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BIMAHONIAN DISTRIBUTIONS

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Abstract. Motivated by permutation statistics, we define for any complex reflection group W a family of bivariate generating functions $W^{\sigma}(t,q)$. They are defined either in terms of Hilbert series for W-invariant polynomials when W acts diagonally on two sets of variables, or equivalently, as sums involving the fake degrees of irreducible representations for W. It is shown that $W^{\sigma}(t,q)$ satisfies a "bicyclic sieving phenomenon" which combinatorially interprets its values when t and q are certain roots of unity.

Contents

1. Introduction

This paper contains one main new definition, of the *bimahonian distributions* for a complex reflection group (Definition 1.1), and three main results about it (Theorems 1.2, 1.3, 1.4), inspired by known results in the theory of permutation statistics. We explain here some background and context for these results.

1.1. **The mahonian distribution.** P.A. MacMahon [19] proved that there are two natural equidistributed statistics on permutations w in the symmetric group $\mathfrak{S}_n$ on n letters, namely the *inversion number* (or equivalently, the Coxeter group length)

$$\text{inv}(w) := |\{(i,j) : 1 \le i < j \le n, w(i) > w(j)\}| \;(= \ell(w)) \tag{1.1}$$

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and the *major index*

$$\operatorname{maj}(w) := \sum_{i \in \operatorname{Des}(w)} i \tag{1.2}$$

where $\operatorname{Des}(w) := \{i : w(i) > w(i+1)\}$ is the *descent set* of w.

Foata [12] dubbed their common distribution

$$\sum_{w \in \mathfrak{S}_n} q^{\operatorname{inv}(w)} = \sum_{w \in \mathfrak{S}_n} q^{\operatorname{maj}(w)} = \prod_{i=1}^{n} (1 + q + q^2 + \cdots + q^{i-1}) \tag{1.3}$$

the *mahonian distribution*. Subsequently many other statistics on $\mathfrak{S}_n$ with this distribution have been discovered; see e.g., Clarke [9].

This distribution has a well-known generalization to any *complex reflection group* W, by means of invariant theory, as we next review. We also review here some known generalizations of the statistics $\operatorname{inv}(w), \operatorname{maj}(w)$, which are unfortunately currently known in less generality.

Recall that a *complex reflection group* is a finite subgroup of $GL(V)$ for $V = \mathbb{C}^n$ generated by *reflections* (= elements whose fixed subspace has codimension 1). Shepard and Todd [22] classified such groups. They also showed (as did Chevalley [8]) that they are characterized among the finite subgroups W of $GL(V)$ as those whose action on the symmetric algebra $S(V^*) = \mathbb{C}[V] \cong \mathbb{C}[x_1, \ldots, x_n]$ has the invariant subalgebra $\mathbb{C}[V]^W$ again a polynomial algebra. In this case one can choose homogeneous generators $f_1, \ldots, f_n$ for $\mathbb{C}[V]^W$, having *degrees* $d_1, \ldots, d_n$, and define the *mahonian distribution* for W, in analogy with the product formula in (1.3), by

$$W(q) := \prod_{i=1}^{n} (1 + q + q^2 + \cdots + q^{d_i - 1}).$$

This polynomial $W(q)$ is a "q-analogue" of $|W|$ in the sense that $W(1) = |W|$. It is a polynomial in q whose degree $N^* := \sum_{i=1}^{n} (d_i - 1)$ is the number of reflections in W.

We now give two equivalent ways to rephrase the definition. $W(q)$ is also the *Hilbert series* for the *coinvariant algebra* $\mathbb{C}[V]/(\mathbb{C}[V]^W_+)$:

$$W(q) = \operatorname{Hilb}(\ \mathbb{C}[V]/(\mathbb{C}[V]^W_+)\ , q) \tag{1.4}$$

where $(\mathbb{C}[V]^W_+) := (f_1, \ldots, f_n)$ is the ideal in $\mathbb{C}[V]$ generated by the invariants $\mathbb{C}[V]^W_+$ of positive degree. In their work, both Shepard and Todd [22] and Chevalley [8] showed that the coinvariant algebra $\mathbb{C}[V]/(\mathbb{C}[V]^W_+)$ carries the *regular representation* of W. Consequently, each irreducible complex character/representation χ^λ of W occurs in it with multiplicity equal to its degree $f^\lambda := \chi^\lambda(e)$. This implies that there is a unique polynomial $f^\lambda(q)$ in q with

$$f^\lambda(1) = f^\lambda := \chi^\lambda(e) \tag{1.5}$$

called the *fake degree polynomial*, which records the homogeneous components of $\mathbb{C}[V]/(\mathbb{C}[V]^W_+)$ in which this irreducible occurs:

$$f^\lambda(q) := \sum_{i \geq 0} q^i \langle\ \chi^\lambda\ ,\ \left(\mathbb{C}[V]/(\mathbb{C}[V]^W_+)\right)_i\ \rangle_W. \tag{1.6}$$

Here R_i denotes the i^{th} homogeneous component of a graded ring R, and $\langle \cdot, \cdot \rangle_W$ denotes the inner product (or intertwining number) of two W-representations or characters. Thus our second rephrasing of the definition of $W(q)$ is:

$$W(q) = \sum_{\lambda} f^{\lambda}(1) f^{\lambda}(q). \tag{1.7}$$

At somewhat lower levels of generality, one finds formulae for $W(q)$ generalizing the formulae involving the statistics $\mathrm{inv}(w), \mathrm{maj}(w)$ that appeared in (1.3):

- When W is a (not necessarily crystallographic [1]) real reflection group, and hence a Coxeter group, with Coxeter length function $\ell(w)$, one has
$$W(q) = \sum_{w \in W} q^{\ell(w)}.$$
- When $W = \mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_n$, the wreath product of a cyclic group of order d with the symmetric group, $W(q)$ is the distribution for a statistic $\mathrm{fmaj}(w)$ defined by Adin and Roichman [1] (see Section 6 below):

$$W(q) = \sum_{w \in W} q^{\mathrm{fmaj}(w)}. \tag{1.8}$$

1.2. **The bimahonian distribution.** Foata and Schützenberger [14] first observed that there is a bivariate distribution on the symmetric group $\mathfrak{S}_n$, shared by several pairs of mahonian statistics, including the pairs

$$(\mathrm{maj}(w), \mathrm{inv}(w)) \text{ and } (\mathrm{maj}(w), maj(w^{-1})) \tag{1.9}$$

This bivariate distribution is closely related to the fake degrees for $W = \mathfrak{S}_n$ and *standard Young tableaux* via the Robinson-Schensted correspondence, as well as to bipartite partitions and invariant theory; see Sections 5 and 6 for more discussion and references.

The goal of this paper is to generalize this "bimahonian" distribution on $W = \mathfrak{S}_n$ to any complex reflection group W. In fact, one is naturally led to consider a family of such bivariate distributions $W^{\sigma}(t,q)$, indexed by field automorphisms σ lying in the Galois group [11, §14.5]

$$\mathrm{Gal}(\mathbb{Q}[e^{\frac{2\pi i}{m}}]/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^{\times}$$

of any cyclotomic extension $\mathbb{Q}[e^{\frac{2\pi i}{m}}]$ of $\mathbb{Q}$ large enough to define the matrix entries for W. It is known [21, §12.3] that one can take m to be the least common multiple of the orders of the elements w in W. When referring to Galois automorphisms σ, we will implicitly assume that $\sigma \in \mathrm{Gal}(\mathbb{Q}[e^{\frac{2\pi i}{m}}]/\mathbb{Q})$ for this choice of m, unless specified otherwise.

Definition 1.1. Given σ, in analogy to (1.7), define the *σ-bimahonian distribution* for W by

$$W^{\sigma}(t,q) := \sum_{\lambda} f^{\sigma(\lambda)}(t) f^{\overline{\lambda}}(q) = \sum_{\lambda} f^{\lambda}(t) f^{\overline{\sigma}^{-1}(\lambda)}(q). \tag{1.10}$$

[1] If one further assumes that W is crystallographic, and hence a Weyl group, the coinvariant algebra $\mathbb{C}[V]/(\mathbb{C}[V]^W_+)$ gives the cohomology ring of the associated *flag manifold* G/B, and $W(q)$ is also the *Poincaré series* for G/B.

