BOOTSTRAPPING AND ASKEY-WILSON POLYNOMIALS

JANG SOO KIM AND DENNIS STANTON

Abstract. The mixed moments for the Askey-Wilson polynomials are found using a bootstrapping method and connection coefficients. A similar bootstrapping idea on generating functions gives a new Askey-Wilson generating function. Modified generating functions of orthogonal polynomials are shown to generate polynomials satisfying recurrences of known degree greater than three. An important special case of this hierarchy is a polynomial which satisfies a four term recurrence, and its combinatorics is studied.

1. Introduction

The Askey-Wilson polynomials [1] $p_n(x;a,b,c,d|q)$ are orthogonal polynomials in x which depend upon five parameters: a, b, c, d and q. In [2, §2] Berg and Ismail use a bootstrapping method to prove orthogonality of Askey-Wilson polynomials by initially starting with the orthogonality of the $a=b=c=d=0$ case, the continuous q-Hermite polynomials, and successively proving more general orthogonality relations, adding parameters along the way.

In this paper we implement this idea in two different ways. First, using successive connection coefficients for two sets of orthogonal polynomials, we will find explicit formulas for generalized moments of Askey-Wilson polynomials, see Theorem 2.4. This method also gives a heuristic for a relation between the two measures of the two polynomial sets, see Remark 2.3, which is correct for the Askey-Wilson hierarchy. Using this idea we give a new generating function (Theorem 2.9) for Askey-Wilson polynomials when $d=0$.

The second approach is to assume the two sets of polynomials have generating functions which are closely related, up to a q-exponential factor. We prove in Theorem 3.1, Theorem 3.6, and Theorem 3.15 that if one set is an orthogonal set, the second set has a recurrence relation of predictable order, which may be greater than three. We give several examples using the Askey-Wilson hierarchy.

Finally we consider a more detailed example of the second approach, using a generating function to define a set of polynomials called the discrete big q-Hermite polynomials. These polynomials satisfy a 4-term recurrence relation. We give the moments for the pair of measures for their orthogonality relations. Some of the combinatorics for these polynomials is given in § 5. Finally we record in Proposition 6.1 a possible q-analogue of the Hermite polynomial addition theorem.

We shall use basic hypergeometric notation, which is in Gasper-Rahman [6] and Ismail [7].

2. Askey-Wilson polynomials and connection coefficients

The connection coefficients are defined as the constants obtained when one expands one set of polynomials in terms of another set of polynomials.

For the Askey-Wilson polynomials [7, 15.2.5, p. 383]

$$p_n(x;a,b,c,d|q) = \frac{(ab,ac,ad)_n}{a^n} {}_4\phi_3\left(\left.\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}\right| q;q\right), \quad x=\cos\theta$$

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we shall use the connection coefficients obtained by successively adding a parameter

$$(a,b,c,d)=(0,0,0,0)\to(a,0,0,0)\to(a,b,0,0)\to(a,b,0,0)\to(a,b,c,0)\to(a,b,c,d).$$

Using a simple general result on orthogonal polynomials, we derive an almost immediate proof of an explicit formula for the mixed moments of Askey-Wilson polynomials.

First we set the notation for an orthogonal polynomial set $p_n(x)$. Let $\mathcal{L}_p$ be the linear functional on polynomials for which orthogonality holds

$$\mathcal{L}_p(p_m(x)p_n(x))=h_n\delta_{mn},\quad 0\le m,n.$$

Definition 2.1. The mixed moments of $\mathcal{L}_p$ are $\mathcal{L}_p(x^n p_m(x))$, $\quad 0\le m,n.$

The main tool is the following Proposition, which allows the computation of mixed moments of one set of orthogonal polynomials from another set if the connection coefficients are known.

Proposition 2.2. *Let $R_n(x)$ and $S_n(x)$ be orthogonal polynomials with linear functionals $\mathcal{L}_R$ and $\mathcal{L}_S$, respectively, such that $\mathcal{L}_R(1)=\mathcal{L}_S(1)=1$. Suppose that the connection coefficients are*

$$R_k(x)=\sum_{i=0}^{k}c_{k,i}S_i(x). \tag{1}$$

Then

$$\mathcal{L}_S(x^nS_m(x))=\sum_{k=0}^{n}\frac{\mathcal{L}_R(x^nR_k(x))}{\mathcal{L}_R(R_k(x)^2)}c_{k,m}\mathcal{L}_S(S_m(x)^2).$$

Proof. If we multiply both sides of (1) by $S_m(x)$ and apply $\mathcal{L}_S$, we have

$$\mathcal{L}_S(R_k(x)S_m(x))=c_{k,m}\mathcal{L}_S(S_m(x)^2).$$

Then by expanding x^n in terms of $R_k(x)$

$$x^n=\sum_{k=0}^{n}\frac{\mathcal{L}_R(x^nR_k(x))}{\mathcal{L}_R(R_k(x)^2)}R_k(x)$$

we find

$$\mathcal{L}_S(x^nS_m(x))=\mathcal{L}_S\left(\sum_{k=0}^{n}\frac{\mathcal{L}_R(x^nR_k(x))}{\mathcal{L}_R(R_k(x)^2)}R_k(x)S_m(x)\right)=\sum_{k=0}^{n}\frac{\mathcal{L}_R(x^nR_k(x))}{\mathcal{L}_R(R_k(x)^2)}c_{k,m}\mathcal{L}_S(S_m(x)^2).$$

□

Remark 2.3. One may also use the idea of Proposition 2.2 to give a heuristic for representing measures of the linear functionals. Putting $m=0$, if representing measures were absolutely continuous, say $w_R(x)dx$ for $R_n(x)$, and $w_S(x)dx$ for $S_n(x)$ then one might guess that

$$w_S(x)=w_R(x)\sum_{k=0}^{\infty}\frac{R_k(x)}{\mathcal{L}_R(R_k(x)^2)}c_{k,0}.$$

For the rest of this section we will compute the mixed moments $\mathcal{L}_p(x^np_m(x))$ for the Askey-Wilson polynomials using Proposition 2.2 starting from the q-Hermite polynomials.

Let $\mathcal{L}_{a,b,c,d}$ be the linear functional for $p_n(x;a,b,c,d|q)$ satisfying $\mathcal{L}_{a,b,c,d}(1)=1$. Then $\mathcal{L}=\mathcal{L}_{0,0,0,0}$, $\mathcal{L}_a=\mathcal{L}_{a,0,0,0}$, $\mathcal{L}_{a,b}=\mathcal{L}_{a,b,0,0}$, and $\mathcal{L}_{a,b,c}=\mathcal{L}_{a,b,c,0}$ are the linear functionals for these polynomials: q-Hermite, $H_n(x|q)=p_n(x;0,0,0,0|q)$, the big q-Hermite $H_n(x;a|q)=p_n(x;a,0,0,0|q)$, the Al-Salam-Chihara $Q_n(x;a,b|q)=p_n(x;a,b,0,0|q)$, and the dual q-Hahn $p_n(x;a,b,c|q)=p_n(x;a,b,c,0|q)$.

The L^2-norms are given by [7, 15.2.4 p.383]

$$\mathcal{L}(H_n(x|q)H_m(x|q)) = (q)_n\delta_{mn}, \tag{2}$$

$$\mathcal{L}_a(H_n(x;a|q)H_m(x;a|q)) = (q)_n\delta_{mn}, \tag{3}$$

$$\mathcal{L}_{a,b}(Q_n(x;a,b|q)Q_m(x;a,b|q)) = (q,ab)_n\delta_{mn}, \tag{4}$$

$$\mathcal{L}_{a,b,c}(p_n(x;a,b,c|q)p_m(x;a,b,c|q)) = (q,ab,ac,bc)_n\delta_{mn}, \tag{5}$$

$$\mathcal{L}_{a,b,c,d}(p_n(x;a,b,c,d|q)p_m(x;a,b,c,d|q)) = \frac{(q,ab,ac,ad,bc,bd,cd,abcdq^{n-1})_n}{(abcd)_{2n}}\delta_{mn}. \tag{6}$$

To apply Proposition 2.2, we need the following connection coefficient formula for the Askey-Wilson polynomials given in [1, (6.4)]

$$\frac{p_n(x;A,b,c,d|q)}{(q,bc,bd,cd)_n} = \sum_{k=0}^{n} \frac{p_k(x;a,b,c,d|q)}{(q,bc,bd,cd)_k} \times \frac{a^{n-k}(A/a)_{n-k}(Abcdq^{n-1})_k}{(abcdq^{k-1})_k(q,abcdq^{2k})_{n-k}}. \tag{7}$$

The following four identities are special cases of (7):

$$H_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q H_k(x;a|q)a^{n-k}, \tag{8}$$

$$H_n(x;a|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q Q_k(x;a,b|q)b^{n-k}, \tag{9}$$

$$Q_n(x;a,b|q) = (ab)_n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{p_k(x;a,b,c|q)}{(ab)_k} c^{n-k}, \tag{10}$$

$$\frac{p_n(x;b,c,d|q)}{(q,bc,bd,cd)_n} = \sum_{k=0}^{n} \frac{p_k(x;a,b,c,d|q)}{(q,bc,bd,cd)_k} \cdot \frac{a^{n-k}}{(abcdq^{k-1})_k(q,abcdq^{2k})_{n-k}}. \tag{11}$$

For the initial mixed moment we need the following result proved independently by Josuat-Vergès [12, Proposition 5.1] and Cigler [3, Proposition 15]

$$\mathcal{L}(x^nH_m(x;q)) = \frac{(q)_m}{2^n}\overline{P}(n,m),$$

where

$$\overline{P}(n,m) = \sum_{k=m}^{n} \left(\binom{n}{\frac{n-k}{2}} - \binom{n}{\frac{n-k}{2}-1} \right) (-1)^{(k-m)/2} q^{\binom{(k-m)/2+1}{2}} \begin{bmatrix} \frac{k+m}{2} \\ \frac{k-m}{2} \end{bmatrix}_q.$$

We shall use the convention $\binom{n}{k} = \begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ if $k < 0$, $k > n$, or k is not an integer. Thus $\overline{P}(n,m) = 0$ if $n \not\equiv m \mod 2$.

