-1}$ so we can give their P and Q directly thanks to unitriangularity. The third theorem extends the first to some rectangular matrices [3, Thm. 3]. That theorem can be obtained by specializing to 0 some variables in the first theorem applied to a suitable shape. The specialization leads to a more general statement (Corollary 13 below). Our direct method works in this case also.

Let $(a, b) \notin \lambda$ be a square in the border strip that runs from the end of the first column of λ to the end of the first row of λ, shown as the shaded region in Figure 5.

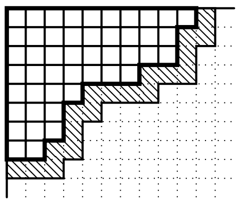

FIGURE 5. λ in bold.

Let ρ be the $a \times b$ rectangle shape with lower right square (a, b). Let A_{ij} be the generating function for the skew shapes $\lambda(i, j) \setminus \mu$. Write

$$A(\lambda, \rho) = (A_{ij})_{1\le i\le a,\, 1\le j\le b}. \tag{34}$$

Put $c = \min(a, b)$ and define

$$U_\rho(i, k) = \{(u, v) \in \lambda : i \le u < k \le v + a - b\} \quad (1 \le i \le k \le a) \tag{35}$$
$$L_\rho(l, j) = \{(u, v) \in \lambda : j \le v < l \le u + b - a\} \quad (1 \le j \le l \le b) \tag{36}$$
$$d_i = \textstyle\prod_{s\in\lambda(a-i+1,b-i+1)} x_s \quad (1 \le i \le c). \tag{37}$$

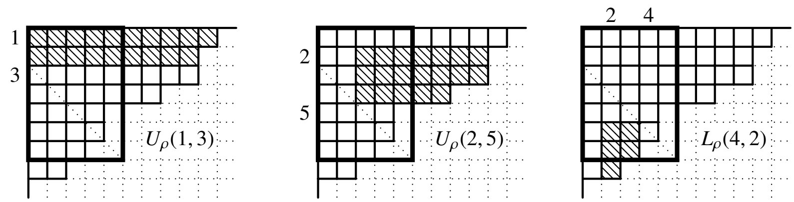

FIGURE 6. ρ in bold.

Theorem 3 is a special case of the following result. It is the case where ρ is square-shaped of size $(d+1) \times (d+1)$.

Theorem 4. *$A(\lambda, \rho) = UDL$ where*

(i) *$U = (U_{ik})_{1 \le i,k \le a}$ is the upper unitriangular matrix given by*

$$U_{ik} = \begin{cases} \text{the generating function for skew shapes in } U_\rho(i,k) & \text{if } i \le k, \\ 0 & \text{otherwise,} \end{cases}$$

(ii) *$D = (D_{kl})_{1 \le k \le a,\, 1 \le l \le b}$ is the matrix given by*

$$D_{kl} = \begin{cases} d_i & \text{if } (k,l) = (a-i+1, b-i+1), \\ 0 & \text{otherwise,} \end{cases}$$

(iii) *$L = (L_{lj})_{1 \le l,j \le b}$ is the lower unitriangular matrix given by*

$$L_{lj} = \begin{cases} \text{the generating function for skew shapes in } L_\rho(l,j) & \text{if } j \le l, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Write $A_{ij} = \sum_{k=1}^{a} \sum_{l=1}^{b} U_{ik} D_{kl} L_{lj}$ as in the proof of Theorem 3. □

Corollary 13. *$A(\lambda, \rho)$ has SSNF $\operatorname{diag}_{a\times b}(1, d_{c-1}, d_{c-2}, \ldots, d_1, 0, \ldots, 0)$ over $\mathbf{Z}[x]$ (the $a \times b$ matrix D with $D_{11}, \ldots, D_{cc}$ as given and 0's elsewhere.)* □