Here both sums run over all irreducible complex W-representations λ, and

$$\chi^{\sigma(\lambda)}(w) = \chi^{\lambda}(\sigma(w)) = \sigma(\chi^{\lambda}(w))$$

denotes the character of the irreducible representation $\sigma(\lambda)$ defined by applying σ entrywise to the matrices representing the group elements in λ.

The two most important examples of σ will be

- $\sigma(z) = z$, for which we denote $W^{\sigma}(t,q)$ by $W(t,q)$, and
- $\sigma(z) = \bar{z}$, where $\sigma(\lambda)$ is equivalent to the representation *contragredient* to λ, and for which we denote $W^{\sigma}(t,q)$ by $\overline{W}(t,q)$.

From Definition 1.1 one can see that

$$W^{\sigma}(q,t) = W^{\sigma^{-1}}(t,q) \tag{1.11}$$

and hence both $W(t,q), \overline{W}(t,q)$ are symmetric polynomials in t,q. Setting $t=1$ or $q=1$ in (1.10) and comparing with (1.7) gives

$$W^{\sigma}(1,q) = W^{\sigma}(q,1) = W(q)$$

for any σ.

It turns out that, by analogy to (1.4), one can also define $W^{\sigma}(t,q)$ as a *Hilbert series* arising in invariant theory. One considers the σ*-diagonal* embedding of W defined by

$$\Delta^{\sigma} W := \{(w, \sigma(w)) : w \in W\} \subset W \times W^{\sigma} \subset GL(V) \times GL(V)$$

where $W^{\sigma} := \{\sigma(w) : w \in W\} \subset GL(V)$. Note that $W \times W^{\sigma}$ acts on the symmetric algebra

$$S(V^* \oplus V^*) = \mathbb{C}[V \oplus V] \cong \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$$

in a way that preserves the $\mathbb{N}^2$-grading given by $\deg(x_i) = (1,0), \deg(y_j) = (0,1)$ for all i,j. Hence one can consider the $\mathbb{N}^2$-graded Hilbert series in t,q for various graded subalgebras and subquotients of $\mathbb{C}[V \oplus V]$. The following analogue of (1.4) is proven in Section 2 below.

Theorem 1.2. *For any complex reflection group W and Galois automorphism σ, $W^{\sigma}(t,q)$ is the $\mathbb{N}^2$-graded Hilbert series for the ring*

$$\mathbb{C}[V \oplus V]^{\Delta^{\sigma} W} / \left(\mathbb{C}[V \oplus V]_+^{W \times W^{\sigma}}\right).$$

Note that when W is a *real* reflection group, one has $\overline{W} = W$ and $\Delta^{\sigma}(W) = W$, so that $\overline{W}(t,q) = W(t,q)$. For example, in the original motivating special case where $W = \mathfrak{S}_n$, the description of $W(t,q)$ via Theorem 1.2 relates to work on *bipartite partitions* by Carlitz [7], Wright [29], Gordon [16], Solomon [23], and Garsia and Gessel [15]; see [25, Example 5.3].

Much of this was generalized from $W = \mathfrak{S}_n$ to the wreath products $W = \mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_n$ in work of Adin and Roichman [1] (see also Foata and Han [13], Bergeron and Biagioli [5]). We discuss some of this in Section 6 below, and prove the following interpretation for $W^{\sigma}(t,q)$, generalizing a result of Adin and Roichman [1].

Theorem 1.3. *For the wreath products $W = \mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_n$ and any Galois automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}[e^{\frac{2\pi i}{d}}]/\mathbb{Q})$, one has*

$$W^{\sigma}(t,q) = \sum_{w \in W} q^{\mathrm{fmaj}(w)} t^{\mathrm{fmaj}(\sigma(w^{-1}))}.$$

1.3. **Bicyclic sieving.** One motivation for the current work came from previous studies [3, 4] of the coefficients $a_{k,\ell}(i,j)$ uniquely defined by

$$W^\sigma(t,q) \equiv \sum_{\substack{0\leq i<k \\ 0\leq j<\ell}} a_{k,\ell}(i,j) t^i q^j \mod (t^k-1, q^\ell-1) \tag{1.12}$$

in the case where $W = \mathfrak{S}_n$ and σ is the identity. Equivalent information is provided by knowing the evaluations $W^\sigma(\omega,\omega')$ where as ω,ω' vary over all k^{th},ℓ^{th} complex roots of unity, respectively; these were studied in the case $W = \mathfrak{S}_n$ by Carlitz [7] and Gordon [16].

In Section 4 below, we prove Theorem 1.4, generalizing some of these results to all complex reflection groups, and providing a combinatorial interpretation for some of these evaluations $W^\sigma(\omega,\omega')$. This is our first instance of a *bicyclic sieving phenomenon*, generalizing the notion of a *cyclic sieving phenomenon* introduced in [20].

Theorem 1.4. *Let C, C' be cyclic subgroups of W generated by regular elements c, c' in W, having regular eigenvalues ω, ω', in the sense of Springer* [24]*. Given the Galois automorphism σ, choose an integer s with the property that $\sigma(\omega) = \omega^s$, and then define a (left-)action of $C \times C'$ on W as follows:*

$$(c,c')w := c^s w \sigma(c')^{-1}.$$

Then for any integers i,j one has

$$W^\sigma(\omega^{-i},(\omega')^{-j}) = |\{x \in X : (c^i,(c')^j)x = x\}|.$$

Alternatively, one can phrase this phenomenon (see Proposition 3.1) as a combinatorial interpretation of the coefficients $a_{k,\ell}(i,j)$ in (1.12): if the regular elements c, c' in our complex reflection group W have orders k,ℓ, then $a_{k,\ell}(i,j)$ is the number of $C\times C'$-orbits on X for which the stabilizer subgroup of any element in the orbit lies in the kernel of the character $\rho^{(i,j)} : C\times C' \to \mathbb{C}^\times$ sending $(c,c') \mapsto \omega^{-i}(\omega')^{-j}$.

2. Proof of Theorem 1.2

After some preliminaries about Galois automorphisms σ, we recall the statement of Theorem 1.2, prove it, and give some combinatorial consequences for the bimahonian distributions $W^\sigma(t,q)$.

For any Galois automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}[e^{\frac{2\pi i}{m}}]/\mathbb{Q})$, the group $W^\sigma = \sigma(W)$ is also a complex reflection group: σ takes a reflection r having characteristic polynomial $(t-\omega)(t-1)^{n-1}$ to a reflection $\sigma(r)$ having characteristic polynomial $(t-\sigma(\omega))(t-1)^{n-1}$. Furthermore, if $\mathbb{C}[V]^W = \mathbb{C}[f_1,\ldots,f_n]$, then

$$\mathbb{C}[V]^{W^\sigma} = \mathbb{C}[\sigma(f_1),\ldots,\sigma(f_n)],$$

and hence W and W^σ share the same degrees $d_1,\ldots,d_n$ for their basic invariants. This implies

$$\mathrm{Hilb}(\mathbb{C}[V]^W,q) = \mathrm{Hilb}(\mathbb{C}[V]^{W^\sigma},q) = \prod_{i=1}^n \frac{1}{1-q^{d_i}}. \tag{2.1}$$

This gives a relation between the graded character of the W-coinvariant algebra $A := \mathbb{C}[V]/(\mathbb{C}[V]^W_+)$ and the W^σ-coinvariant algebra $A^\sigma := \mathbb{C}[V]/(\mathbb{C}[V]^{W^\sigma}_+)$. Let A_j, A^σ_j denote their j^{th}-graded components.