Theorem 2.4. *We have*

$$\mathcal{L}_a(x^nH_m(x;a|q)) = \frac{(q)_m}{2^n} \sum_{\alpha \geq 0} \overline{P}(n,\alpha+m) \begin{bmatrix} \alpha+m \\ m \end{bmatrix}_q a^\alpha, \tag{12}$$

$$\mathcal{L}_{a,b}(x^nQ_m(x;a,b|q)) = \frac{(q,ab)_m}{2^n} \sum_{\alpha,\beta \geq 0} \overline{P}(n,\alpha+\beta+m) \begin{bmatrix} \alpha+\beta+m \\ \alpha,\beta,m \end{bmatrix}_q a^\alpha b^\beta, \tag{13}$$

$$\begin{aligned} \mathcal{L}_{a,b,c}(x^np_m(x;a,b,c|q)) = \frac{(q,ac,bc)_m}{2^n} &\sum_{\alpha,\beta,\gamma \geq 0} \overline{P}(n,\alpha+\beta+\gamma+m) \begin{bmatrix} \alpha+\beta+\gamma+m \\ \alpha,\beta,\gamma,m \end{bmatrix}_q \\ &\times a^\alpha b^\beta c^\gamma (ab)_{\gamma+m}, \end{aligned} \tag{14}$$

$$\begin{aligned} \mathcal{L}_{a,b,c,d}(x^np_m(x;a,b,c,d|q)) = \frac{1}{2^n} &\sum_{\alpha,\beta,\gamma,\delta \geq 0} a^\alpha b^\beta c^\gamma d^\delta \overline{P}(n,\alpha+\beta+\gamma+\delta) \begin{bmatrix} \alpha+\beta+\gamma+\delta \\ \alpha,\beta,\gamma,\delta \end{bmatrix}_q \\ &\times \frac{(bd)_\alpha(cd)_\alpha(bc)_{\alpha+\delta}}{(abcd)_\alpha} \cdot \frac{(ab,ac,ad)_m(q^\alpha;q^{-1})_m}{a^m(abcdq^\alpha)_m}. \end{aligned} \tag{15}$$

Proof. By (8), Proposition 2.2 and (2),

$$\begin{aligned}\mathcal{L}_a(x^n H_m(x;a|q)) &= \sum_{k=0}^{n} \frac{\mathcal{L}(x^n H_k(x|q))}{\mathcal{L}(H_k(x)^2)} \begin{bmatrix} k \\ m \end{bmatrix}_q a^{k-m} \mathcal{L}_a(H_m(x;a|q)^2) \\ &= \frac{(q)_m}{2^n} \sum_{k=0}^{n} \overline{P}(n,k) \begin{bmatrix} k \\ m \end{bmatrix}_q a^{k-m}.\end{aligned}$$

Equations (13), (14), and (15) can be proved similarly using the connection coefficient formulas (9), (10), and (11). □

We note that Ismail and Rahman [8, Theorem 3.2] found a formula for

$$\mathcal{L}_{a,b,c,d}((ve^{i\theta}, ve^{-i\theta})_n p_k(x;a,b,c,d|q)).$$

Letting $m=0$ in (15) we obtain a formula for the nth moment of the Askey-Wilson polynomials.

Corollary 2.5. *We have*

$$\mathcal{L}_{a,b,c,d}(x^n) = \frac{1}{2^n} \sum_{\alpha,\beta,\gamma,\delta\ge 0} a^\alpha b^\beta c^\gamma d^\delta \overline{P}(n,\alpha+\beta+\gamma+\delta) \begin{bmatrix} \alpha+\beta+\gamma+\delta \\ \alpha,\beta,\gamma,\delta \end{bmatrix}_q \frac{(bd)_\alpha (cd)_\alpha (bc)_{\alpha+\delta}}{(abcd)_\alpha}. \tag{16}$$

In [14] the authors found a slightly different formula

$$\mathcal{L}_{a,b,c,d}(x^n) = \frac{1}{2^n} \sum_{\alpha,\beta,\gamma,\delta\ge 0} a^\alpha b^\beta c^\gamma d^\delta \overline{P}(n,\alpha+\beta+\gamma+\delta) \begin{bmatrix} \alpha+\beta+\gamma+\delta \\ \alpha,\beta,\gamma,\delta \end{bmatrix}_q \frac{(ad)_{\beta+\gamma} (ac)_\beta (bd)_\gamma}{(abcd)_{\beta+\gamma}},$$

which can be rewritten using the symmetry in a,b,c,d as

$$\mathcal{L}_{a,b,c,d}(x^n) = \frac{1}{2^n} \sum_{\alpha,\beta,\gamma,\delta\ge 0} a^\alpha b^\beta c^\gamma d^\delta \overline{P}(n,\alpha+\beta+\gamma+\delta) \begin{bmatrix} \alpha+\beta+\gamma+\delta \\ \alpha,\beta,\gamma,\delta \end{bmatrix}_q \frac{(bc)_{\alpha+\delta} (bd)_\alpha (ac)_\delta}{(abcd)_{\alpha+\delta}}. \tag{17}$$

One can obtain (17) from (16) by applying the ${}_3\phi_1$-transformation [6, (III.8)] to the α-sum after fixing γ, δ, and $N=\alpha+\beta$.

We next check if the heuristic in Remark 2.3 leads to correct results in these cases. The absolutely continuous Askey-Wilson measure $w(x;a,b,c,d|q)$ with total mass 1 for $0<q<1$, $\max(|a|,|b|,|b|,|d|)<1$ is, if $x=\cos\theta$, $\theta\in[0,\pi]$,

$$\begin{aligned} w(\cos\theta;a,b,c,d|q) &= \frac{(q,ab,ac,ad,bc,bd,cd)_\infty}{2\pi(abcd)_\infty} \\ &\quad\times \frac{(e^{2i\theta},e^{-2i\theta})_\infty}{(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta},ce^{i\theta},ce^{-i\theta},de^{i\theta},de^{-i\theta})_\infty}.\end{aligned} \tag{18}$$

Then the measures for the q-Hermite $H_n(x|q)$, the big q-Hermite $H_n(x;a|q)$, the Al-Salam-Chihara $Q_n(x;a,b|q)$, and the dual q-Hahn $p_n(x;a,b,c|q)$ are, respectively, $w(\cos\theta;0,0,0,0|q)$, $w(\cos\theta;a,0,0,0|q)$, $w(\cos\theta;a,b,0,0|q)$, and $w(\cos\theta;a,b,c,0|q)$. Notice that each successive measure comes from the previous measure by inserting infinite products.

Example 2.6. Let $R_k(x)=H_k(x|q)$ and $S_k(x)=H_k(x;a|q)$ so that

$$w_S(\cos\theta) = w_R(\cos\theta)\frac{1}{(ae^{i\theta},ae^{-i\theta})_\infty}.$$

In this case, we have $\mathcal{L}_R(R_k(x)^2)=(q)_k$ and

$$R_k(x) = \sum_{i=0}^{k} c_{k,i} S_i(x),$$

where $c_{k,i}=\begin{bmatrix} k \\ i \end{bmatrix}_q a^{k-i}$. By the heuristic in Remark 2.3,

$$w_S(x) = w_R(x)\sum_{k=0}^{\infty} \frac{R_k(x)}{(q)_k} a^k = w_R(x)\frac{1}{(ae^{i\theta},ae^{-i\theta})_\infty},$$

where we have used the q-Hermite generating function [13, (14.26.11), p.542].