Proof. Corollary 12 handles $a = b$. Assume $a > b$. Let D be as in Theorem 4. Then $XD = \operatorname{diag}_{a\times b}(d_1, d_2, \ldots, d_{c-1}, 1, 0, \ldots, 0)$ for some X that is a permutation matrix with last row possibly scaled by -1 so that $\det X = 1$. The proof of Theorem 4 provides P, Q such that $\det(P) = \det(Q) = 1$ and

$$P \operatorname{diag}(d_1, d_2, d_3, \ldots, d_{c-1}, 1)\, Q = \operatorname{diag}(1, d_{c-1}, d_{c-2}, \ldots, d_1).$$

Consider the block matrix $P' = \operatorname{diag}(P, I_{a-b})$. Then $\det P' = 1$ and

$$P' \operatorname{diag}_{a\times b}(d_1, d_2, \ldots, d_{c-1}, 1, 0, \ldots, 0)\, Q = \operatorname{diag}_{a\times b}(1, d_{c-1}, d_{c-2}, \ldots, d_1, 0, \ldots, 0).$$

The case $a < b$ follows by transposing matrices. □

6. Analogues of Theorem 1

In this section we give two analogues of Theorem 1: one for biorthogonal polynomials (Theorem 5), and the other for finite lattices (Theorem 6).

6.1. Biorthogonal version of Theorem 1. There is a version of Theorem 1 for Toeplitz matrices. In an integer polynomial ring take two sequences $b = (b_0, b_1, \dots)$ and $\lambda = (\lambda_1, \lambda_2, \dots)$ subject to $b_i \neq 0$ and define $q_0(z), q_1(z), \dots$ by

$$q_{n+1}(z) = (z - b_n)q_n(z) - z\lambda_n q_{n-1}(z), \quad q_{-1}(z) = 0, q_0(z) = 1. \tag{38}$$

Let $\mathcal{L}$ be the unique linear functional on $R[z, z^{-1}]$ determined by $\mathcal{L}(1) = 1$ and

$$\mathcal{L}(z^m q_n(1/z)) = 0 \quad (0 \le m < n). \tag{39}$$

Then

$$\mathcal{L}(p_m(z)q_n(1/z)) = (-1)^n \frac{\lambda_1\lambda_2 \dots \lambda_n}{b_1 b_2 \dots b_n} \delta_{mn} \tag{40}$$

for $p_0(z) = 1$ and

$$p_m(z) = \frac{z^m q_{m+1}(1/z) - z^{m-1} q_m(1/z)}{(-1)^{m+1} b_0 b_1 \dots b_m} \quad (m \ge 1) \tag{41}$$

so that $p_m(z)$ is a monic polynomial of degree m over $\mathbf{Z}[b_0^{\pm 1}, \dots, b_m^{\pm 1}, \lambda_1, \dots, \lambda_m]$.

Kamioka [10] gave a combinatorial approach to these *Laurent biorthogonal polynomials* and the moments of $\mathcal{L}$ are in terms of *Schröder paths*. These are the lattice paths $\omega = (\omega_1, \omega_2, \dots)$ from the origin to the x-axis that stay at or above the x-axis with steps ω_i chosen from the following types:

ω_i		$\overline{\mathrm{wt}}(\omega_i)$	$\mathrm{wt}(\omega_i)$
NE	$(x,k) \to (x+1/2, k+1)$	1	1
E	$(x,k) \to (x+1, k)$	b_k	$1/b_k$
SE	$(x,k) \to (x+1/2, k-1)$	λ_k	$\lambda_k/(b_{k-1}b_k)$

(42)

Put $f(\omega) = f(\omega_1)f(\omega_2)\dots$ for $f = \mathrm{wt}, \overline{\mathrm{wt}}$ defined above. Then the *n-th positive moment* $\mathcal{L}(x^n)$ $(n \ge 0)$ is the weighted generating function

$$\mu_n = \sum_{\omega} \mathrm{wt}(\omega) \tag{43}$$

over Schröder paths ω ending at $(0, n)$, and $\mathcal{L}(x^{-n})$ is the weighted generating function

$$\mu_{-n} = \sum_{\omega} \overline{\mathrm{wt}}(\omega) \tag{44}$$

over Schröder paths ω ending at $(0, n)$ with first step ω_1 horizontal (E); see Figure 7.