Proposition 2.1. *For every j, one has*

$$\chi_{A_j^\sigma}(\sigma(w)) = \sigma\left(\chi_{A_j}(w)\right)$$

Proof. Since $\mathbb{C}[V]^W, \mathbb{C}[V]^{W^\sigma}$ are generated by homogeneous systems of parameters for $\mathbb{C}[V]$, and since $\mathbb{C}[V]$ is a *Cohen-Macaulay* ring [25, §3], $\mathbb{C}[V]$ is free as a module over either of these subalgebras. Hence

$$\begin{aligned}
\mathrm{Hilb}(\mathbb{C}[V]^W, q)\left(\sum_j \chi_{A_j}(w)q^j\right) &= \sum_j \chi_{\mathbb{C}[V]_j}(w)q^j \\
\mathrm{Hilb}(\mathbb{C}[V]^{W^\sigma}, q)\left(\sum_j \chi_{A_j^\sigma}(\sigma(w))q^j\right) &= \sum_j \chi_{\mathbb{C}[V]_j}(\sigma(w))q^j \\
&= \sigma\left(\sum_j \chi_{\mathbb{C}[V]_j}(w)q^j\right)
\end{aligned}$$

so that, dividing by $\mathrm{Hilb}(\mathbb{C}[V]^W, q)(= \mathrm{Hilb}(\mathbb{C}[V]^{W^\sigma}, q))$, one obtains

$$\begin{aligned}
\sum_j \chi_{A_j^\sigma}(\sigma(w))q^j &= \sigma\left(\frac{\sum_j \chi_{\mathbb{C}[V]_j}(w)q^j}{\mathrm{Hilb}(\mathbb{C}[V]^{W^\sigma}, q)}\right) \\
&= \sigma\left(\frac{\sum_j \chi_{\mathbb{C}[V]_j}(w)q^j}{\mathrm{Hilb}(\mathbb{C}[V]^{W}, q)}\right) \\
&= \sigma\left(\sum_j \chi_{A_j}(w)q^j\right) \\
&= \sum_j \sigma(\chi_{A_j}(w))q^j.
\end{aligned}$$

Here we have used throughout that the coefficients of any Hilbert series are integers, and hence are fixed by σ, while the second equality uses (2.1). Comparing coefficients of q^j gives the assertion. □

We next recall the statement of Theorem 1.2, and prove it.

Theorem 1.2. *For any complex reflection group W and $\sigma \in \mathrm{Gal}(\mathbb{Q}[e^{\frac{2\pi i}{m}}]/\mathbb{Q})$, $W^\sigma(t, q)$ is the $\mathbb{N}^2$-graded Hilbert series for the ring*

$$\mathbb{C}[V \oplus V]^{\Delta^\sigma W} / \left(\mathbb{C}[V \oplus V]_+^{W \times W^\sigma}\right).$$

Proof. Note that the $W \times W^\sigma$-coinvariant algebra

$$A_{W \times W^\sigma} = \mathbb{C}[V \oplus V]/(\mathbb{C}[V \oplus V]_+^{W \times W^\sigma})$$

satisfies

$$A_{W \times W^\sigma} \cong A \otimes A^\sigma$$

as $W \times W^\sigma$-representations, and hence for each i, j one has

$$(A_{W \times W^\sigma})_{i,j} \cong A_i \otimes A_j^\sigma. \tag{2.2}$$

Thus one has

$$\begin{aligned}
&\mathrm{Hilb}\left(\mathbb{C}[V\oplus V]^{\Delta^\sigma W}/(\mathbb{C}[V\oplus V]_+^{W\times W^\sigma});t,q\right)\\
&=\mathrm{Hilb}\left((A_{W\times W^\sigma})^{\Delta^\sigma W};t,q\right)\\
&=\sum_{i,j} t^i q^j \left\langle (A_{W\times W^\sigma})_{i,j}\,,\mathbf{1}\right\rangle_{\Delta^\sigma W}.
\end{aligned}$$

Here the first equality uses the fact that the *averaging (or Reynolds) operator*

$$f\mapsto \frac{1}{|W|}\sum_{w\in W} w(f)$$

gives an $\mathbb{C}[V\oplus V]^{W\times W^\sigma}$-module projection $\mathbb{C}[V\oplus V]\twoheadrightarrow \mathbb{C}[V\oplus V]^{\Delta^\sigma W}$ that splits[2] the inclusion $\mathbb{C}[V\oplus V]^{\Delta^\sigma W}\hookrightarrow \mathbb{C}[V\oplus V]$, so that

$$\begin{aligned}
(A_{W\times W^\sigma})^{\Delta^\sigma W} &:= \left(\mathbb{C}[V\oplus V]/(\mathbb{C}[V\oplus V]_+^{W\times W^\sigma})\right)^{\Delta^\sigma W}\\
&\cong \mathbb{C}[V\oplus V]^{\Delta^\sigma W}/(\mathbb{C}[V\oplus V]_+^{W\times W^\sigma}).
\end{aligned}$$

One can then compute

$$\begin{aligned}
\left\langle (A_{W\times W^\sigma})_{i,j}\,,\mathbf{1}\right\rangle_{\Delta^\sigma W} &= \left\langle \mathrm{Res}^{W\times W^\sigma}_{\Delta^\sigma W}\left(\chi_{A_i}\otimes \chi_{A_j^\sigma}\right),\mathbf{1}\right\rangle_{\Delta^\sigma W}\\
&=\frac{1}{|W|}\sum_{w\in W}\chi_{A_i}(w)\cdot \chi_{A_j^\sigma}(\sigma(w))\cdot 1\\
&=\frac{1}{|W|}\sum_{w\in W}\chi_{A_i}(w)\cdot \sigma\left(\chi_{A_j}(w)\right)\\
&=\left\langle \chi_{A_i},\overline{\sigma}\chi_{A_j}\right\rangle_W
\end{aligned}$$

where the first equality uses (2.2) and the third equality uses Proposition 2.1.

To finish the proof, it remains to show that $\langle \chi_{A_i},\overline{\sigma}\chi_{A_j}\rangle_W$ is the coefficient of $t^i q^j$ in

$$W^\sigma(t,q)=\sum_\lambda f^\lambda(t) f^{\overline{\sigma}^{-1}(\lambda)}(q).$$

But this follows from Definition 1.6: the class function χ_{A_i} is the coefficient of t^i in

$$\chi_A=\sum_\lambda f^\lambda(t)\cdot\chi^\lambda,$$

while the class function $\overline{\sigma}\chi_{A_j}$ is the coefficient of q^j in

$$\overline{\sigma}\chi_A=\sum_\lambda f^\lambda(q)\cdot\chi^{\overline{\sigma}(\lambda)}=\sum_\lambda f^{\overline{\sigma}^{-1}(\lambda)}(q)\cdot\chi^\lambda.$$

□

[2]We are being careful here– the relation between the invariant rings $k[V]^G, k[V]^H$ and $k[V]$ and coinvariants for subgroups $H\subset G\subset GL(V)$ is not as straightforward when working over a field k of positive characteristic. One always has a natural map $k[V]^H/(k[V]^G_+)\to \left(k[V]/(k[V]^G_+)\right)^H$, but it can fail to be an isomorphism when one lacks a $k[V]^G$-module splitting (such as the Reynolds operator above) for the inclusion $k[V]^H\hookrightarrow k[V]$.

Theorem 1.2 immediately gives a "Molien-type" formula for $W^\sigma(t,q)$. Note that

$$\mathrm{Hilb}(\mathbb{C}[V \oplus V]^{W \times W^\sigma};\, t, q) = \frac{1}{\prod_{i=1}^n (1-t^{d_i})(1-q^{d_i})}. \tag{2.3}$$

Corollary 2.2. *For any complex reflection group W, and Galois automorphism σ, one has*

$$W^\sigma(t,q) = \frac{1}{|W|} \sum_{w \in W} \frac{\prod_{i=1}^n (1-t^{d_i})(1-q^{d_i})}{\det(1-tw)\det(1-q\sigma(w))}$$

Proof. Molien's Theorem [25, §2] tells us that

$$\mathrm{Hilb}(\mathbb{C}[V \oplus V]^{\Delta^\sigma W}; t, q) = \frac{1}{|W|} \sum_{w \in W} \frac{1}{\det(1-tw)\det(1-q\sigma(w))}. \tag{2.4}$$

On the other hand, $\mathbb{C}[V \oplus V]^{\Delta^\sigma W}$ is a Cohen-Macaulay ring [25, §3], and hence a free module over the polynomial subalgebra $\mathbb{C}[V \oplus V]^{W \times W^\sigma}$. Thus

$$\begin{aligned} W^\sigma(t,q) &= \mathrm{Hilb}(\mathbb{C}[V \oplus V]^{\Delta^\sigma W}/(\mathbb{C}[V \oplus V]_+^{W \times W^\sigma});\, t, q) \\ &= \frac{\mathrm{Hilb}(\mathbb{C}[V \oplus V]^{\Delta^\sigma W};\, t, q)}{\mathrm{Hilb}(\mathbb{C}[V \oplus V]^{W \times W^\sigma};\, t, q)}. \end{aligned} \tag{2.5}$$

The desired formula for $W^\sigma(t,q)$ results from taking the quotient of the right side of (2.4) by the right side of (2.3). □

It was already noted in (1.11) of the Introduction that $W^\sigma(q,t) = W^{\sigma^{-1}}(t,q)$, and hence that $W(t,q), \overline{W}(t,q)$ are symmetric polynomials in t, q. It was further noted there that $W^\sigma(1,q) = W^\sigma(q,1) = W(q)$, so the maximal q-degree and maximal t-degree in $W^\sigma(t,q)$ are both equal to the degree of $W(q)$, namely $N^* = \sum_{i=1}^n (d_i - 1)$, the number of reflections in W. We note here one further symmetry property enjoyed by $\overline{W}(t,q)$.