Example 2.7. Let $R_k(x) = H_k(x; a|q)$ and $S_k(x) = Q_k(x; a, b|q)$ so that

$$w_S(\cos\theta) = w_R(\cos\theta)\frac{(ab)_\infty}{(be^{i\theta}, be^{-i\theta})_\infty}.$$

In this case, we have $\mathcal{L}_R(R_k(x)^2) = (q)_k$ and

$$R_k(x) = \sum_{i=0}^{k} c_{k,i} S_i(x),$$

where $c_{k,i} = \begin{bmatrix} k \\ i \end{bmatrix}_q b^{k-i}$. By the heuristic in Remark 2.3,

$$w_S(x) = w_R(x)\sum_{k=0}^{\infty}\frac{R_k(x)}{(q)_k}c^k = w_R(x)\frac{(ab)_\infty}{(be^{i\theta}, be^{-i\theta})_\infty},$$

where we have used the big q-Hermite generating function [13, (14.18.13), p.512].

Example 2.8. Let $R_k(x) = Q_k(x; a, b|q)$ and $S_k(x) = p_k(x; a, b, c|q)$ so that

$$w_S(\cos\theta) = w_R(\cos\theta)\frac{(ac, bc)_\infty}{(ce^{i\theta}, ce^{-i\theta})_\infty}.$$

In this case, we have $\mathcal{L}_R(R_k(x)^2) = (q, ab)_k$ and

$$R_k(x) = \sum_{i=0}^{k} c_{k,i} S_i(x),$$

where $c_{k,i} = \begin{bmatrix} k \\ i \end{bmatrix}_q \frac{(ab)_k}{(ab)_i} c^{k-i}$. By the heuristic in Remark 2.3,

$$w_S(x) = w_R(x)\sum_{k=0}^{\infty}\frac{R_k(x)}{(q, ab)_k}(ab)_k c^k = w_R(x)\frac{(ac, bc)_\infty}{(ce^{i\theta}, ce^{-i\theta})_\infty},$$

where we have used the Al-Salam-Chihara generating function [13, (14.8.13), p.458].

Notice that in the above example we used the known generating function for the Al-Salam-Chihara polynomials $Q_n(x; a, b|q)$. If we apply the same steps to $R_k(x) = p_k(x; a, b, c, 0|q)$ and $S_k(x) = p_k(x; a, b, c, d|q)$, a new generating function appears.

Theorem 2.9. *We have*

$$(abct)_\infty \sum_{k=0}^{\infty}\frac{p_k(x; a, b, c, 0|q)}{(q, abct)_k}t^k = \frac{(at, bt, ct)_\infty}{(te^{i\theta}, te^{-i\theta})_\infty}.$$

Proof. We must show

$$(abct)_\infty \sum_{n=0}^{\infty}\frac{t^n}{(q, abct)_n}p_n(x; a, b, c, 0|q) = \frac{(bt, ct)_\infty}{(te^{i\theta}, te^{-i\theta})_\infty}(at)_\infty. \tag{19}$$

Using the Al-Salam-Chihara generating function and the q-binomial theorem [6, (II.3), p. 354], (19) is equivalent to

$$\sum_{n=0}^{N}\frac{p_n(x; b, c, 0, 0|q)}{(q)_n}\frac{(-a)^{N-n}q^{\binom{N-n}{2}}}{(q)_{N-n}} = \sum_{n=0}^{N}\frac{p_n(x; a, b, c, 0|q)}{(q)_n}\frac{(-abcq^n)^{N-n}q^{\binom{N-n}{2}}}{(q)_{N-n}}. \tag{20}$$

Now use the connection coefficients

$$p_n(x; b, c, 0, 0|q) = (bc)_n \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q p_k(x; a, b, c, 0|q)\frac{a^{n-k}}{(bc)_k},$$

to show that (20) follows from

$$\sum_{n=k}^{N}\frac{(bc)_n}{(q)_n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{a^{N-k}}{(bc)_k}\frac{(-1)^{N-n}q^{\binom{N-n}{2}}}{(q)_{N-n}} = \frac{1}{(q)_k}\frac{(-abcq^k)^{N-k}}{(q)_{N-k}}q^{\binom{N-k}{2}}.$$

This summation is a special case of the q-Vandermonde theorem [6, (II.6), p. 354]. $\square$

A generalization of Theorem 2.9 to Askey-Wilson polynomials is given in [11].

A natural generalization of the mixed moments in (15) is

$$\mathcal{L}_{a,b,c,d}(x^n p_m(x;a,b,c,d|q)p_\ell(x;a,b,c,d|q)).$$

For general orthogonal polynomials Viennot has given a combinatorial interpretation for $\mathcal{L}(x^n p_m p_\ell)$ in terms of weighted Motzkin paths. An explicit formula when $p_n = p_n(x;a,b,c,d|q)$ may be given using (7) and a q-Taylor expansion [10], but we do not state the result here.

3. Generating functions

In § 2 we noted the following generating functions for our bootstrapping polynomials: continuous q-Hermite $H_n(x|q)$, continuous big q-Hermite $H_n(x;a|q)$, and Al-Salam-Chihara $Q_n(x;a,b|q)$

$$\sum_{n=0}^{\infty} \frac{H_n(x|q)}{(q)_n} t^n = \frac{1}{(te^{i\theta}, te^{-i\theta})_\infty}, \tag{21}$$

$$\sum_{n=0}^{\infty} \frac{H_n(x;a|q)}{(q)_n} t^n = \frac{(at)_\infty}{(te^{i\theta}, te^{-i\theta})_\infty}, \tag{22}$$

$$\sum_{n=0}^{\infty} \frac{Q_n(x;a,b|q)}{(q)_n} t^n = \frac{(at,bt)_\infty}{(te^{i\theta}, te^{-i\theta})_\infty}. \tag{23}$$

Note that (22) is obtained from (21) by multiplying by $(at)_\infty$ and (23) is obtained from (22) by multiplying by $(bt)_\infty$. However, if we multiply (23) by $(ct)_\infty$, we no longer have a generating function for orthogonal polynomials. It is the generating function for polynomials which satisfy a recurrence relation of finite order, but longer than order three, which orthogonal polynomials have.

The purpose of this section is to explain this phenomenon. We consider polynomials whose generating function are obtained by multiplying the generating function of orthogonal polynomials by $(yt)_\infty$ or $1/(-yt)_\infty$.

We say that polynomials $p_n(x)$ *satisfy a d-term recurrence relation* if there exist a real number A and sequences $\{b_n^{(0)}\}_{n\ge0}, \{b_n^{(1)}\}_{n\ge1}, \dots, \{b_n^{(d-2)}\}_{n\ge d-2}$ such that, for $n \ge 0$,

$$p_{n+1}(x) = (Ax - b_n^{(0)})p_n(x) - b_n^{(1)} p_{n-1}(x) - \cdots - b_n^{(d-2)} p_{n-d+2}(x),$$

where $p_i(x) = 0$ for $i < 0$.

Theorem 3.1. *Let $p_n(x)$ be polynomials satisfying $p_{n+1}(x) = (Ax - b_n)p_n(x) - \lambda_n p_{n-1}(x)$ for $n \ge 0$, where $p_{-1}(x) = 0$ and $p_0(x) = 1$. If b_k and $\frac{\lambda_k}{1-q^k}$ are polynomials in q^k of degree r and s, respectively, which are independent of y, then the polynomials $P_n^{(1)}(x,y)$ in x defined by*

$$\sum_{n=0}^{\infty} P_n^{(1)}(x,y) \frac{t^n}{(q)_n} = (yt)_\infty \sum_{n=0}^{\infty} p_n(x) \frac{t^n}{(q)_n}$$

satisfy a d-term recurrence relation for $d = \max(r+2, s+3)$.

We use two lemmas to prove Theorem 3.1. In the following lemmas we use the same notations as in Theorem 3.1.

Lemma 3.2. *We have*

$$P_n^{(1)}(x,y) = P_n^{(1)}(x,yq) - y(1-q^n)P_{n-1}^{(1)}(x,yq).$$

Proof. This is obtained by equating the coefficients of t^n in

$$\sum_{n=0}^{\infty} P_n^{(1)}(x,y)\frac{t^n}{(q)_n} = (1-yt)\sum_{n=0}^{\infty} P_n^{(1)}(x,yq)\frac{t^n}{(q)_n}.$$

□

Lemma 3.3. *Suppose that b_k and $\frac{\lambda_k}{1-q^k}$ are polynomials in q^k of degree r and s, respectively, i.e.,*

$$b_k = \sum_{j=0}^{r} c_j (q^k)^j, \qquad \frac{\lambda_k}{1-q^k} = \sum_{j=0}^{s} d_j (q^k)^j.$$

Then

$$P_{n+1}^{(1)}(x,y) = (Ax-y)P_n^{(1)}(x,yq) - \sum_{j=0}^{r} c_j q^{nj} P_n^{(1)}(x,yq^{1-j}) - (1-q^n)\sum_{j=0}^{s} d_j q^{nj} P_{n-1}^{(1)}(x,yq^{1-j}).$$

Proof. Expanding $(yt)_\infty$ using the q-binomial theorem, we have

$$P_n^{(1)}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k y^k q^{\binom{k}{2}} p_{n-k}(x).$$

