CRANKS AND T-CORES

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Abstract. New statistics on partitions (called *cranks*) are defined which combinatorially prove Ramanujan's congruences for the partition function modulo 5, 7, 11, and 25. Explicit bijections are given for the equinumerous crank classes. The cranks are closely related to the t-core of a partition. Using q-series, some explicit formulas are given for the number of partitions which are t-cores. Some related questions for self-conjugate and distinct partitions are discussed.

1. Introduction.

Let $p(n)$ be the number of partitions of n [1]. Dyson [7] proposed a combinatorial proof of Ramanujan's congruence

$$p(5n+4) \equiv 0 \mod 5.$$

He defined an integral statistic on partitions, called the rank, whose value mod 5 split the set of partitions of $5n+4$ into 5 equal classes. He further conjectured that the rank also proved

$$p(7n+5) \equiv 0 \mod 7,$$

and hypothesized a statistic, called the crank, which would similarly prove

$$p(11n+6) \equiv 0 \mod 11.$$

Atkin and Swinnerton-Dyer [5] proved Dyson's conjecture for 5 and 7. In [9], a crank for 11 as well as new cranks for 5 and 7 were found relative to vector partitions. Later, Andrews and Garvan [3] were able to define a crank, in terms of ordinary partitions, for 5, 7 and 11, thus completing the solution of Dyson's crank conjecture. New relations for the crank of partitions mod 8, 9 and 10 have been found in [10].

Explicit bijections between these equinumerous classes have not been found in any of these three cases. This should be related to another possible approach to these problems: split the set of partitions of $5n+4$ into $p(5n+4)/5$ classes, each of size 5. Hopefully, the bijections among the five classes should be realized by a 5-cycle which acts on all partitions of $5n+4$ with no fixed points. The orbits of

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the 5 cycle are the $p(5n+4)/5$ classes. A crank statistic should be defined which is $\{0, 1, 2, 3, 4\}$ on an orbit. A 5-cycle, but no crank, was found by computer in [11]. In this paper, we complete this program by finding dihedral groups of size 10, 14, and 22 which act on the partitions of $5n+4$, $7n+5$, and $11n+6$, respectively. The 5 (7 or 11) cycles in these groups have no fixed points, thus proving the congruences. We also define new statistics on partitions, i.e. new cranks, which split the set of partitions into equinumerous classes. A cycle in the dihedral groups increases the crank by 1, thus supplying an explicit bijection between the crank classes. Our definitions of the cycles and the cranks are completely combinatorial.

The key ingredients to our proof are two bijections for partitions and t-cores, which are given in §2. They allow us to find dihedral groups as symmetry groups of quadratic forms. There are interesting q-series attached to these forms, and these q-series imply some remarkable explicit formulas (Theorems 4 and 5 in §5) for the number of partitions which are t-cores, $t = 5$ or 7. The explicit cranks are given in Theorems 2 and 3, and in Proposition 1 of §3 and §4. A crank for $25n+24$ is given in §6. Similar bijections and generating functions for self-conjugate partitions and partitions with distinct parts are given in §7 and §8.

We shall use the notation for generating function from q-series, e.g.

$$(a\,;q)_k = \prod_{i=0}^{k-1}(1-aq^i),$$

so

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q\,;q)_\infty}.$$

2. Two bijections.

In this section we give two bijections relating partitions and t-cores.

First we set the notation. Let P be the set of all partitions. For any $\lambda \in P$, let $|\lambda|$ denote the number that λ partitions. Fix a positive integer t. Let P_{t-core} be the set of partitions which are t-cores. Recall that there are two equivalent definitions of a t-core: a partition λ is a t-core if, and only if, λ has no hook numbers that are multiples of t [13, 2.7.40]; or if, and only if, λ has no rim hooks that are multiples of t [13, p. 75]. We let $a_t(n)$ be the number of partitions of n which are t-cores.

The first bijection is well-known [13, 2.7.17].

Bijection 1. *There is a bijection* $\phi_1 : P \to P_{t-core} \times P \times P \times \cdots \times P$,

$$\phi_1(\lambda) = (\tilde{\lambda}, \lambda_0, \lambda_1, \ldots, \lambda_{t-1}),$$

such that $|\lambda| = |\tilde{\lambda}| + t\sum_{i=0}^{t-1}|\lambda_i|$.

The generating function identity equivalent to Bijection 1 is

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q^t\,;q^t)_\infty^t}\sum_{n=0}^{\infty} a_t(n)q^n. \tag{2.1}$$

The second bijection is on t-cores only.