Corollary 2.3. *For any complex reflection group W,*

$$(tq)^{N^*}\overline{W}(t^{-1}, q^{-1}) = \overline{W}(t,q).$$

In other words, for all i, j, the monomials $t^i q^j$ and $t^{N^*-i} q^{N^*-j}$ carry the same coefficient in $\overline{W}(t,q)$.

Proof. We offer two proofs.
Proof 1. Let R denote the operator on rational functions $f(t,q)$, defined by

$$R(f) := (tq)^{N^*} f(t^{-1}, q^{-1}).$$

We wish to show that R fixes $\overline{W}(t,q)$. In fact, one can check that in the formula for $\overline{W}(t,q)$ given by Corollary 2.2, the operator R will exchange the summand corresponding to w

$$\frac{\prod_{i=1}^n (1-t^{d_i})(1-q^{d_i})}{\det(1-tw)\det(1-q\overline{w})} = \frac{\prod_{i=1}^n (1-t^{d_i})(1-q^{d_i})}{\det(1-tw)\det(1-qw^{-1})} \tag{2.6}$$

with the summand for w^{-1}. To see this, apply R to (2.6) by first sending $t, q \mapsto t^{-1}, q^{-1}$, and then getting rid of all the negative powers of t, q in the numerator and denominator by multiplying in $(tq)^{N^*} = \frac{(tq)^{\sum_i d_i}}{(tq)^n}$. The result is

$$\frac{\prod_{i=1}^n (t^{d_i}-1)(q^{d_i}-1)}{\det(t-w)\det(q-w^{-1})}.$$

which one can see equals the summand for w^{-1} using these facts:

$$\begin{aligned}\det(t-w) &= \det(w)\det(tw^{-1}-1)\\ \det(q-w^{-1}) &= \det(w^{-1})\det(qw-1)\\ 1 &= \det(w)\det(w^{-1}).\end{aligned}$$

Proof 2. $\mathbb{C}[V\oplus V]^{\Delta^\sigma W}$ turns out to be a *Gorenstein ring*, as we now explain. In this case, $\Delta^\sigma W$ is a subgroup of $SL(V\oplus V)$, because

$$\det(w\oplus\overline{w}) = \det(w)\overline{\det(w)} = 1.$$

Hence the invariants when $\Delta^\sigma W$ acts on $\mathbb{C}[V\oplus V]$ form a Gorenstein ring by a result of Watanabe; see [25, Corollary 8.2].

This means that the Gorenstein quotient

$$\mathbb{C}[V\oplus V]^{\Delta^\sigma W}/(\mathbb{C}[V\oplus V]_+^{W\times W^\sigma})$$

of Krull dimension 0 is a Poincaré duality algebra. It shares the same socle bidegree (N^*,N^*) as the larger Gorenstein quotient $\mathbb{C}[V\oplus V]/(\mathbb{C}[V\oplus V]_+^{W\times W^\sigma})$, since there is an antidiagonally-invariant element $J(\mathbf{x})J(\mathbf{y})$ living in this bidegree. Here $J(\mathbf{x})$ is the *Jacobian*

$$J(\mathbf{x}) := \det\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j=1}^{n} = \prod_H (\ell_H)^{s_H},$$

where H runs through the reflecting hyperplanes for W, with ℓ_H any linear form defining H, and s_H is the number of reflections that fix H pointwise; see [25, Proposition 4.7]. □

Example 2.4. We analyze here in detail the case where $n=1$. Here $V=\mathbb{C}^1$ and $W=<c>=\mathbb{Z}/d\mathbb{Z}$ is cyclic, acting by a representation $\rho: W\to GL(V)$ that sends c to some primitive d^{th} root-of-unity ω. Then W acts on $\mathbb{C}[V]=\mathbb{C}[x]$ by $c(x)=\omega^{-1}x$, and $\mathbb{C}[V]^W=\mathbb{C}[x^d]$ so there is one fundamental degree $d_1=d$.

The coinvariant algebra is

$$A=\mathbb{C}[V]/(\mathbb{C}[V]_+^W)\cong\mathbb{C}[x]/(x^d)=\mathbb{C}\{1,x,x^2,\ldots,x^{d-1}\}$$

so its Hilbert series gives the Mahonian distribution

$$W(q)=\mathrm{Hilb}(\mathbb{C}[x]/(x^d);q)=1+q+q^2+\cdots+q^{d-1}.$$

Indexing the irreducible representations/characters as the powers $\{\rho^i\}_{i\in\mathbb{Z}/d\mathbb{Z}}$ of the primitive representation ρ, one can see that for $i=0,1,\ldots,d-1$ the homogeneous component $A_i=\mathbb{C}\{x^i\}$ carries the representation ρ^{d-i}, and hence the fake degree polynomials are given by $f^{\mathbf{1}}=1$ and $f^{\rho^i}(q)=q^{d-i}$ for $i=1,\ldots,d-1$.

Note that W is defined over the cyclotomic extension $\mathbb{Q}[\omega]=\mathbb{Q}[e^{\frac{2\pi i}{d}}]$, for which the Galois automorphisms take the form $\sigma(\omega)=\omega^s$ for some $s\in(\mathbb{Z}/d\mathbb{Z})^\times$.

Working in $\mathbb{C}[V\oplus V]=\mathbb{C}[x,y]$, regardless of the choice of σ, one has

$$\mathbb{C}[V\oplus V]^{W\times W^\sigma}=\mathbb{C}[x^d,y^d].$$

The diagonal invariants $\mathbb{C}[x,y]^{\Delta^\sigma W}$ have $\mathbb{C}$-basis given by the monomials

$$\{x^ay^b: a,b\in\mathbb{N}\text{ and } a+sb\equiv 0 \mod d\}.$$

Furthermore, $\mathbb{C}[x,y]^{\Delta^\sigma W}$ is a free $\mathbb{C}[x^d,y^d]$-module with $\mathbb{C}[x^d,y^d]$-basis given by the d monomials

$$\{x^ay^b: a,b\in\{0,1,\ldots,d-1\}\text{ and } a+sb\equiv 0 \mod d\}.$$

These monomials form a basis for the quotient

$$\mathbb{C}[V \oplus V]^{\Delta^\sigma W} / (\mathbb{C}[V \oplus V]_+^{W \times W^\sigma}),$$

and hence

$$W^\sigma(t,q) = \sum_{\substack{a,b \in \{0,1,\ldots,d-1\} \\ a+sb \equiv 0 \mod n}} t^a q^b.$$

In the two most important special cases, one has explicitly

$$\begin{aligned} \overline{W}(t,q) &= 1 + tq + t^2q^2 + \cdots + t^{d-1}q^{d-1} \\ &= \sum_\lambda f^\lambda(t) f^\lambda(q) \\ W(t,q) &= 1 + t^{d-1}q + t^{d-2}q^2 + \cdots + t^2q^{d-2} + tq^{d-1} \\ &= \sum_\lambda f^\lambda(t) f^{\overline{\lambda}}(q). \end{aligned}$$

Note that within this family $W = \mathbb{Z}/d\mathbb{Z}$, one has $\overline{W}(t,q) = W(t,q)$ if and only if $d \leq 2$, that is, exactly when W is actually a *real* reflection group– either $W = \mathbb{Z}/2\mathbb{Z}$ or the trivial group $W = \mathbb{Z}/1\mathbb{Z}$.