Using the relation $\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n \\ k \end{bmatrix}_q$, we have

$$\begin{aligned} P_{n+1}^{(1)}(x,y) &= \sum_{k=0}^{n+1} \left(\begin{bmatrix} n \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n \\ k \end{bmatrix}_q \right) (-1)^k y^k q^{\binom{k}{2}} p_{n+1-k}(x) \\ &= -yP_n^{(1)}(x,yq) + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k (yq)^k q^{\binom{k}{2}} p_{n+1-k}(x). \end{aligned}$$

By $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{1-q^n}{1-q^{n-k}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$ and the 3-term recurrence

$$p_{n+1-k}(x) = (Ax - b_{n-k})p_{n-k}(x) - \lambda_{n-k} p_{n-1-k}(x),$$

we get

$$\begin{aligned} P_{n+1}^{(1)}(x,y) = (Ax-y)P_n^{(1)}(x,yq) &- \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k (yq)^k q^{\binom{k}{2}} p_{n-k}(x) b_{n-k} \\ &- (1-q^n)\sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q (-1)^k (yq)^k q^{\binom{k}{2}} p_{n-1-k}(x) \frac{\lambda_{n-k}}{1-q^{n-k}}. \end{aligned} \tag{24}$$

Since

$$b_{n-k} = \sum_{j=0}^{r} c_j q^{nj} (q^k)^{-j}, \qquad \frac{\lambda_{n-k}}{1-q^{n-k}} = \sum_{j=0}^{s} q^{nj} d_j (q^k)^{-j},$$

and

$$P_n^{(1)}(x,yq^{1-j}) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k (yq)^k q^{\binom{k}{2}} p_{n-k}(x) (q^k)^{-j},$$

we obtain the desired recurrence relation. □

Now we can prove Theorem 3.1.

Proof of Theorem 3.1. By Lemma 3.3, we can write

$$P_{n+1}^{(1)}(x,y) = (Ax-y)P_n^{(1)}(x,yq) - \sum_{j=0}^{r} c_j q^{nj} P_n^{(1)}(x,yq^{1-j}) - (1-q^n)\sum_{j=0}^{s} d_j q^{nj} P_{n-1}^{(1)}(x,yq^{1-j}).$$

Using Lemma 3.2 we can express $P_k^{(1)}(x,yq^{1-j})$ as a linear combination of

$$P_k^{(1)}(x,yq), P_{k-1}^{(1)}(x,yq), \ldots, P_{k-j}^{(1)}(x,yq).$$

Replacing y by y/q, we obtain a $\max(r+2, s+3)$-term recurrence relation for $P_n^{(1)}(x,y)$. □

Remark 3.4. One may verify that the order of recurrence for $P_n^{(1)}(x,y)$ is exactly $\max(2+r, 3+s)$ in the following way. Lemma 3.2 is applied s times to the term $P_{n-1}^{(1)}(x, yq^{1-s})$ to obtain a linear combination of $P_{n-1}^{(1)}(x,yq), P_{n-2}^{(1)}(x,yq), \cdots, P_{n-s-1}^{(1)}(x,yq)$. The coefficient of $P_{n-s-1}^{(1)}(x,yq)$ in this expansion is $(-1)^s(q^{n-1};q^{-1})_s y^s q^{\binom{s}{2}}$. Similarly, considering $P_n^{(1)}(x,yq^{1-r})$, the coefficient of $P_{n-r}^{(1)}(x,yq)$ in the expansion is $(-1)^r(q^n;q^{-1})_r y^r q^{\binom{r}{2}}$. These terms are non-zero, give a recurrence of order $\max(r+2, s+3)$, and could only cancel if $r = s+1$. In this case, the coefficient of $P_{n-s-1}^{(1)}(x,yq)$ is

$$(q^n;q^{-1})_{s+1}(-1)^{s+1}y^s q^{\binom{s}{2}} q^{ns}\left(d_s - yc_r q^{r+s}\right).$$

Since d_s and c_r are non-zero and independent of y, this is non-zero.

Remark 3.5. Theorem 3.1 can be generalized for polynomials $p_n(x)$ satisfying a finite term recurrence relation of order greater than 3. For instance, if $p_{n+1}(x) = (Ax-b_n)p_n(x) - \lambda_n p_{n-1}(x) - \nu_n p_{n-2}(x)$, then using $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{1-q^n}{1-q^{n-k}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$ twice one can see that Equation (24) has the following extra sum in the right hand side:

$$-(1-q^n)(1-q^{n-1}) \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q (-1)^k (yq)^k q^{\binom{k}{2}} p_{n-2-k}(x) \frac{\nu_{n-k}}{(1-q^{n-k})(1-q^{n-k-1})}.$$

Thus if $\frac{\nu_k}{(1-q^k)(1-q^{k-1})}$ is a polynomial in q^k then $P_n^{(1)}(x,y)$ satisfy a finite term recurrence relation.

Note that by using Lemmas 3.2 and 3.3, one can find a recurrence relation for $P_n^{(1)}(x,y)$ in Theorem 3.1.

An analogous theorem holds for polynomial in q^{-k}. We state the result without proof.

Theorem 3.6. *Let $p_n(x)$ be polynomials satisfying $p_{n+1}(x) = (Ax - b_n)p_n(x) - \lambda_n p_{n-1}(x)$ for $n \ge 0$, where $p_{-1}(x) = 0$ and $p_0(x) = 1$. If b_k and $\frac{\lambda_k}{1-q^k}$ are polynomials in q^{-k} of degree r and s, respectively, which are independent of y, and the constant term of $\frac{\lambda_k}{1-q^k}$ is zero, then the polynomials $P_n^{(2)}(x,y)$ defined by*

$$\sum_{n=0}^{\infty} P_n^{(2)}(x,y) \frac{q^{\binom{n}{2}} t^n}{(q)_n} = \frac{1}{(-yt)_\infty} \sum_{n=0}^{\infty} p_n(x) \frac{q^{\binom{n}{2}} t^n}{(q)_n}$$

satisfy a d-term recurrence relation for $d = \max(r+1, s+2)$.

We now give several applications of Theorem 3.1 and Theorem 3.6. In the following examples, we use the notation in these theorems.

Example 3.7. Let $p_n(x)$ be the continuous q-Hermite polynomial $H_n(x|q)$. Then $A = 2, b_n = 0$, and $\lambda_n = 1 - q^n$. Since $r = -\infty$ and $s = 0$, $P_n^{(1)}(x,y)$ satisfies a 3-term recurrence relation. By Lemma 3.3, we have

$$P_{n+1}^{(1)}(x,y) = (2x-y)P_n^{(1)}(x,yq) - (1-q^n)P_{n-1}^{(1)}(x,yq).$$

By Lemma 3.2 we have

$$P_{n+1}^{(1)}(x,y) = P_{n+1}^{(1)}(x,yq) - y(1-q^n)P_n^{(1)}(x,yq).$$

Thus

$$P_{n+1}^{(1)}(x,yq) = (2x - yq^n)P_n^{(1)}(x,yq) - (1-q^n)P_{n-1}^{(1)}(x,yq).$$

Replacing y by y/q we obtain

$$P_{n+1}^{(1)}(x,y) = (2x - yq^{n-1})P_n^{(1)}(x,y) - (1-q^n)P_{n-1}^{(1)}(x,y).$$

Thus $P_n(x,y)$ are orthogonal polynomials, which are the continuous big q-Hermite polynomials $H_n(x;y|q)$.

Example 3.8. Let $p_n(x)$ be the continuous big q-Hermite polynomials $H_n(x;a|q)$. Then $A = 2, b_n = aq^n$, and $\lambda_n = 1-q^n$. Since $r=1$ and $s=0$, $P_n^{(1)}(x,y)$ satisfies a 3-term recurrence relation. Using the same method as in the previous example, we obtain

$$P_{n+1}^{(1)}(x,y) = (2x-(a+y)q^n)P_n^{(1)}(x,y) - (1-q^n)(1-ayq^{n-1})P_{n-1}^{(1)}(x,y).$$

Thus $P_n^{(1)}(x,y)$ are orthogonal polynomials, which are the Al-Salam-Chihara polynomials $Q_n(x;a,y|q)$.

Example 3.9. Let $p_n(x)$ be the Al-Salam-Chihara polynomials $Q_n(x;a,b|q)$. Then $A=2, b_n = (a+b)q^n$, and $\lambda_n = (1-q^n)(1-abq^{n-1})$. Since $r=1$ and $s=1$, $P_n(x,y)$ satisfies a 4-term recurrence relation. By Lemma 3.3, we have

$$P_{n+1}^{(1)}(x,y) = (2x-y)P_n^{(1)}(x,yq)-(a+b)q^nP_n^{(1)}(x,y)-(1-q^n)(-abq^{n-1}P_{n-1}^{(1)}(x,y)+P_{n-1}^{(1)}(x,yq)).$$

Using Lemma 3.2 we get

$$P_{n+1}^{(1)} = (2x-(a+b+y)q^n)P_n^{(1)}-(1-q^n)(1-(ab+ay+by)q^{n-1})P_{n-1}^{(1)}-abyq^{n-2}(1-q^n)(1-q^{n-1})P_{n-2}^{(1)}.$$

Example 3.10. Let $p_n(x)$ be the continuous dual q-Hahn polynomials $p_n(x;a,b,c|q)$. Then $A=2$ and

$$\begin{aligned} b_n &= (a+b+c)q^n - abcq^{2n} - abcq^{2n-1}, \\ \lambda_n &= (1-q^n)(1-abq^{n-1})(1-bcq^{n-1})(1-caq^{n-1}). \end{aligned}$$

Since $r=2$ and $s=3$, $P_n(x,y)$ satisfies a 6-term recurrence relation. It is possible to find an explicit recurrence relation using the same idea as in the previous example.