FIGURE 7. $\omega = (E, NE, NE, SE, SE)$ has $\overline{\mathrm{wt}}(\omega) = b_0\lambda_2\lambda_1$. $\omega = (NE, E, NE, SE, SE)$ has $\mathrm{wt}(\omega) = \frac{1}{b_1}\frac{\lambda_2}{b_1 b_2}\frac{\lambda_1}{b_0 b_1}$.

Theorem 5. $(\mu_{i-j})_{0 \le i,j \le n}$ *has SSNF* $\mathrm{diag}(1, -\frac{\lambda_1}{b_1}, \frac{\lambda_1\lambda_2}{b_1 b_2}, \dots, \pm\frac{\lambda_1\lambda_2\dots\lambda_n}{b_1 b_2 \dots b_n})$ *over* $\mathbf{Z}[b, b^{-1}, \lambda]$.

Proof. Write P_{ik} for the coefficient of z^k in $p_i(z)$, and write Q_{ik} for the coefficient of z^k in $q_i(z)$. Let $P = (P_{ik})_{0\le i,k\le n}$ and $Q = (Q_{ik})_{0\le i,k\le n}$. Then by (40)

$$PTQ^t = \operatorname{diag}\left(1, -\frac{\lambda_1}{b_1}, \frac{\lambda_1\lambda_2}{b_1b_2}, \dots, (-1)^n\frac{\lambda_1\lambda_2\dots\lambda_n}{b_1b_2\dots b_n}\right), \quad T = (\mu_{i-j})_{0\le i,j\le n}. \tag{45}$$

Since the polynomials $p_m(z)$ and $q_m(z)$ are monic of degree m over $\mathbf{Z}[b, b^{-1}, \lambda]$, the matrices P and Q are lower unitriangular over $\mathbf{Z}[b, b^{-1}, \lambda]$. □

Theorem 5 gives an interesting Schröder analogue of Corollary 1. Let R_n be the number of Schröder paths ending at $(0, n)$ so that in terms of Catalan numbers C_n then R_n equals $\sum_{k=0}^n \binom{n+k}{n-k} C_k$. Put $R_n(q) = \sum_{k=0}^n \binom{n+k}{n-k} C_k(q)$ and consider the Hankel-like matrix

$$\left(R_{i+j-1_{\{j>i\}}}(q)\right)_{0\le i,j\le n} = \begin{pmatrix} R_0(q) & R_0(q) & R_1(q) & R_2(q) & \dots & R_{n-1}(q) \\ R_1(q) & R_0(q) & R_0(q) & & \ddots & \vdots \\ R_2(q) & R_1(q) & & \ddots & & R_2(q) \\ R_3(q) & & \ddots & & R_0(q) & R_1(q) \\ \vdots & \ddots & & R_1(q) & R_0(q) & R_0(q) \\ R_n(q) & \dots & R_3(q) & R_2(q) & R_1(q) & R_0(q) \end{pmatrix}. \tag{46}$$

Corollary 14. *The matrix in* (46) *has SSNF* $\operatorname{diag}(1, -q^{\binom{1}{2}}, q^{\binom{2}{2}}, \dots, \pm q^{\binom{n}{2}})$ *over* $\mathbf{Z}[q]$.

Proof. Put $b_n = 1$ and $\lambda_n = q^{n-1}$ to get from (43)–(44) that

$$\mu_n = \begin{cases} R_n(q) & \text{if } n \ge 0, \\ R_{-n-1}(q) & \text{if } n < 0, \end{cases} \tag{47}$$

and then apply Theorem 5. □

6.2. The Hankel matrix as a Gram matrix. In §5 we gave a factorization for the Bessenrodt–Stanley matrix. We now factor the Hankel matrix $H = (\mu_{i+j})_{0\le i,j\le n}$. Then we combinatorially describe H as a Gram matrix. We maintain the notation of §3.