3. Bicyclic sieving phenomena

The *cyclic sieving phenomenon* for a triple $(X, X(q), C)$ was defined in [20]. Here X is a finite set with an action of a cyclic group C, and $X(q)$ is a polynomial in q with integer coefficients. We wish to generalize this notion to actions of bicyclic groups $C \times C'$ and bivariate polynomials $X(t,q)$, so we define this carefully here.

Let X be a finite set with a permutation action of a finite *bicyclic group*, that is a product $C \times C' \cong \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z}$. Fix embeddings

$$\begin{aligned} \omega &: C \hookrightarrow \mathbb{C}^\times \\ \omega' &: C' \hookrightarrow \mathbb{C}^\times \end{aligned}$$

into the complex roots-of-unity. Assume we are given a bivariate polynomial $X(t,q) \in \mathbb{Z}[t,q]$, with nonnegative integer coefficients.

Proposition 3.1. *(cf.* [20, Proposition 2.1]*) In the above situation, the following two conditions on the triple $(X, X(t,q), C \times C')$ are equivalent:*

(i) *For any $(c, c') \in C \times C'$*

$$X(\omega(c), \omega'(c')) = |\{x \in X : (c,c')x = x\}|.$$

(ii) *The coefficients $a(i,j)$ uniquely defined by the expansion*

$$X(t,q) \equiv \sum_{\substack{0 \leq i < k \\ 0 \leq j < \ell}} a(i,j) t^i q^j \mod (t^k - 1, q^\ell - 1)$$

have the following interpretation: $a(i,j)$ is the number of orbits of $C \times C'$ on X for which the $C \times C'$-stabilizer subgroup of any element in the orbit lies in the kernel of the $C \times C'$-character $\rho^{(i,j)}$ defined by

$$\rho^{(i,j)}(c,c') = \omega(c)^i \omega'(c')^j.$$

Definition 3.2. Say that the triple $(X, X(t,q), C \times C')$ *exhibits the bicyclic sieving phenomenon* (or *biCSP* for short) if it satisfies the conditions of Proposition 3.1.

In other words, whenever $(X, X(t,q), C \times C')$ exhibits the biCSP, not only is the evaluation $X(1,1)$ of $X(t,q)$ telling us the cardinality $|X|$, but the other root-of-unity evaluations $X(\omega,\omega')$ are telling us all the $C \times C'$-character values for the permutation action of $C \times C'$ on X, thus determining the representation up to isomorphism. Furthermore, the reduction of $X(q,t) \mod (q^k-1, t^\ell-1)$ has a simple interpretation for its coefficients.

Proof. (of Proposition 3.1) We follow the proof of [20, Proposition 2.1], by introducing a third equivalent condition.

Given $X(t,q)$ define an $\mathbb{N}^2$-graded complex vector space $A_X = \oplus_{i,j\geq 0}(A_X)_{ij}$ with $\dim_{\mathbb{C}}(A_X)_{ij}$ equal to the coefficient of t^iq^j in $X(t,q)$, so that by definition one has

$$\mathrm{Hilb}(A_X;\, t,q) = X(t,q).$$

Make $C \times C'$ act on A_X by having (c,c') act on the homogeneous component $(A_X)_{ij}$ via the scalar $\rho^{(i,j)}(c,c') = \omega(c)^i\omega'(c')^j$.

We claim that conditions (i) and (ii) in Proposition 3.1 are equivalent to

(iii) $A_X \cong \mathbb{C}[X]$ as (ungraded) $C \times C'$-representations.

The equivalence of (i) and (iii) follows immediately from the observation that $C \times C'$-representations are determined by their character values for each element (c,c'); in A_X this character value is $X(\omega(c),\omega'(c'))$ by construction, and in $\mathbb{C}[X]$ this character value is $|\{x \in X : (c,c')x = x\}|$.

For the equivalence of (ii) and (iii), first note that the complete set of irreducible representations or characters of $C \times C'$ are given by $\{\rho^{(i,j)}\}$ for $0 \leq i < k$ and $0 \leq j < \ell$. Consequently, (ii) holds if and only if for all such i,j one has

$$\langle \rho^{(i,j)}, A_X\rangle_{C\times C'} = \langle \rho^{(i,j)}, \mathbb{C}[X]\rangle_{C\times C'}. \tag{3.1}$$

We compute the left side of (3.1):

$$\begin{aligned}\langle \rho^{(i,j)}, A_X\rangle_{C\times C'} &= \frac{1}{|C\times C'|}\sum_{(c,c')\in C\times C'} \rho^{(i,j)}((c,c')^{-1})\chi_{A_X}(c,c')\\ &= \frac{1}{k\ell}\sum_{\substack{(\omega,\omega'):\\ \omega^k=1\\ (\omega')^\ell=1}} \omega^{-i}(\omega')^{-j}X(\omega,\omega')\\ &= a(i,j).\end{aligned}$$

To compute the right side of (3.1), first decompose X into its various $C \times C'$-orbits $\mathcal{O}$. Denote by $G_{\mathcal{O}}$ the $C \times C'$-stabilizer subgroup of any element in the orbit $\mathcal{O}$. One then has

$$\begin{aligned}\langle \rho^{(i,j)}, \mathbb{C}[X]\rangle_{C\times C'} &= \sum_{\mathcal{O}} \langle \rho^{(i,j)}, \mathbb{C}[\mathcal{O}]\rangle_{C\times C'}\\ &= \sum_{\mathcal{O}} \langle \rho^{(i,j)}, \mathrm{Ind}_{G_{\mathcal{O}}}^{C\times C'}\mathbf{1}\rangle_{C\times C'}\\ &= \sum_{\mathcal{O}} \langle \mathrm{Res}_{G_{\mathcal{O}}}^{C\times C'}\rho^{(i,j)}, \mathbf{1}\rangle_{G_{\mathcal{O}}}\end{aligned}$$

and each term in this last sum is either 1 or 0 depending upon whether $G_{\mathcal{O}}$ lies in the kernel of $\rho^{(i,j)}$, or not. Thus (ii) holds if and only if (3.1) holds for all i,j, that is, if and only if (iii) holds. $\square$

4. Springer's regular elements and proof of Theorem 1.4

In order to prove Theorem 1.4, we first must recall Springer's notion of a regular element in a complex reflection group.

Definition 4.1. Given a complex reflection group W in $GL(V)$, a vector $v \in V$ is called *regular* if it lies on none of the reflecting hyperplanes for reflections in W. An element c in W is called *regular* if it has a regular eigenvector v; the eigenvalue ω for which $c(v) = \omega v$ will be called the accompanying *regular* eigenvalue. One can show [24, p. 170] that ω will have the same multiplicative order in $\mathbb{C}^\times$ as the multiplicative order of c in W.

Note that the eigenvalues of a regular element need not lie in the *smallest* cyclotomic extension over which W is defined; Example 4.2 below illustrates this. However, if one chooses m to be the least common multiple of the orders of the elements of W, then any such eigenvalue *will* lie in the cyclotomic extension $\mathbb{Q}[e^{\frac{2\pi i}{m}}]$. Hence any σ in $\mathrm{Gal}(\mathbb{Q}[e^{\frac{2\pi i}{m}}]/\mathbb{Q})$ acts on any such eigenvalue ω, with $\sigma(\omega) = \omega^s$ for some unique $s \in (\mathbb{Z}/d\mathbb{Z})^\times$, where d is the order of the regular element c. For the remainder of this section we will always assume that m is this least common multiple.

Example 4.2. Springer [24, §5] classified the regular elements in the real reflection groups. When $W = \mathfrak{S}_n$, regular vectors are those vectors in $\mathbb{C}^n$ for which all coordinates are distinct. Any n-cycle or $(n-1)$-cycle is a regular element– for example $c = (1\,2\,\cdots\,n)$ has regular eigenvalue ω and regular eigenvector $v = (1, \omega, \omega^2, \ldots, \omega^{n-1})$, if ω is any primitive n^{th} root-of-unity. Similarly, $c' = (1\,2\,\cdots\,n-1)(n)$ has regular eigenvalue ζ and regular eigenvector $v' = (1, \zeta, \zeta^2, \ldots, \zeta^{n-1}, 0)$ for any primitive $(n-1)^{st}$ root-of-unity ζ. Any power of an n-cycle will therefore also be regular, with the same regular eigenvector v, as will any power of an $(n-1)$-cycle.