Example 3.11. Let $p_n(x)$ be the discrete q-Hermite I polynomial $h_n(x;q)$. Then $A=1, b_n=0$, and $\lambda_n = q^{n-1}(1-q^n)$. Since $r=-\infty$ and $s=1$, $P_n^{(1)}(x,y)$ satisfies a 4-term recurrence relation which is

$$P_{n+1}^{(1)}(x,y) = (x-yq^n)P_n^{(1)}(x,y) - q^{n-1}(1-q^n)P_{n-1}^{(1)}(x,y) + yq^{n-2}(1-q^n)(1-q^{n-1})P_{n-2}^{(1)}(x,y).$$

In §4 we will study $P_n^{(1)}(x,y) = h_n(x,y;q)$, the discrete big q-Hermite I polynomials $h_n(x,y;q)$. This is a proof of Theorem 4.3.

Example 3.12. Let $p_n(x)$ be the discrete q-Hermite II polynomial $\tilde{h}_n(x;q)$. Then $A=1, b_n=0$, and $\lambda_n = q^{-2n+1}(1-q^n)$. Since b_n and $\lambda_n/(1-q^n)$ are polynomials in q^{-n} of degrees $-\infty$ and 2, respectively, and the constant term of $\lambda_n/(1-q^n)$ is 0, so $P_n^{(2)}(x,y)$ satisfies a 4-term recurrence relation. It is

$$P_{n+1}^{(2)}(x,y) = (x-yq^{-n})P_n^{(2)}(x,y)-q^{-2n+1}(1-q^n)P_{n-1}^{(2)}(x,y)-yq^{3-3n}(1-q^n)(1-q^{n-1})P_{n-2}^{(2)}(x,y).$$

$P_n^{(2)}(x,y)$ are the discrete big q-Hermite II polynomials $\tilde{h}_n(x,y;q)$ of § 5.

Example 3.13. The *Al-Salam–Carlitz I* polynomials $U_n^{(a)}(x;q)$ are defined by

$$\sum_{n=0}^{\infty} \frac{U_n^{(a)}(x;q)}{(q)_n} t^n = \frac{(t)_\infty (at)_\infty}{(xt)_\infty}.$$

They have the 3-term recurrence relation

$$U_{n+1}^{(a)}(x;q) = (x-(1+a)q^n)U_n^{(a)}(x;q) + aq^{n-1}(1-q^n)U_{n-1}^{(a)}(x;q).$$

Let $p_n(x)$ be the polynomials with generating function

$$\sum_{n=0}^{\infty} \frac{p_n(x)}{(q)_n} t^n = \frac{(t)_\infty}{(xt)_\infty} = \sum_{n=0}^{\infty} \frac{x^n(1/x)_n}{(q)_n} t^n.$$

Then $p_n(x) = x^n(1/x)_n$. Thus $p_{n+1}(x) = (x-q^n)p_n(x)$, and we have $A=1, b_n=q^n$, and $\lambda_n=0$, and $U_n^{(a)}(x;q) = P_n^{(1)}(x,a)$.

Example 3.14. The *Al-Salam–Carlitz II* polynomials $V_n^{(a)}(x;q)$ are defined by

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q)_n} V_n^{(a)}(x;q) t^n = \frac{(xt)_\infty}{(t)_\infty (at)_\infty}.$$

They have the 3-term recurrence relation

$$V_{n+1}^{(a)}(x;q) = (x-(1+a)q^{-n})V_n^{(a)}(x;q) - aq^{-2n+1}(1-q^n)V_{n-1}^{(a)}(x;q). \tag{25}$$

Let $p_n(x)$ be the polynomials with generating function

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q)_n} p_n(x) t^n = \frac{(xt)_\infty}{(t)_\infty} = \sum_{n=0}^{\infty} \frac{(x)_n}{(q)_n} t^n = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} x^n (1/x)_n}{(q)_n} t^n.$$

Then $p_n(x) = (-1)^n x^n (1/x)_n$. Thus $p_{n+1}(x) = (-x+q^{-n})p_n(x)$, and we have $A=-1, b_n = -q^{-n}$, and $\lambda_n = 0$ and we obtain $V_n^{(a)}(x;q) = (-1)^n P_n^{(2)}(-x,-a)$ and (25).

Garrett, Ismail, and Stanton [5, Section 7] considered the polynomials $\hat{H}_n(x|q)$ defined by the generating function

$$\sum_{n=0}^{\infty} \hat{H}_n(x|q) \frac{t^n}{(q)_n} = \frac{(t^2)_\infty}{(te^{i\theta}, te^{-i\theta})_\infty} = (t^2)_\infty \sum_{n=0}^{\infty} H_n(x|q) \frac{t^n}{(q)_n}.$$

It turns out that $p_n = \hat{H}_n(x|q)$ satisfies the 5-term recurrence relation

$$p_{n+1} = 2xp_n + (q^{2n} + q^{2n-1} - q^{n-1} - 1)p_{n-1} + q^{n-2}(1-q^n)(1-q^{n-1})(1-q^{n-2})p_{n-3}.$$

The following generalization of Theorem 3.1 explains this phenomenon for $m=2$, $r=0$, and $s=0$. We omit the proof, which is similar to that of Theorem 3.1.

Theorem 3.15. *Let m be a positive integer. Let $p_n(x)$ be polynomials satisfying $p_{n+1}(x) = (Ax-b_n)p_n(x) - \lambda_n p_{n-1}(x)$ for $n \ge 0$, where $p_{-1}(x) = 0$ and $p_0(x) = 1$. If b_k and $\frac{\lambda_k}{1-q^k}$ are polynomials in q^k of degree r and s, respectively, which are independent of y, then the polynomials $P_n(x,y)$ in x defined by*

$$\sum_{n=0}^{\infty} P_n(x,y) \frac{t^n}{(q)_n} = (yt^m)_\infty \sum_{n=0}^{\infty} p_n(x) \frac{t^n}{(q)_n}$$

satisfy a d-term recurrence relation for $d = \max(rm^2+2, sm^2+3, m^2+1)$.

4. Discrete big q-Hermite polynomials

In this section we study a set of polynomials which satisfy a 4-term recurrence relation, called the discrete big q-Hermite polynomials (see Definition 4.1). These polynomials generalize the discrete q-Hermite polynomials and appear in Example 3.11.

Recall [7] that the *continuous q-Hermite polynomials* $H_n(x|q)$ are defined by

$$\sum_{n=0}^{\infty} \frac{H_n(x|q)}{(q)_n} t^n = \frac{1}{(te^{i\theta}, te^{-i\theta})_\infty},$$

and the *continuous big q-Hermite polynomials* $H_n(x;a|q)$ are defined by

$$\sum_{n=0}^{\infty} \frac{H_n(x;a|q)}{(q)_n} t^n = \frac{(at)_\infty}{(te^{i\theta}, te^{-i\theta})_\infty}.$$

Observe that the generating function for $H_n(x;a|q)$ is the generating function for $H_n(x|q)$ multiplied by $(at)_\infty$. In this section we introduce *discrete big q-Hermite polynomials* in an analogous way.

The *discrete q-Hermite I polynomials* $h_n(x;q)$ have generating function

$$\sum_{n=0}^{\infty} \frac{h_n(x;q)}{(q)_n} t^n = \frac{(t^2;q^2)_\infty}{(xt)_\infty}.$$

Definition 4.1. The *discrete big q-Hermite I polynomials* $h_n(x,y;q)$ are given by

$$\sum_{n=0}^{\infty} h_n(x,y;q)\frac{t^n}{(q)_n} = \frac{(t^2;q^2)_\infty (yt)_\infty}{(xt)_\infty}. \tag{26}$$

Expanding the right hand side of (26) using the q-binomial theorem, we find the following expression for $h_n(x,y;q)$.

Proposition 4.2. *For $n \geq 0$,*

$$h_n(x,y;q) = \sum_{k=0}^{\lfloor n/2 \rfloor} \begin{bmatrix} n \\ 2k \end{bmatrix}_q (q;q^2)_k q^{2\binom{k}{2}} (-1)^k x^{n-2k} (y/x)_{n-2k}.$$

The polynomials $h_n(x,y;q)$ are orthogonal polynomials in neither x nor y. However they satisfy the following simple 4-term recurrence relation which was established in Example 3.11.