6.2.1. Recall the orthogonality relation (5) for the polynomials $p_i(x)$. It says that

$$PHP^t = D, \quad H = (\mu_{i+j})_{0\le i,j\le n}, \quad P = (\text{ coeff. of } x^j \text{ in } p_i(x))_{0\le i,j\le n},$$
$$D = \operatorname{diag}(1, \lambda_1, \lambda_1\lambda_2, \dots, \lambda_1\lambda_2\dots\lambda_n).$$

Write

$$H = QDQ^t, \quad Q = P^{-1}. \tag{48}$$

Since P is lower unitriangular over $\mathbf{Z}[b, \lambda]$, the same is true for Q. In fact [24, Ch. III] the entries of Q are the following weighted generating functions:

$$Q_{ij} = \sum_{\omega} \operatorname{wt}(\omega) \quad (0 \le i, j \le n) \tag{49}$$

over all Motzkin paths ω of length i such that $\omega(1) = 0$ and $\omega(i + 1) = j$.

Remark 2. The factorization (48) has a direct combinatorial proof, and is the Hankel analogue of our factorization (Theorem 3) for the Bessenrodt–Stanley matrix.

6.2.2. The Hankel matrix of moments can be viewed as a Gram matrix:

$$H = (\langle x^i, x^j \rangle)_{0\le i,j\le n}, \quad \langle f(x), g(x)\rangle = \mathcal{L}(f(x)g(x)). \tag{50}$$

Since $Q = P^{-1}$,

$$x^i = \sum_{j=0}^{n} Q_{ij} p_j(x). \tag{51}$$

Together (49) and (51) give a combinatorial description of x^i as a weighted generating function for Motzkin paths ω of length i. Moreover, the orthogonality of the p_i's results in a simple description for the pairing on these generating functions. The end result is a combinatorial description of the Hankel matrix H as a Gram matrix.

6.3. Other Gram matrices. Here are three more examples using Gram matrices.

6.3.1. *First example.* Let L be a finite ranked lattice with an arbitrary fixed ordering $L = (x_1, x_2, \ldots, x_n)$. Take a function $f : L \to R$ and put $\langle x, y\rangle = f(x \vee y)$. Define

$$G = (f(x \vee y))_{x,y\in L} = (f(x_i \vee x_j))_{1\le i,j\le n}. \tag{52}$$

Write

$$Z = (\zeta(x_i, x_j))_{1\le i,j\le n}, \qquad \zeta(x, y) = \begin{cases} 1 & \text{if } x \le y, \\ 0 & \text{otherwise.} \end{cases} \tag{53}$$

Let $\mu(x, y)$ be the Möbius function given by $(\mu(x_i, x_j))_{1\le i,j\le n} = Z^{-1}$. Then the function $g(x) = \sum_{y\ge x} \mu(x, y) f(y)$ satisfies $f(x) = \sum_{y\ge x} g(y)$.

Proposition 3. (a) $G = Z\,\mathrm{diag}(g(x_1), g(x_2), \ldots, g(x_n))\, Z^t$ *and* $Z \in \mathrm{SL}(n, R)$.
(b) *If the ordering* $L = (x_1, x_2, \ldots, x_n)$ *is chosen so that* $i \le j$ *whenever* $x_i \le x_j$ *(resp.* $x_j \le x_i$*), then* Z *is upper (resp. lower) unitriangular.*

Proof. $f(x \vee y) = \sum_{z\ge x,y} g(z) = \sum_{z\in L} \zeta(x, z)g(z)\zeta(y, z)$. The rest is clear, since Z is conjugate (by a suitable permutation matrix) to an upper unitriangular matrix. □

Corollary 15 (Lindström [17])**.** $\det G = \prod_{x\in L} g(x)$. □

Corollary 16. *If* π *is a permutation such that* $g(x_{\pi(i)})$ *is a multiple of* $g(x_{\pi(j)})$ *whenever* $i \ge j$*, then the matrix* G *has SSNF* $\mathrm{diag}(g(x_{\pi(1)}), g(x_{\pi(2)}), \ldots, g(x_{\pi(n)}))$ *over* R. □

The next theorem is a direct consequence of Proposition 3 and Corollary 16 for $R = \mathbf{Z}[q]$ and $f(x) = q^{\mathrm{rank}(L)-\mathrm{rank}(x)}$. In this case $g(x)$ is the *characteristic polynomial* $\chi([x, 1], q)$ of the interval $[x, 1] = \{y \in L : x \le y\}$.