It turns out (and is not hard to see directly) that there are no other regular elements besides powers of n-cycles and $(n-1)$-cycles in $\mathfrak{S}_n$.

Springer proved the following.

Theorem 4.3. [24, Prop. 4.5] *For a regular element c in a complex reflection group W, with an accompanying regular eigenvalue ω, one has*

$$\chi^\lambda(c) = f^\lambda(\omega^{-1}).$$

This leads to an interesting interaction between Galois conjugates σ and a regular element c. Recall that there is a unique $s \in (\mathbb{Z}/d\mathbb{Z})^\times$ such that the regular eigenvalue ω satisfies $\sigma(\omega) = \omega^s$.

Proposition 4.4. *With the above notation, for any W-irreducible λ, one has*

$$\chi^{\sigma(\lambda)}(c) = \chi^\lambda(c^s).$$

Proof. Note that since c is regular with regular eigenvalue ω, one has that c^s is regular with regular eigenvalue ω^s. Then one has

$$\begin{aligned}\chi^{\sigma(\lambda)}(c) &= \sigma(\chi^\lambda(c)) \\ &= \sigma\left(f^\lambda(\omega^{-1})\right) \\ &= f^\lambda\left(\sigma(\omega^{-1})\right) \\ &= f^\lambda(\omega^{-s}) \\ &= \chi^\lambda(c^s)\end{aligned}$$

where the second and fifth equalities use Theorem 4.3. □

From this we can now prove Theorem 1.4, which we rephrase here as a biCSP, after establishing the appropriate notation.

Given two regular elements c, c' in W, with regular eigenvalues ω, ω', embed the cyclic groups $C = \langle c \rangle, C' = \langle c' \rangle$ into $\mathbb{C}^\times$ by sending

$$\begin{aligned} c &\mapsto \omega^{-1} \\ c' &\mapsto (\omega')^{-1}. \end{aligned}$$

As above, let d denote the multiplicative order of c and of ω, and given the Galois automorphism σ, define $s \in (\mathbb{Z}/d\mathbb{Z})^\times$ by the property that $\sigma(\omega) = \omega^s$

Let $X = W$, carrying the following σ-twisted left-action of $C \times C'$:

$$(c, c') \cdot w := c^s w (c')^{-1}.$$

Let $X(t, q) = W^\sigma(t, q)$.

Theorem 1.4 (rephrased). *In the above setting,* $(X, X(t, q), C \times C')$ *exhibits the biCSP.*

Proof. As noted earlier, if c is a regular element c of W with regular eigenvalue ω, then any power c^i is also regular, with regular eigenvalue ω^i. Hence it suffices for us to show that

$$W^\sigma(\omega^{-1}, (\omega')^{-1}) = |\{w \in W : c^s w (c')^{-1} = w\}|.$$

This follows from a string of equalities, explained below:

$$\begin{aligned} W^\sigma(\omega^{-1}, (\omega')^{-1}) &= \sum_\lambda f^{\sigma(\lambda)}(\omega^{-1}) f^{\overline{\lambda}}((\omega')^{-1}) \\ &= \sum_\lambda \chi^{\sigma(\lambda)}(c) \chi^{\overline{\lambda}}(c') \\ &= \sum_\lambda \chi^\lambda(c^s) \overline{\chi^\lambda(c')} \\ &= \begin{cases} |\mathrm{Cent}_W(c^s)| & \text{if } c', c^s, \text{ are } W\text{-conjugate} \\ 0 & \text{otherwise.} \end{cases} \\ &= |\{w \in W : w^{-1} c^s w = c'\}| \\ &= |\{w \in W : c^s w (c')^{-1} = w\}| \end{aligned}$$

The first equality is by Definition 1.1 for $W^\sigma(t,q)$. The second equality uses Theorem 4.3. The third equality uses Proposition 4.4. The fourth equality uses the column orthogonality relation for the character table of W. $\square$

5. Type A: bipartite partitions and work of Carlitz, Wright, Gordon

We discuss briefly here the known root-of-unity evaluations for $W^\sigma(t,q)$ in the much-studied case $W = \mathfrak{S}_n$. Since this W is defined over $\mathbb{Q}$, one may assume σ is the identity and study only $\mathfrak{S}_n(t,q) = W(t,q)$.

Theorem 1.4 combinatorially interprets $\mathfrak{S}_n(\omega,\omega')$ when the roots of unity in question have orders that divide either $n-1$ or n, since these are exactly the orders of regular elements in $W = \mathfrak{S}_n$; see Example 4.2. This combinatorial interpretation is consistent with evaluations done by Gordon [16], building on work of Carlitz [7] and Wright [29]. In fact, Gordon's work actually leads to some further evaluations of $\mathfrak{S}_n(\omega,\omega')$ more generally when the orders of ω,ω' are at most n, as we now explain.

The starting point both for Carlitz and Wright is a generating function for bipartite partitions. Every $\Delta\mathfrak{S}_n$-orbit of monomials $\mathbf{x}^{\mathbf{i}}\mathbf{y}^{\mathbf{j}} = \prod_{m=1}^n x_m^{i_m} y_m^{j_m}$ in $\mathbb{C}[V \oplus V] = \mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]$ corresponds to an (unordered) multiset of n exponent vectors $\{(i_m,j_m)\}_{m=1,\ldots,n}$. A canonically chosen ordering for this multiset, say in lexicographic order, is known as a *bipartite partition*; for further background on this topic, see Solomon [23], Garsia and Gessel [15], and Stanley [25, Example 5.3].

Since such orbits of monomials also form a $\mathbb{C}$-basis for $\mathbb{C}[V\oplus V]^{\Delta\mathfrak{S}_n}$, one obtains the following generating function for the $\mathfrak{S}_n(t,q)$:

$$
\begin{aligned}
\prod_{i,j\geq 0} \frac{1}{1-t^iq^ju} &= \sum_{n\geq 0} u^n \mathrm{Hilb}(\mathbb{C}[V\oplus V]^{\Delta\mathfrak{S}_n};\, t,q) \\
&= \sum_{n\geq 0} u^n \frac{\mathfrak{S}_n(t,q)}{(t;t)_n(q;q)_n}
\end{aligned} \tag{5.1}
$$

where we have used the notation

$$
(q;q)_n := (1-q)(1-q^2)\cdots(1-q^n) = \frac{1}{\mathrm{Hilb}(\mathbb{C}[V]^{\mathfrak{S}_n},q)}
$$

to rewrite (2.5) in this situation. Logarithmically differentiating (5.1) gives the following recurrence used by Wright [29], Carlitz [7], and then Gordon [16].

Proposition 5.1. $\mathfrak{S}_n(t,q)$ *is uniquely defined by the recurrence*

$$
\mathfrak{S}_n(t,q) = \frac{1}{n}\sum_{m=1}^{n} \frac{(t;t)_n(q;q)_n}{(t;t)_{n-m}(q;q)_{n-m}} \frac{\mathfrak{S}_{n-m}(t,q)}{(1-t^m)(1-q^m)}
$$

with initial condition $\mathfrak{S}_0(t,q) = 1$.

From Proposition 5.1, Gordon deduced via induction on n the following lemma about the behavior of $\mathfrak{S}_n(t,q)$ when one of the two variables is specialized to a root of unity.

Lemma 5.2. [16, §3] *Let ω be a primitive ℓ^{th} root of unity, and write $n = m\ell + r$ with $0 \leq r < \ell$. Then*

$$\mathfrak{S}_n(\omega, q) = \frac{(q;q)_n}{(q;q)_r(1-q^\ell)^m}\mathfrak{S}_r(\omega, q).$$

The next corollary gives two interesting evaluations of $\mathfrak{S}_n(\omega, \omega')$ obtainable with a little work by taking $q = \omega'$ in Lemma 5.2. The first was already deduced by Gordon as one of his main results. Although the second he seems not to have written down, we omit its relatively straightforward proof here.

Corollary 5.3. *Let ω, ω' be roots of unity.*

(i) *If they have unequal orders, with both orders at most n, then*

$$\mathfrak{S}_n(\omega, \omega') = 0.$$

(ii) *If their orders are both ℓ, then*

$$\mathfrak{S}_n(\omega, \omega') = \ell^m m! \mathfrak{S}_r(\omega, \omega')$$

where one has written uniquely $n = m\ell + r$ with $0 \leq r < \ell$.