Theorem 4.3. *For $n \geq 0$,*

$$h_{n+1}(x,y;q) = (x - yq^n)h_n(x,y;q) - q^{n-1}(1-q^n)h_{n-1}(x,y;q) + yq^{n-2}(1-q^n)(1-q^{n-1})h_{n-2}(x,y;q).$$

Note that when $y = 0$, the 4-term recurrence relation reduces to the 3-term recurrence relation for the discrete q-Hermite I polynomials. The polynomials $h_n(x,y;q)$ are not symmetric in x and y. If we consider $h_n(x,y;q)$ as a polynomial in y, then it does not satisfy a finite term recurrence relation, see Proposition 4.7.

Since $h_n(x,y;q)$ satisfies a 4-term recurrence, it is a multiple orthogonal polynomial in x. Thus there are two linear functionals $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(1)}$ such that, for $i \in \{0,1\}$,

$$\mathcal{L}^{(i)}(h_m) = \delta_{mi}, \quad m \geq 0,$$

$$\mathcal{L}^{(i)}(h_m(x,y;q)h_n(x,y;q)) = 0 \quad \text{if } m > 2n+i, \text{ and } \quad \mathcal{L}^{(i)}(h_{2n+i}(x,y;q)h_n(x,y;q)) \neq 0.$$

We have explicit formulas for the moments for $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(1)}$.

Theorem 4.4. *The moments for the discrete big q-Hermite polynomials are*

$$\mathcal{L}^{(0)}(x^n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \begin{bmatrix} n \\ 2k \end{bmatrix}_q (q;q^2)_k y^{n-2k},$$

$$\mathcal{L}^{(1)}(x^n) = (1-q^n) \sum_{k=0}^{\lfloor n/2 \rfloor} \begin{bmatrix} n-1 \\ 2k \end{bmatrix}_q (q;q^2)_k y^{n-2k-1}.$$

Before proving Theorem 4.4 we show that in general there is a way to find the linear functionals of d-orthogonal polynomials if we know how to expand certain orthogonal polynomials in terms of these d-orthogonal polynomials. This is similar to Proposition 2.2.

Theorem 4.5. *Let $R_n(x)$ be orthogonal polynomials with linear functionals $\mathcal{L}_R$ such that $\mathcal{L}_R(1) = 1$. Let $S_n(x)$ be d-orthogonal polynomials with linear functionals $\{\mathcal{L}_S^{(i)}\}_{i=0}^{d-1}$ such that $\mathcal{L}_S^{(i)}(S_n(x)) = \delta_{n,i}$. Suppose*

$$R_k(x) = \sum_{m=0}^{k} c_{km} S_m(x). \tag{27}$$

Then

$$\mathcal{L}_S^{(i)}(x^n) = \sum_{k=0}^{n} \frac{\mathcal{L}_R(x^n R_k(x))}{\mathcal{L}_R(R_k(x)^2)} d_{k,i},$$

where

$$d_{k,i} = \begin{cases} c_{k,i} & \text{if } k \geq i, \\ 0 & \text{if } k < i. \end{cases}$$

Proof. If we apply $\mathcal{L}_S^{(i)}$ to both sides of (27), we have

$$\mathcal{L}_S^{(i)}(R_k(x)) = d_{k,i}.$$

Then by expanding x^n in terms of $R_k(x)$ we get

$$\mathcal{L}_S^{(i)}(x^n) = \mathcal{L}_S^{(i)}\left(\sum_{k=0}^{n} \frac{\mathcal{L}_R(x^n R_k(x))}{\mathcal{L}_R(R_k(x)^2)} R_k(x)\right) = \sum_{k=0}^{n} \frac{\mathcal{L}_R(x^n R_k(x))}{\mathcal{L}_R(R_k(x)^2)} d_{k,i}.$$

□

We will apply Theorem 4.5 with $R_n(x) = h_n(x;q)$ and $S_n(x) = h_n(x,y;q)$ to prove Theorem 4.4. The first ingredient is (27), which follows from the generating function (26)

$$h_k(x;q) = \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_q y^{k-m} h_m(x,y;q).$$

The second ingredient is the value of $\mathcal{L}_h(x^n h_k)$.

Proposition 4.6. *Let $\mathcal{L}_h$ be the linear functional for $h_n(x;q)$ with $\mathcal{L}_h(1) = 1$. Then*

$$\mathcal{L}_h(x^n h_m(x;q)) = \begin{cases} 0 \text{ if } m > n \text{ or } n \not\equiv m \mod 2, \\ \frac{q^{\binom{m}{2}}(q)_n}{(q^2;q^2)_{\frac{n-m}{2}}} \text{ if } n \geq m, n \equiv m \mod 2. \end{cases}$$

Proof. Clearly we may assume that $n \geq m$ and $n \equiv m \mod 2$. Using the explicit formula

$$h_m(x;q) = x^m {}_2\phi_0 \left(\left. \begin{matrix} q^{-m}, q^{-m+1} \\ - \end{matrix} \right| q^2, \frac{q^{2m-1}}{x^2} \right),$$

and the fact

$$\mathcal{L}_h(x^k) = \begin{cases} 0 & \text{if } k \text{ is odd}, \\ (q;q^2)_{k/2} & \text{if } k \text{ is even}, \end{cases}$$

we obtain

$$\mathcal{L}_h(x^n h_m(x;q)) = (q;q^2)_{\frac{n+m}{2}} {}_2\phi_1 \left(\left. \begin{matrix} q^{-m}, q^{-m+1} \\ q^{-n-m+1} \end{matrix} \right| q^2, q^{m-n} \right),$$

$$\mathcal{L}_h(x^n h_m(x;q)) = (q;q^2)_{\frac{n+m}{2}}$$

which is evaluable by the q-Vandermonde theorem [6, (II.5), p, 354]. □

The discrete q-Hermite polynomials have the following orthogonality:

$$\mathcal{L}_h(h_m(x;q)h_n(x;q)) = q^{\binom{n}{2}}(q)_n \delta_{mn}. \tag{28}$$

Using Theorem 4.5, Proposition 4.6, and (28) we have proven Theorem 4.4. We do not know representing measures for the moments in Theorem 4.4.

One may also find a recurrence relation for $h_n(x,y;q)$ as a polynomial in y, whose proof is routine.

Proposition 4.7. *For $n \geq 0$, we have*

$$yq^n h_n(x,y;q) = -h_{n+1}(x,y;q) + \sum_{k=0}^{n} (q^n;q^{-1})_k (-1)^k h_{n-k}(x,y,;q) \times \begin{cases} x \text{ if } k \text{ is even} \\ 1 \text{ if } k \text{ is odd.} \end{cases}$$

We can also consider discrete q-Hermite II polynomials. The *discrete q-Hermite II polynomials* $\tilde{h}_n(x,y;q)$ have the generating function

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} \tilde{h}_n(x;q)}{(q)_n} t^n = \frac{(-xt)_\infty}{(-t^2;q^2)_\infty}.$$

We define the *discrete big q-Hermite II polynomials* $\tilde{h}_n(x, y; q)$ by

$$\sum_{n=0}^{\infty} \tilde{h}_n(x, y; q) \frac{q^{\binom{n}{2}} t^n}{(q)_n} = \frac{1}{(-t^2; q^2)_\infty} \frac{(-xt)_\infty}{(-yt)_\infty}.$$

Then $\tilde{h}_n(x, 0|q)$ is the discrete q-Hermite II polynomial.

The following proposition is straightforward to check.

Proposition 4.8. *For $n \geq 0$, we have*

$$\tilde{h}_n(x, y; q) = i^{-n} h_n(ix, iy; q^{-1}).$$

5. Combinatorics of the discrete big q-Hermite polynomials

In this section we give some combinatorial information about the discrete big q-Hermite polynomials. This includes a combinatorial interpretation of the polynomials (Theorem 5.2), and a combinatorial proof of the 4-term recurrence relation. Viennot's interpretation of the moments as weighted generalized Motzkin paths is also considered.

For the purpose of studying $h_n(x, y; q)$ combinatorially we will consider the following rescaled continuous big q-Hermite polynomials $h_n^*(x, y; q)$:

$$h_n^*(x, y; q) = (1 - q)^{-n/2} h_n(x\sqrt{1 - q}, y\sqrt{1 - q}|q).$$

By (26) we have

$$h_n^*(x, y; q) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{2\binom{k}{2}} [2k - 1]_q!! \begin{bmatrix} n \\ 2k \end{bmatrix}_q x^{n-2k} (y/x)_{n-2k}. \tag{29}$$

Because $h_n^*(x, y; 1) = H_n(x - y)$, which is a generating function for bicolored matchings of $[n] := \{1, 2, \ldots, n\}$, we need to consider q-statistics on matchings.

A *matching* of $[n] = \{1, 2, \ldots, n\}$ is a set partition of $[n]$ in which every block is of size 1 or 2. A block of a matching is called a *fixed point* if its size is 1, and an *edge* if its size is 2. When we write an edge $\{u, v\}$ we will always assume that $u < v$. A *fixed point bi-colored matching* or *FB-matching* is a matching for which every fixed point is colored with x or y. Let $\mathcal{FBM}(n)$ be the set of FB-matchings of $[n]$.