Theorem 6. *Let* L *be a finite ranked lattice with an ordering* $L = (x_1, x_2, \ldots, x_n)$. *Let* $G = (q^{\mathrm{rank}(L)-\mathrm{rank}(x_i\vee x_j)})_{1\le i,j\le n}$. *Then the following hold.*

(a) $G = Z\,\mathrm{diag}(\chi([x_1, 1], q), \chi([x_2, 1], q), \ldots, \chi([x_n, 1], q))\, Z^t$.
(b) *Suppose that* $\chi([x, 1], q)$ *depends only on the rank of* x. *Define* $l = \mathrm{rank}(L)$, $L_k = \{x : \mathrm{rank}(x) = l - k\}$, *and* $\chi_k(q) = \chi([x, 1], q)$ *for* $x \in L_k$ $(0 \le k \le l)$. *If* $\chi_i(q)$ *is a multiple of* $\chi_j(q)$ *whenever* $i > j$*, then the matrix* G *has SSNF* $\mathrm{diag}(\chi_0(q)I_{|L_0|}, \chi_1(q)I_{|L_1|}, \ldots, \chi_l(q)I_{|L_l|})$ *over* $\mathbf{Z}[q]$. □

Take for example the lattice Π_n of set partitions of $[n]$. Here $x \le y$ in Π_n if and only if each block in x is a subset of some block in y. In particular, the bottom element of Π_n is the partition with n blocks. The top element is the partition with only 1 block. Denote by $|x|$ the number of blocks in x so that $|x| = \text{block}(x)$.

Corollary 17. *Over $\mathbf{Z}[q]$ the matrix $(q^{|x\vee y|})_{x,y\in\Pi_n}$ has SSNF*

$$\mathrm{diag}(qI_{S(n,1)}, q(q-1)I_{S(n,2)}, \ldots, q(q-1)\ldots(q-n+1)I_{S(n,n)}) \tag{54}$$

where $S(n,k)$ is the Stirling number of the second kind given by

$$q^n = \sum_{k=0}^{n} q(q-1)\ldots(q-k+1)S(n,k). \tag{55}$$

Proof. There are exactly $S(n,k)$ elements x in Π_n such that $|x| = k$. For each one $\chi([x,1],q) = (q-1)\ldots(q-k+1)$. Hence by Theorem 6(b) with $L = \Pi_n$ the matrix $(q^{|x\vee y|-1})_{x,y\in\Pi_n}$ has SSNF $\mathrm{diag}(I_{S(n,1)}, (q-1)I_{S(n,2)}, \ldots, (q-1)\ldots(q-n+1)I_{S(n,n)})$. Scaling by q gives the result. □

6.3.2. *Second example.* Let $x \in \mathrm{NC}_n$ be a noncrossing partition of $[n]$. Associate to x the permutation $\sigma(x) \in S_n$ that has one cycle $(i_1\, i_2\, \ldots\, i_k)$ for each block $\{i_1 < i_2 < \ldots < i_k\} \in x$. The partition $\{1, 2, \ldots, n\}$ corresponds to the long cycle $c = (1\,2\,\ldots\,n)$. The *dual partition* $x' \in \mathrm{NC}_n$ corresponds to $\sigma(x)^{-1}c$. Define

$$J_n(q,\delta) = \left(q^{|x\vee_{\Pi_n} y|}\delta^{|x'\vee_{\Pi_n} y'|}\right)_{x,y\in\mathrm{NC}_n} \tag{56}$$

and

$$J_n(q) = J_n(q,1) = \left(q^{|x\vee_{\Pi_n} y|}\right)_{x,y\in\mathrm{NC}_n}. \tag{57}$$