In particular, if $n \equiv 0, 1 \mod \ell$ then $\mathfrak{S}_n(\omega, \omega') = \ell^m m!$.

We explain here how Corollary 5.3 recovers (more than) the assertion of Theorem 1.4 for $W = \mathfrak{S}_n$. Assume that c, c' in $W = \mathfrak{S}_n$ are regular elements, and ω, ω' their corresponding regular eigenvalues. According to Example 4.2, c, c' must each be a power of an $(n-1)$-cycle or n-cycle. Note that this means that c, c' will be W-conjugate (that is, have the same cycle type) if and only if they have the same multiplicative order.

Case 1. c, c' are not W-conjugate.

In this case, Theorem 1.4 predicts $\mathfrak{S}_n(\omega, \omega') = 0$. By the above comment, ω, ω' have unequal orders, and hence Corollary 5.3(i) also predicts $\mathfrak{S}_n(\omega, \omega') = 0$.

Case 2. c, c' are W-conjugate.

In this case, Theorem 1.4 predicts that $\mathfrak{S}_n(\omega', \omega)$ will be the number of elements $w \in W$ with $cw(c')^{-1} = w$, or equivalently, $w^{-1}cw = c'$, which is counted by the centralizer-order $|\mathrm{Cent}_W(c)| = \ell^m m!$. If c, c' have order ℓ, their being regular forces $n = m\ell + r$ with remainder $r = 0$ or $r = 1$, and Corollary 5.3(ii) predicts

$$\mathfrak{S}_n(\omega', \omega) = \ell^m m! \mathfrak{S}_r(\omega', \omega) = \ell^m m!$$

since $\mathfrak{S}_0(t, q) = \mathfrak{S}_1(t, q) = 1$.

We close this section with a few remarks.

Remark 5.4. The results in [4, §4, 5] are assertions about the coefficients $a_{d,d}(i, j)$ defined in (1.12) for $W = \mathfrak{S}_n$ with $n \cong 0, 1 \mod d$. The approach taken there is to deduce them from an explicit number-theoretic formula [4, Theorem 4.1] for $a_{n,n}(i, j)$. This formula is in turn derived using a result of Stanley and Kraskiewicz-Weyman (equivalent to Theorem 4.3 in type A) by first reinterpreting the coefficients $a_{n,n}(i, j)$ as certain intertwining numbers:

$$a_{n,n}(i,j) = \langle \mathrm{Ind}_{C_n}^{\mathfrak{S}_n} \rho^i, \mathrm{Ind}_{C_n}^{\mathfrak{S}_n} \rho^j, \rangle_{\mathfrak{S}_n}. \tag{5.2}$$

Here C_n denotes the cyclic subgroup of $\mathfrak{S}_n$ generated by an n-cycle c, and $\rho : C_n \to \mathbb{C}^\times$ is the primitive character sending c to a primitive n^{th} root-of-unity ω.

It is not hard to check that the combinatorial interpretation for $a_{n,n}(i,j)$ given by Theorem 1.4 (counting $C_n \times C_n$-orbits $\mathcal{O}$ on $\mathfrak{S}_n$ whose stabilizer $G_{\mathcal{O}} \subset \ker \rho^{(i,j)}$) is equivalent to (5.2) via Mackey's formula. Thus the combinatorial interpretation for $a_{n,n}(i,j)$ gives another route to the results of [4, §4,5].

Remark 5.5. The identity (5.1) gives a concise generating function that compiles bivariate distributions $(\mathrm{maj}(w), \mathrm{maj}(w^{-1}))$ for all of the symmetric groups $W = \mathfrak{S}_n$. It is a specialization of a stronger identity due to Garsia and Gessel [15] that does the same for the four-variate distribution of $(\mathrm{maj}(w), \mathrm{maj}(w^{-1}), \mathrm{des}(w), \mathrm{des}(w^{-1}))$ where $\mathrm{des}(w) := \{i : w(i) > w(i+1)\}$ is the number of *descents* in w.

More recently, Foata and Han [13, Eqn. (1.8)] generalized this by giving such a generating function for the *hyperoctahedral groups* $W = W(B_n) = \mathbb{Z}/2\mathbb{Z} \wr \mathfrak{S}_n$ of signed permutations. Their result incorporates the five-variate distribution of certain statistics $(\mathrm{fmaj}(w), \mathrm{fmaj}(w^{-1}), \mathrm{fdes}(w), \mathrm{fdes}(w^{-1}), \mathrm{neg}(w))$. Here $\mathrm{neg}(w)$ is the number of negative signs appearing in the signed permutation, so this distribution for $W(B_n)$ can be specialized to the previous one for $\mathfrak{S}_n$.

Remark 5.6. Corollary 5.3 leaves us with the question of what can one say about $\mathfrak{S}_r(\omega, \omega')$ for roots of unity ω, ω' of the *same* order d when $2 \leq r \leq d-1$. In general, such evaluations $\mathfrak{S}_r(\omega, \omega')$ can be negative real numbers, or can lie in $\mathbb{C} \setminus \mathbb{R}$.

There is however at least one more (somewhat trivial) thing one can say about the "antidiagonal" values.

Proposition 5.7. *For any root of unity ω, one has that $\mathfrak{S}_n(\omega, \omega^{-1})$ lies in $\mathbb{R}$.*

Proof. The fact that $\mathfrak{S}_n(t,q)$ has real coefficients and is symmetric in t, q implies that $\mathfrak{S}_n(\omega, \omega^{-1})$ is fixed under complex conjugation:

$$\overline{\mathfrak{S}_n(\omega, \omega^{-1})} = \mathfrak{S}_n(\overline{\omega}, \overline{\omega^{-1}}) = \mathfrak{S}_n(\omega^{-1}, \omega) = \mathfrak{S}_n(\omega, \omega^{-1}).$$

□

6. The wreath products $\mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_n$, tableaux, and the flag-major index

We review here for the classical complex reflection groups $W = \mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_n$, how one expresses the fake degrees and bimahonian distributions in terms of major-index-like statistics, both on tableaux and on W. The one new result here is Theorem 1.3, which generalizes a result of Adin and Roichman [1].

We first review the motivating special case when $W = \mathfrak{S}_n$; see also [14]. It appears that Lusztig first computed (see [25, Prop. 4.11], and [17]) the following formula for the fake degree polynomial $f^\lambda(q)$ for the irreducible representation of $\mathfrak{S}_n$ indexed by a partition λ of n:

$$f^\lambda(q) = \sum_{Q \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(Q)}.$$

Here $\mathrm{SYT}(\lambda)$ is the set of *standard Young tableaux* of shape λ, that is, fillings of the Ferrers/Young diagram for λ with each number $1, 2, \ldots, n$ occurring exactly once, increasing left-to-right in rows and top-to-bottom in columns. The *major index* statistic $\mathrm{maj}(Q)$ is defined by

$$\mathrm{maj}(Q) := \sum_{i \in \mathrm{Des}(Q)} i,$$

where the *descent set* $\mathrm{Des}(Q)$ is the set of values i for which $i+1$ occurs in a lower row of Q. The relation to the bimahonian distribution is provided by the *Robinson-Schensted correspondence*, which gives a bijection between permutations w in $\mathfrak{S}_n$ and pairs (P,Q) of standard Young tableaux of the same shape. Some fundamental properties of this bijection are that if $w \mapsto (P,Q)$, then

$$\begin{aligned} w^{-1} &\mapsto (Q,P) \\ \mathrm{Des}(w) &= \mathrm{Des}(Q) \\ \mathrm{Des}(w^{-1}) &= \mathrm{Des}(P). \end{aligned} \tag{6.1}$$

Hence one obtains

$$\begin{aligned} W(t,q) &= \sum_{\lambda} f^{\lambda}(t) f^{\lambda}(q) \\ &= \sum_{(P,Q)} t^{\mathrm{maj}(Q)} q^{\mathrm{maj}(P)} \\ &= \sum_{w \in \mathfrak{S}_n} t^{\mathrm{maj}(w)} q^{\mathrm{maj}(w^{-1})} \end{aligned} \tag{6.2}$$

and where the second sum is over all pairs (P,Q) of standard Young tableaux of size n of the same shape.