Let $\pi \in \mathcal{FBM}(n)$. A *crossing* of π is a pair of two edges $\{a, b\}$ and $\{c, d\}$ such that $a < c < b < d$. A *nesting* of π is a pair of two edges $\{a, b\}$ and $\{c, d\}$ such that $a < c < d < b$. An *alignment* of π is a pair of two edges $\{a, b\}$ and $\{c, d\}$ such that $a < b < c < d$. The *block-word* $\mathrm{bw}(\pi)$ of π is the word $w_1 w_2 \ldots w_n$ such that $w_i = 1$ if i is a fixed point and $w_i = 0$ otherwise. An *inversion* of a word $w_1 w_2 \ldots w_n$ is a pair of integers $i < j$ such that $w_i > w_j$. The number of inversions of w is denoted by $\mathrm{inv}(w)$.

Suppose that π has k edges and $n - 2k$ fixed points. The *weight* $\mathrm{wt}(\pi)$ of π is defined by

$$\mathrm{wt}(\pi) = (-1)^k q^{2\binom{k}{2} + 2\,\mathrm{al}(\pi) + \mathrm{cr}(\pi) + \mathrm{inv}(\mathrm{bw}(\pi))} z_1 z_2 \ldots z_{n-2k}, \tag{30}$$

where $z_i = x$ if the ith fixed point is colored with x, and $z_i = -yq^{i-1}$ if the ith fixed point is colored with y.

A *complete matching* is a matching without fixed points. Let $\mathcal{CM}(2n)$ denote the set of complete matchings of $[2n]$.

Proposition 5.1. *We have*

$$\sum_{\pi \in \mathcal{CM}(2n)} q^{2\,\mathrm{al}(\pi) + \mathrm{cr}(\pi)} = [2n - 1]_q!!.$$

Proof. It is known that

$$\sum_{\pi \in \mathcal{CM}(2n)} q^{\mathrm{cr}(\pi) + 2\,\mathrm{ne}(\pi)} = \sum_{\pi \in \mathcal{CM}(2n)} q^{2\,\mathrm{cr}(\pi) + \mathrm{ne}(\pi)} = [2n - 1]_q!!.$$

Since a pair of two edges is either an alignment, a crossing, or a nesting we have $\mathrm{al}(\pi)+\mathrm{ne}(\pi)+\mathrm{cr}(\pi)=\binom{n}{2}$. Thus

$$\sum_{\pi\in\mathcal{CM}(2n)} q^{2\,\mathrm{al}(\pi)+\mathrm{cr}(\pi)} = q^{2\binom{n}{2}} \sum_{\pi\in\mathcal{CM}(2n)} q^{-2\,\mathrm{ne}(\pi)-\mathrm{cr}(\pi)} = q^{2\binom{n}{2}}[2n-1]_{q^{-1}}!! = [2n-1]_q!!.$$

□

Theorem 5.2. *We have*

$$h_n^*(x,y;q) = \sum_{\pi\in\mathcal{FBM}(n)} \mathrm{wt}(\pi).$$

Proof. Let $M(n)$ be the set of 4-tuples (k,w,σ,X) such that $0\le k\le\lfloor n/2\rfloor$, w is a word of length n consisting of k 0's and $n-2k$ 1's, $\sigma\in\mathcal{CM}(2k)$, and $Z=(z_1,z_2,\dots,z_{n-2k})$ is a sequence such that z_i is either x or $-yq^{i-1}$ for each i.

For $\pi\in\mathcal{FBM}(n)$ we define $g(\pi)$ to be the 4-tuple $(k,w,\sigma,Z)\in M(n)$, where k is the number of edges of π, $w=\mathrm{bw}(\pi)$, σ is the induced complete matching of π, and $Z=(z_1,z_2,\dots,z_{n-2k})$ is the sequence such that $z_i=x$ if the ith fixed point is colored with x, and $z_i=-yq^{i-1}$ if the ith fixed point is colored with y. Here, the *induced complete matching* of π is the complete matching of $[2k]$ for which i and j form an edge if and only if the ith non-fixed point and the jth non-fixed point of π form an edge.

It is easy to see that g is a bijection from $\mathcal{FBM}(n)$ to $M(n)$ such that if $g(\pi)=(k,w,\sigma,Z)$ with $Z=(z_1,z_2,\cdots,z_{n-2k})$ then

$$\mathrm{wt}(\pi) = (-1)^k q^{2\binom{k}{2}} q^{2\,\mathrm{al}(\sigma)+\mathrm{cr}(\sigma)} q^{\mathrm{inv}(w)} z_1z_2\cdots z_{n-2k}.$$

Thus

$$\sum_{\pi\in\mathcal{FBM}(n)} \mathrm{wt}(\pi) = \sum_{(k,w,\sigma,Z)\in M(n)} (-1)^k q^{\binom{k}{2}} q^{2\,\mathrm{al}(\sigma)+\mathrm{cr}(\sigma)} q^{\mathrm{inv}(w)} z_1z_2\cdots z_{n-2k}.$$

Here once k is fixed σ can be any complete matching of $[2k]$, w can be any word consisting of k 0's and $n-2k$ 1's, and for $Z=(z_1,z_2,\cdots,z_{n-2k})$ each z_i can be either x or $-yq^{i-1}$. Thus the sum of $q^{2\,\mathrm{al}(\sigma)+\mathrm{cr}(\sigma)}$ for all such σ's gives $[2k-1]_q!!$, the sum of $\mathrm{inv}(w)$ for all such w gives $\begin{bmatrix} n\\ 2k\end{bmatrix}_q$, the sum of $z_1z_2\cdots z_{n-2k}$ for all such Z gives $(x/y)_{n-2k}$. This finishes the proof. □

Proposition 5.3. *For $n\ge 0$, we have*

$$h_{n+1}^* = (x-yq^n)h_n^* - q^{n-1}[n]_q h_{n-1}^* + yq^{n-2}[n-1]_q(1-q^n)h_{n-2}^*.$$

Proof of Proposition 5.3. Let $W_-(n)$ be the sum of $\mathrm{wt}(\pi)$ for all $\pi\in\mathcal{FBM}(n)$ such that n is not a fixed point. Let $W_x(n)$ (respectively $W_y(n)$) be the sum of $\mathrm{wt}(\pi)$ for all $\pi\in\mathcal{FBM}(n)$ such that n is a fixed point colored with x (respectively y). Then

$$h_{n+1}^*(x,y;q) = \sum_{\pi\in\mathcal{FBM}(n)} \mathrm{wt}(\pi) = W_-(n+1)+W_x(n+1)+W_y(n+1).$$

We claim that

$$W_x(n+1) = xh_n^*(x,y;q), \tag{31}$$

$$W_y(n+1) = -yq^n(W_x(n)+W_y(n)) - yW_-(n), \tag{32}$$

$$W_-(n+1) = -q^{n-1}[n]_q h_{n-1}^*(x,y;q). \tag{33}$$

From (30) we easily get (31).

For (33), consider a matching $\pi\in\mathcal{FBM}(n+1)$ such that $n+1$ is connected with i where $1\le i\le n$. Suppose that π has k edges and $n+1-2k$ fixed points. Let us compute the contribution of an edge of a fixed point together with the edge $\{i,n+1\}$ to $2\,\mathrm{al}(\pi)+\mathrm{cr}(\pi)+\mathrm{inv}(\mathrm{bw}(\pi))$. An edge with two integers less than i contributes 2 to $2\,\mathrm{al}(\pi)$. An edge with exactly one integer less than i contributes 1 to $\mathrm{cr}(\pi)$. An edge with two integers greater than i contributes nothing. Each fixed point of π less than i contributes 2 to $\mathrm{inv}(\mathrm{bw}(\pi))$ together with the edge $\{i,n+1\}$. Each fixed point of π greater than i contributes 1 to $\mathrm{inv}(\mathrm{bw}(\pi))$ together with the edge $\{i,n+1\}$. Thus the contribution of the edge $\{i,n+1\}$ to $2\,\mathrm{al}(\pi)+\mathrm{cr}(\pi)+\mathrm{inv}(\mathrm{bw}(\pi))$ is equal to $i-1+(n+1-2k)$. Let σ be the matching obtained from π by removing the edge $\{i,n+1\}$. Then

$$2\,\mathrm{al}(\pi)+\mathrm{cr}(\pi)+\mathrm{inv}(\mathrm{bw}(\pi)) = 2\,\mathrm{al}(\sigma)+\mathrm{cr}(\sigma)+\mathrm{inv}(\mathrm{bw}(\sigma))+i-1+(n+1-2k).$$

Thus, using (30), the above identity and $2\binom{k}{2}=2\binom{k-1}{2}+2k-2$, we have $\mathrm{wt}(\pi)=-q^{n-1}q^{i-1}\,\mathrm{wt}(\sigma)$. Since i can be any integer from 1 to n and $\sigma\in\mathcal{FBM}(n-1)$ we get (33).