Dahab [7] expressed the determinant of $J_n(q,\delta)$ in terms of *Beraha factors* $f_k(z)$. Define polynomials $p_1(z), p_2(z), \ldots$ by the three-term recurrence relation

$$p_{k+1}(z) = b_k p_k(z) - p_{k-1}(z), \qquad p_{-1}(z) = 0,\ p_0(z) = 1, \tag{58}$$

$$b_k = \begin{cases} z & \text{if } k \text{ is even,} \\ 1 & \text{if } k \text{ is odd.} \end{cases} \tag{59}$$

Then $f_k(z)$ $(k \ge 1)$ is the unique irreducible factor of $p_k(z)$ over $\mathbf{Z}[z]$ that is a factor of no previous $p_j(z)$ $(j < k)$. More explicitly, $f_k(z)$ $(k \ge 1)$ is the minimal polynomial of $4\cos^2(\frac{\pi}{k+1})$, and is given by

$$f_k(z) = \prod_{\substack{1\le j\le (k+1)/2 \\ (j,k+1)=1}} \left(z - 4\cos^2\frac{\pi j}{k+1}\right), \qquad (k \ge 1). \tag{60}$$

Dahab proved that [7, Thm. 1.8.1]

$$\det J_n(z) = \prod_{k=1}^{n} f_k(z)^{m_k} \tag{61}$$

and [7, Thm. 2.5.2]

$$\det J_n(q,\delta) = \det J_n(q\delta) \tag{62}$$

where

$$m_k = \#\{\text{Dyck paths of length } 2n \text{ and height} \ge k\}. \tag{63}$$

We conjecture the following refinement of Dahab's determinantal evaluations.

Conjecture 1. $J_n(q,\delta)$ has SSNF $\mathrm{diag}(s_1(q,\delta)I_{h_1}, s_2(q,\delta)I_{h_2}, \ldots, s_n(q,\delta)I_{h_n})$ over $\mathbf{Z}[q,\delta]$, where

$$h_k = \#\{\text{Dyck paths of length } 2n \text{ and height } k\} \tag{64}$$

and

$$s_k(q,\delta) = f_1(q\delta)f_2(q\delta)\ldots f_k(q\delta) = \prod_{1\le j\le k}\ \prod_{\substack{1\le i\le (j+1)/2\\ (i,j+1)=1}} \left(q\delta - 4\cos^2\frac{\pi i}{j+1}\right). \tag{65}$$

In particular:

(a) $J_n(q)$ has SSNF $\mathrm{diag}(s_1(q)I_{h_1}, s_2(q)I_{h_2}, \ldots, s_n(q)I_{h_n})$ over $\mathbf{Z}[q]$, where

$$s_k(q) = s_k(q,1) = f_1(q)f_2(q)\ldots f_k(q) = \prod_{1\le j\le k}\ \prod_{\substack{1\le i\le (j+1)/2\\ (i,j+1)=1}} \left(q - 4\cos^2\frac{\pi i}{j+1}\right). \tag{66}$$

(b) $J_n(q,q)$ has SSNF $\mathrm{diag}(s_1'(q)I_{h_1}, s_2'(q)I_{h_2}, \ldots, s_n'(q)I_{h_n})$ over $\mathbf{Z}[q]$, where

$$s_k'(q) = s_k(q,q) = f_1(q^2)f_2(q^2)\ldots f_k(q^2) = q\prod_{\substack{1\le i\le j\le k\\ (i,j+1)=1}} \left(q - 2\cos\frac{\pi i}{j+1}\right). \tag{67}$$

6.3.3. *Third example.* Take two noncrossing perfect matchings x, y on $[2n]$ and put $\langle x,y\rangle = q^{c(x,y)}$ where $c(x,y)$ is the number of connected components in the graph on $[2n]$ whose multiset of edges is $x \cup y$. This is Lickorish's form [16] and the determinant of the Gram matrix $M_n(q) = (\langle x,y\rangle)_{x,y}$ has been studied [8, 13, 16], see [14]. But a straightforward calculation shows that

$$M_n(q) = q^{-1}J_n(q,q) \tag{68}$$

(for some compatible ordering of the matchings and the non-crossing partitions). Therefore Conjecture 1 implies the following conjecture for $M_n(q)$.