Bijection 2. *There is a bijection* $\phi_2 : P_{t-core} \to \{\vec{n} = (n_0, n_1, \ldots, n_{t-1}) : n_i \in \mathbb{Z}, n_0 + \cdots + n_{t-1} = 0\}$*, where*

$$|\tilde{\lambda}| = t\|\vec{n}\|^2/2 + \vec{b} \cdot \vec{n}, \qquad \vec{b} = (0, 1, \cdots, t-1).$$

The generating function identity equivalent to Bijection 2 is

$$\sum_{n=0}^{\infty} a_t(n) q^n = \sum_{\substack{\vec{n}\cdot\vec{1}=0 \\ \vec{n}\in\mathbb{Z}^t}} q^{\frac{t}{2}\|\vec{n}\|^2 + \vec{b}\cdot\vec{n}}. \tag{2.2}$$

Clearly (2.1) and (2.2) imply

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q^t; q^t)_\infty^t} \sum_{\substack{\vec{n}\cdot\vec{1}=0 \\ \vec{n}\in\mathbb{Z}^t}} q^{\frac{t}{2}\|\vec{n}\|^2 + \vec{b}\cdot\vec{n}}. \tag{2.3}$$

We now give Bijection 2, and sketch Bijection 1.

Let $\tilde{\lambda}$ be a t-core. Define the vector $\vec{n} = \phi_2(\tilde{\lambda})$ in the following way. Label a cell in the i th row and j th column of $\tilde{\lambda}$ by $j - i \mod t$. (This is called the t-residue diagram in [13, p. 84].) We also label the cells in column 0 in the same way, and call the resulting diagram the extended t-residue diagram. A cell is called exposed if it is at the end of a row. Region r of the extended t-residue diagram of $\tilde{\lambda}$ is the set of cells (i, j) satisfying $t(r-1) \le j - i < tr$. We now define n_i to be the maximum region of $\tilde{\lambda}$ which contains an exposed cell labeled i. Since column 0 contains infinitely many exposed cells, n_i is well-defined.

If an exposed cell labeled i lies in region r, then there is an exposed cell labeled i in each region $< r$. For example, in region $r-1$, if i is not exposed, then the rim hook whose northeast head is i in region r, and whose tail is $i+1$ in region $r-1$ has length t. (Note: If $i = t-1$, the rim hook will lie entirely in region r.) This contradicts the assumption that $\tilde{\lambda}$ is a t-core.

Note that the size of the Durfee square of $\tilde{\lambda}$ is the sum of the positive n_i's, since this is the number of exposed cells in positive regions.

We next verify that $n_0 + n_1 + \cdots + n_{t-1} = 0$. Let $\tilde{\lambda}'$ be the conjugate of $\tilde{\lambda}$. First we show that if $\phi_2(\tilde{\lambda}) = (n_0, n_1, \cdots, n_{t-1})$, then $\phi_2(\tilde{\lambda}') = (-n_{t-1}, -n_{t-2}, \cdots, -n_0)$. Let a cell labeled i be exposed in region r of $\tilde{\lambda}$, so that $i+1$ lies above i. In $\tilde{\lambda}'$, the cell corresponding to $i+1$ will be labeled $t-i-1$, lie in region $1-r$, and not be the last cell in its row. Thus $t-i-1$ is not exposed in region $1-r$ of $\tilde{\lambda}'$. This shows that if i is exposed in region r of $\tilde{\lambda}$, $r \le n_i$; then $t-1-i$ is not exposed in region r of $\tilde{\lambda}'$, $r \ge 1 - n_i$. A similar argument shows that if a boundary cell labeled $t-i-1$ is not exposed in region $1-r$, then i is exposed in region r. This verifies the claim about $\phi_2(\tilde{\lambda}')$. Since the size of the Durfee square is unchanged by conjugation, the sum of the positive n_i's must be equal to minus the sum of the negative n_i's; that is, $n_0 + n_1 + \cdots + n_{t-1} = 0$.

The inverse map to ϕ_2 is easy to find: if $(n_0, n_1, \cdots, n_{t-1})$ is given, then the exposed cells in each region are known. This in turn gives the lengths of each row of the Ferrers diagram of $\tilde{\lambda}$, and thus $\tilde{\lambda}$.

Finally we show that if $\phi_2(\tilde{\lambda}) = \vec{n}$, then $\tilde{\lambda}$ is a partition of the exponent in Bijection 2. The number of cells to the right of the main diagonal of the Ferrers diagram of λ is

$$\sum_{n_i>0} in_i + t\binom{n_i}{2}.$$

For the cells below the main diagonal, we take $\tilde{\lambda}'$, to find

$$-\sum_{n_j<0}(t-1-j)n_j + t\binom{-n_j}{2}.$$

There are

$$\sum_{n_i>0} n_i$$

cells on the main diagonal. The sum of these three terms is the exponent of q in (2.3).

We quickly sketch ϕ_1 using the notation in Bijection 2. Given λ, we construct t biinfinite words in the letters N and E, $w_0, \ldots, w_{t-1}$. The j th letter of w_i is E if i is exposed in region j of λ, otherwise the j th letter is N (not exposed). If λ is a t-core, then each word w_i is an infinite sequence of E's followed by an infinite sequence of N's. In this case, $\lambda = \tilde{\lambda}$, and $\lambda_i = \varnothing$ for all i. Otherwise there is an E to the right of some N. Find the rightmost E, say in position j. Find the rightmost N to the left of this E, say in position $k < j$. Delete a rim hook in λ whose northeast head is the cell corresponding to E, and whose southwest tail is adjacent to the cell corresponding to N. Place a part of size $j-k$ in λ_i. The new partition has N's in w_i from position j on, thus the E's have been pushed to the left. This operation can be done in any order whatsoever, to finally obtain the t-core of λ, $\tilde{\lambda}$. The parts of each λ_i appear in increasing order.