Now we prove (32). Consider a matching $\pi\in\mathcal{FBM}(n+1)$ such that $n+1$ is a fixed point colored with y. Suppose that π has k edges with $2k$ non-fixed points $b_1<b_2<\cdots<b_{2k}$. For $0\le i\le 2k+1$, let $a_i=b_i-b_{i-1}-1$, where $b_0=0$ and $b_{2k+1}=n$. Then $a_0+a_1+\cdots+a_{2k+1}=n-2k$. Let σ be the matching obtained from π by removing $n+1$. Then we have $\mathrm{wt}(\pi)=-yq^{n-2k}\,\mathrm{wt}(\sigma)$. We consider two cases.

Case 1: $a_0\ne 0$. Let τ be the matching obtained from σ by changing 1 into n and decreasing the other integers by 1. We color the ith fixed point of τ with the same color of the ith fixed point of σ. Then $\mathrm{wt}(\sigma)=q^{2k}\,\mathrm{wt}(\tau)$ and $\mathrm{wt}(\pi)=-yq^n(\tau)$. Since n is a fixed point in τ the sum of $\mathrm{wt}(\pi)$ in this case gives $-yq^n(W_x(n)+W_y(n))$.

Case 2: $a_0=0$. Note that

$$\mathrm{bw}(\sigma)=0\overbrace{1\cdots1}^{a_1}0\overbrace{1\cdots1}^{a_2}0\cdots0\overbrace{1\ldots1}^{a_{2k}}0\overbrace{1\ldots1}^{a_{2k+1}}.$$

We define τ to be the matching with

$$\mathrm{bw}(\tau)=\overbrace{1\cdots1}^{a_1}0\overbrace{1\cdots1}^{a_2}0\overbrace{1\cdots1}^{a_3}1\cdots0\overbrace{1\ldots1}^{a_{2k+1}}0$$

and the ith fixed point of τ is colored with the same color of the ith fixed point of σ. Then $\mathrm{wt}(\sigma)=q^{-n+2k}\,\mathrm{wt}(\tau)$ and $\mathrm{wt}(\pi)=-y\,\mathrm{wt}(\tau)$. Since n is a non-fixed point in τ, the sum of $\mathrm{wt}(\pi)$ in this case gives $-yW_-(n)$.

It is easy to see that (31), (32), and (33) implies the 4-term recurrence relation. $\square$

Since the polynomials $h_n(x,y;q)$ satisfy a 4-term recurrence relation, they are 2-fold multiple orthogonal polynomials in x. By Viennot's theory, we can express the two moments $\mathcal{L}^{(0)}(x^n)$ and $\mathcal{L}^{(1)}(x^n)$ as a sum of weights of certain lattice paths.

A *2-Motzkin path* is a lattice path consisting of an up step $(1,1)$, a horizontal step $(1,0)$, a down step $(1,-1)$, and a double down step $(1,-2)$, which starts at the origin and never goes below the x-axis.

For $i=0,1$ let $\mathrm{Mot}_i(n)$ denote the set of 2-Motzkin paths of length n with final height i. The *weight* of $M\in\mathrm{Mot}_i(n)$ is the product of weights of all steps, where the weight of each step is defined as follows.

- An up step has weight 1.
- A horizontal step starting at level i has weight yq^i.
- A down step starting at level i has weight $q^{i-1}(1-q^i)$.
- A double down step starting at level i has weight $-yq^{i-2}(1-q^i)(1-q^{i-1})$.

Then by Viennot's theory we have

$$\mathcal{L}_i(y^n)=\sum_{M\in\mathrm{Mot}_i(n)}\mathrm{wt}(M).$$

Thus we obtain the following corollary from Theorem 4.4.

Corollary 5.4. *For $n\ge 0$, we have*

$$\sum_{M\in\mathrm{Mot}_0(n)}\mathrm{wt}(M)=\sum_{k=0}^{\lfloor n/2\rfloor}\begin{bmatrix}n\\2k\end{bmatrix}_q(q;q^2)_k y^{n-2k},$$

$$\sum_{M\in\mathrm{Mot}_1(n)}\mathrm{wt}(M)=(1-q^n)\sum_{k=0}^{\lfloor n/2\rfloor}\begin{bmatrix}n-1\\2k\end{bmatrix}_q(q;q^2)_k y^{n-2k-1}.$$

It would be interesting to prove the above corollary combinatorially.

6. An addition theorem

A Hermite polynomial addition theorem is

$$H_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}H_k(x/a)a^kH_{n-k}(y/b)b^{n-k} \tag{34}$$

where $a^2+b^2=1$. We give a q-analogue of this result (Proposition 6.1) using the discrete big q-Hermite polynomials.

We will use $h_n(x,y;q)$ as our q-version of $H_n(x-y)$,

$$\lim_{q\to 1} h_n^*(x,y;q)=\lim_{q\to 1}\frac{h_n(x\sqrt{1-q},y\sqrt{1-q};q)}{(1-q)^{n/2}}=H_n(x-y).$$

and $h_n(x/a,0;q)$, the discrete q-Hermite, as our version of $H_n(x/a)$

$$\lim_{q\to 1} h_n^*(x,0;q)=\lim_{q\to 1}\frac{h_n(x\sqrt{1-q},0;q)}{(1-q)^{n/2}}=H_n(x).$$

Another q-version of $b^{n-k}H_{n-k}(y/b)$, $a^2+b^2=1$ is given by $p_{n-k}(y,a;q)$, where

$$p_t(y,a;q)=\sum_{m=0}^{[t/2]}\begin{bmatrix}t\\2m\end{bmatrix}_q(q;q^2)_m a^{2m}(1/a^2;q^2)_m y^{t-2m}q^{\binom{t-2m}{2}}.$$

$$\lim_{q\to 1}\frac{p_t(y\sqrt{1-q},a;q)}{(1-q)^{t/2}}=b^tH_n(x/b).$$

The result is

Proposition 6.1. *For $n\geq 0$,*

$$h_n(x,y;q)=(-1)^n\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q h_k(x/a,0|q)(-a)^kp_{n-k}(y,a|q).$$

Proof. The generating function of p_n is

$$F(y,a,w)=\sum_{n=0}^{\infty}\frac{p_n(y,a;q)}{(q)_n}w^n=\frac{(w^2;q^2)_\infty(-yw)_\infty}{(a^2w^2;q^2)_\infty}.$$

If

$$G(x,y,t)=\frac{(t^2;q^2)_\infty(yt)_\infty}{(xt)_\infty}$$

is the discrete big q-Hermite generating function, then

$$G(x,y,-t)=G(x/a,0,-at)F(y,a,t),$$

which gives Proposition 6.1. □

References

[1] R. Askey and J. Wilson. Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials. *Mem. Amer. Math. Soc.*, 54(319):iv+55, 1985.

[2] C. Berg and M. E. H. Ismail. q-Hermite polynomials and classical orthogonal polynomials. *Canad. J. Math.*, 9 (1996), 43-63.

[3] J. Cigler and J. Zeng. A curious q-analogue of Hermite polynomials. *J. Combin. Theory Ser. A*, 118(1):9–26, 2011.

[4] S. Corteel, R. Stanley, D. Stanton, and L. Williams. Formulae for Askey-Wilson moments and enumeration of staircase tableaux. *Transactions of the American Mathematical Society*, 364(11):6009–6037, 2012.

[5] K. Garrett, M. E. Ismail, and D. Stanton. Variants of the rogers–ramanujan identities. *Advances in Applied Mathematics*, 23(3):274–299, 1999.

[6] G. Gasper and M. Rahman. Basic hypergeometric series, volume 96 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 2004. With a foreword by Richard Askey.

[7] M. E. H. Ismail, Classical and Quantum Orthogonal Polynomials in One Variable. Cambridge University Press, Cambridge, 2005.

[8] M. E. H. Ismail and M. Rahman. Connection relations and expansions. *Pacific J. Math.*, 252(2):427–446, 2011.

[9] M. E. H. Ismail and P. Simeonov. Complex Hermite polynomials: Their combinatorics and integral operators. preprint.
[10] M. E. H. Ismail and D. Stanton. q-Taylor theorems, polynomial expansions, and interpolation of entire functions. *J. Approx. Theory*, 123(1):125–146, 2003.
[11] M. E. H. Ismail and D. Stanton. Expansions in the Askey-Wilson polynomials. preprint.
[12] M. Josuat-Vergès. Rook placements in young diagrams and permutation enumeration. *Adv. in Appl. Math.*, 47:1–22, 2011.
[13] R. Koekoek, P. A. Lesky, and R. R. F. Swarttouw. Hypergeometric orthogonal polynomials and their q-analogues. Springer, 2010.
[14] J. S. Kim and D. Stanton. Moments of Askey-Wilson polynomials. *to appear in J. Combin. Theory Ser. A.*
[15] G. Viennot. Une théorie combinatoire des polynômes orthogonaux. Lecture Notes, UQAM, 1983.

(Jang Soo Kim) Department of Mathematics, Sungkyunkwan University, Suwon 440-746, South Korea
E-mail address: `jangsookim@skku.edu`

(Dennis Stanton) School of Mathematics, University of Minnesota, Minneapolis, MN 55455
E-mail address: `stanton@math.umn.edu`