Conjecture 2. M_n has SSNF $\mathrm{diag}(s_1(q)I_{h_1}, s_2(q)I_{h_2}, \ldots, s_n(q)I_{h_n})$ over $\mathbf{Z}[q]$ where h_k is the number of Dyck paths of length $2n$ and height k, and

$$s_k(q) = \prod_{\substack{1\le i\le j\le k\\ (i,j+1)=1}} \left(q - 2\cos\frac{\pi i}{j+1}\right). \tag{69}$$

7. Remarks

There are new and interesting results for other types of matrices as well. Some recent examples are found in [22, 23]; they again refine some well-known determinantal evaluations. But many determinantal evaluations (e.g. [14, 15]) have not been considered. Here for example is a new result we found for the Vandermonde matrix.

Theorem 7. *Let*

$$g_i(x) = \sum_{k=0}^{i} A_{ik}a^k x^k, \quad A_{ik} \in \mathbf{Z}[a,q],\ A_{ii} = 1. \tag{70}$$

Then over $\mathbf{Z}[a,q]$ *the matrix* $(g_i([j]_q))_{0\le i,j\le n}$ *has SSNF*

$$\mathrm{diag}(1, a^1q^{\binom{1}{2}}[1]!_q, a^2q^{\binom{2}{2}}[2]!_q, \ldots, a^nq^{\binom{n}{2}}[n]!_q). \tag{71}$$

In particular:

(a) *Over* $\mathbf{Z}[a, q]$ *the Vandermonde matrix* $((1 + a[j]_q)^i)_{0\le i,j\le n}$ *has SSNF*

$$\mathrm{diag}(1, a^1 q^{\binom{1}{2}}[1]!_q, a^2 q^{\binom{2}{2}}[2]!_q, \ldots, a^n q^{\binom{n}{2}}[n]!_q). \tag{72}$$

(b) *Over* $\mathbf{Z}[q]$ *the Vandermonde matrix* $([j + 1]_q^i)_{0\le i,j\le n}$ *has SSNF*

$$\mathrm{diag}(1, q^{\binom{2}{2}}[1]!_q, q^{\binom{3}{2}}[2]!_q, \ldots, q^{\binom{n+1}{2}}[n]!_q). \tag{73}$$

Theorem 7 is a special case of the following generalization of Theorem 1.

Theorem 8. *Maintain the notation of §3 so that* $\mathcal{L}$ *is the linear functional for the polynomials* $p_k(x)$ *defined by the three-term recurrence relation*

$$p_{n+1}(x) = (x - b_n)p_n(x) - \lambda_n p_{n-1}(x), \quad p_{-1}(x) = 0, p_0(x) = 1.$$

Let $Y_0(x), Y_1(x), \ldots, Y_n(x), Z_0(x), Z_1(x), \ldots, Z_n(x)$ *be monic polynomials over* $\mathbf{Z}[b, \lambda]$ *such that* $Y_k(x)$ *and* $Z_k(x)$ *have degree* k *for* $0 \le k \le n$*. Then the matrix*

$$\Big(\mathcal{L}(Y_i(x)Z_j(x))\Big)_{0\le i,j\le n} \tag{74}$$

has SSNF

$$\mathrm{diag}(1, \lambda_1, \lambda_1\lambda_2, \ldots, \lambda_1\lambda_2 \ldots \lambda_n) \tag{75}$$

over $\mathbf{Z}[b, \lambda]$. □

Theorem 1 is the special case of Theorem 8 where $Y_k(x) = Z_k(x) = x^k$ for all k. Theorem 7 is the case where the polynomials $p_k(x)$ are the q-Charlier polynomials $C_k^a(x; q)$ from §4.3 and

$$Z_j(x) = \sum_{u=0}^{j} \begin{bmatrix} j \\ u \end{bmatrix}_q p_u(x), \tag{76}$$

$$Y_i(x) = \sum_{t=0}^{i} \sum_{k=t}^{i} A_{ik} S_q(k, t) a^{k-t} p_t(x), \tag{77}$$

where $\begin{bmatrix} j \\ u \end{bmatrix}_q = [j]_q[j-1]_q \ldots [j-u+1]_q/[u]!_q$.

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