For W a Weyl group of type B_n, that is, $W = \mathbb{Z}/2\mathbb{Z} \wr \mathfrak{S}_n$, Adin and Roichman [1], defined a *flag major index* statistic $\mathrm{fmaj}(w)$, equidistributed with the Coxeter group length. More generally, they define such a statistic for $W = \mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_n$ with the property that

$$\begin{aligned} \mathrm{Hilb}(\mathbb{C}[V \oplus V]^{\Delta W} / \mathbb{C}[V \oplus V]^{W \times W}; q, t) \, (= W(t,q)) \\ = \sum_{w \in \mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_n} t^{\mathrm{fmaj}(w)} q^{\mathrm{fmaj}(w^{-1})}. \end{aligned} \tag{6.3}$$

A variation on the $d=2$ (type B) case of this appears in Biagioli and Bergeron [5]. For W a Weyl group of type D_n, a similar $\mathrm{fmaj}(w)$ statistic was defined by Biagioli and Caselli [6], who proved an analogous result to (6.3).

As one might expect, there is an approach to (6.3) as in (6.2): start with the fake-degree Definition 1.1 for $W(t,q)$, use a tableau formula for the fake-degrees, and then apply something like a Robinson-Schensted correspondence– such a proof is sketched in [2, §5]. We pursue here a similar approach[3] to the more general Theorem 1.3.

Let ω be a primitive d^{th} root-of-unity. A typical element w in $W = \mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_n$, sending e_i to $\omega^k e_j$, can be represented by a string of letters $w_1 \cdots w_n$, where $w_i = \omega^k j$. Here the letters come from an alphabet that we linearly order as follows:

$$\begin{aligned} &\omega^{d-1}1 \prec \cdots \prec \omega^{d-1}n \\ &\prec \cdots \\ &\prec \omega^1 1 \prec \cdots \prec \omega^1 n \\ &\prec \omega^0 1 \prec \cdots \prec \omega^0 n. \end{aligned} \tag{6.4}$$

[3]The second author thanks M. Taşkin for explaining how this works in type B_n.

Call the letters $\omega^k 1, \ldots, \omega^k n$ the k^{th} *subalphabet*, and let $r_k(w)$ denote the number of letters w_i that lie in this subalphabet. Let

$$\mathrm{maj}(w) := \sum_{\substack{i=1,\ldots,n-1: \\ w_{i+1} \prec w_i}} i$$

denote the usual major index of w with respect to this order, as defined in (1.2) above. Although $\mathrm{fmaj}(w)$ can be defined in a somewhat more algebraic way, it is shown in [1, Theorem 3.1] that it is equivalent to this combinatorial expression:

$$\mathrm{fmaj}(w) = d \cdot \mathrm{maj}(w) + \sum_{k=0}^{d-1} k \cdot r_k(w).$$

The smallest cyclotomic extension over which $W = \mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_n$ can be defined is $\mathbb{Q}[e^{\frac{2\pi i}{d}}]$. Note that since a Galois automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}[e^{\frac{2\pi i}{d}}]/\mathbb{Q})$ will always be of the form $\sigma(\omega) = \omega^s$ for some $s \in (\mathbb{Z}/d\mathbb{Z})^\times$, one has the interesting feature here that $\sigma(w) \in W$ for all $w \in W$. Thus $W^\sigma = \sigma(W) = W$, although σ does not fix w pointwise.

Tableaux expressions for the fake degrees of Weyl groups of type B_n, D_n were computed originally by Lusztig [18], and generalized to the Shepard-Todd infinite family $G(de, e, n)$ of complex reflection groups by Stembridge [28]. Irreducibles for $W = \mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_n$ can be indexed by skew diagrams

$$\lambda = (\lambda^{d-1} \oplus \cdots \oplus \lambda^1 \oplus \lambda^0)$$

having n cells total, consisting of d-tuples of Ferrers diagrams λ^k, arranged in the plane so that λ^{k-1} is northeast of λ^k (using English notation for Ferrers diagrams). Define a *standard Young tableau* of shape λ to be a filling of the skew diagram λ with the numbers $1, 2, \ldots, n$, increasing left-to-right in rows and top-to-bottom in columns. Given such a standard Young tableau Q of the skew shape λ define its descent set $\mathrm{Des}(Q)$ and major index $\mathrm{maj}(w)$ with respect to the ordering (6.4), and then define

$$\mathrm{fmaj}(Q) = d \cdot \mathrm{maj}(Q) + \sum_{k=0}^{d-1} k \cdot |\lambda^k|. \tag{6.5}$$

After reviewing this indexing of irreducibles, Stembridge proves the following.

Theorem 6.1. *(*[28, Theorem 5.3]*) For $W = \mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_n$, one has the fake degree expression*

$$f^\lambda = \sum_Q q^{\mathrm{fmaj} Q}.$$

where Q runs over all standard tableaux of shape λ,

Lastly, we recall from [27] one of the standard generalizations of the Robinson-Schensted correspondence to $W = \mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_n$. Given $w \in W$, one produces a pair (P, Q) of standard tableaux of the same skew shape λ having

$$|\lambda_k| = r_k(w) \tag{6.6}$$

in which the letters w_i coming from the k^{th} subalphabet are inserted using the usual Robinson-Schensted algorithm into the subtableaux P^k that occupies the subshape λ^k, and their position i is recorded in the subtableaux Q^k. As one varies over all λ with n cells as above, one gets a bijection to W.

Proposition 6.2. *This bijection $w \mapsto (P,Q)$ between $W = \mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_n$ and pairs of standard Young tableaux of the same shape $\lambda = (\lambda^{d-1} \oplus \cdots \oplus \lambda^1 \oplus \lambda^0)$ having n cells, has the following properties:*

(i)
$$\overline{w}^{-1} \mapsto (Q,P).$$

(ii)
$$\begin{aligned}\mathrm{Des}(w) &= \mathrm{Des}(Q)\\ \mathrm{fmaj}(Q) &= \mathrm{fmaj}(w)\\ \mathrm{fmaj}(P) &= \mathrm{fmaj}(\overline{w}^{-1}).\end{aligned}$$

(iii) *For any Galois automorphism σ,*
$$\sigma(w) \mapsto (\sigma(P), \sigma(Q)).$$

Here $\sigma(Q)$ denotes the skew tableaux obtained from Q by re-indexing its subtableaux Q^i according to the permutation by which σ, thought of as an element of $(\mathbb{Z}/d\mathbb{Z})^\times$, acts on the indices $i \in \mathbb{Z}/d\mathbb{Z}$.

Proof. The first two properties follow immediately from (6.1), (6.5), and (6.6), while the third is an easy consequence of the definitions. □

We can now recall the statement of Theorem 1.3 and prove it.

Theorem 1.3. *For the wreath products $W = \mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_n$ and any Galois automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}[e^{\frac{2\pi i}{d}}]/\mathbb{Q})$, one has*
$$W^\sigma(t,q) = \sum_{w \in W} q^{\mathrm{fmaj}(w)} t^{\mathrm{fmaj}(\sigma(w^{-1}))}.$$

Proof. One calculates as follows, using the above bijection $w \mapsto (P,Q)$ and its properties from Proposition 6.2:
$$\begin{aligned}\sum_{w \in W} q^{\mathrm{fmaj}(w)} t^{\mathrm{fmaj}(\sigma(w^{-1}))} &= \sum_{w \in W} q^{\mathrm{fmaj}(\overline{\sigma}^{-1}(w))} t^{\mathrm{fmaj}(\overline{w}^{-1})}\\ &= \sum_{(P,Q)} q^{\mathrm{fmaj}(\overline{\sigma}^{-1}(Q))} t^{\mathrm{fmaj}(P)}\\ &= \sum_{\lambda} f^{\overline{\sigma}^{-1}(\lambda)}(q) f^\lambda(t)\\ &= W^\sigma(t,q)\end{aligned}$$
□

Question 6.3. For which complex reflection groups can one produce an fmaj statistic that allows one to generalize Theorem 1.3?

As a first step, one might try to do this for the infinite family $G(de,e,n)$ of complex reflection groups. Presumably such a generalization would involve the fake-degree formulae of Stembridge [28] for $G(de,e,n)$, as well as generalizing the work of Biagioli and Caselli [6] for Weyl groups of type D_n $(= G(2,2,n))$.

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