3. Cranks.

In this section we give the cranks (Theorem 2) and the cycles which give bijections for these crank classes.

Let r be a residue class of t, and consider $p(tn+r)$. On the right side of (2.3), all exponents of q which are equivalent to r mod t lie in the multisum. Since $\vec{n}\cdot\vec{1} = 0$, we have $\frac{t}{2}\|\vec{n}\|^2 \equiv 0 \mod t$. Thus we have

$$\sum_{n=0}^{\infty} p(tn+r)q^{tn+r} = \frac{1}{(q^t;q^t)_\infty^t} \sum_{\substack{\vec{n}\cdot\vec{b}\equiv r \mod t \\ \vec{n}\cdot\vec{1}=0 \\ \vec{n}\in\mathbb{Z}^t}} q^{\frac{t}{2}\|\vec{n}\|^2+\vec{b}\cdot\vec{n}}. \tag{3.1}$$

The symmetry groups of the quadratic forms in (3.1) have been computed by computer for $t \leq 8$ in [11]. Until now, we have assumed that the class t and residue r are arbitrary. We shall see that the choices of $5n+4$, $7n+5$, and $11n+6$ simplify (3.1).

For $5n+4$ we now change variables so that the symmetry group is obviously the dihedral group D_5 of order 10. The five vectors $\vec{n}$ for $\phi_2(p(4))$ are

(1) $\vec{v}_0 = (1,-1,0,0,0)$,
(2) $\vec{v}_1 = (0,1,-1,0,0)$,

(3) $\vec{v}_2 = (0,0,1,-1,0)$,
(4) $\vec{v}_3 = (0,0,0,1,-1)$, and
(5) $\vec{v}_4 = (1,1,0,-1,-1)$.

They form a non-planar pentagon whose center $\vec{c} = (2/5, 1/5, 0, -1/5, -2/5)$.

We now change variables: write $\vec{n} = \alpha_0\vec{v}_0 + \cdots + \alpha_4\vec{v}_4$. A vector $\vec{n} \in \mathbb{Z}^5$ satisfies $\vec{n}\cdot\vec{1} = 0$ and $\vec{b}\cdot\vec{n} \equiv 4 \mod 5$ if, and only if, $\vec{n} = \alpha_0 v_0 + \cdots + \alpha_4 v_4$ for some $\vec{\alpha} \in \mathbb{Z}^5$ such that $\vec{\alpha}\cdot\vec{1} = 1$.

It is easy to see that the quadratic form in (3.1) in terms of the vector $\vec{\alpha}$ is

$$5\|\vec{\alpha}\|^2 - 5(\alpha_0\alpha_1 + \alpha_1\alpha_2 + \cdots + \alpha_4\alpha_0) - 1.$$

Thus we obtain

$$\sum_{n=0}^{\infty} a_5(5n+4)q^{n+1} = \sum_{\substack{\vec{\alpha}\cdot\vec{1}=1 \\ \vec{\alpha}\in\mathbb{Z}^t}} q^{Q(\vec{\alpha})},$$

where $Q(\vec{\alpha}) = \|\vec{\alpha}\|^2 - (\alpha_0\alpha_1 + \alpha_1\alpha_2 + \cdots + \alpha_4\alpha_0)$. Clearly the dihedral group D_5 is the automorphism group.

It is also clear that the 5-cycle which cyclically permutes the α_i's has no fixed points. Any fixed point $\vec{\alpha}$ must have all entries equal. Since $\vec{\alpha}\cdot\vec{1} = 1$, this would imply that the entries are not integral.

The construction of the quadratic form and the group are completely analogous for $7n+5$ and $11n+6$. We collect these results in a theorem.

Theorem 1. *For* $(t, r) = (5, 4), (7, 5)$ *or* $(11, 6)$,

$$\sum_{n=0}^{\infty} p(tn+r)q^{n+1} = \frac{1}{(q\,;q)_\infty^t} \sum_{\substack{\vec{\alpha}\cdot\vec{1}=1 \\ \vec{\alpha}\in\mathbb{Z}^t}} q^{Q(\vec{\alpha})}, \tag{3.2}$$

where $Q(\vec{\alpha}) = \|\vec{\alpha}\|^2 - \sum_{i=0}^{t-1} \alpha_i\alpha_{i+1}$.

It should be noted that other identities for the generating functions of $p(tn+r)$, which yield Ramanujan's congruences, have been found by others. The most famous of these is Ramanujan's [19, (17) p.213] identity for $p(5n+4)$, which was considered by Hardy in agreement with MacMahon as Ramanujan's most beautiful identity [19, p.xxxv]. Ramanujan [19, (18) p.213] also found an analog for $p(7n+5)$. In fact for $t = 5^\alpha, 7^\alpha$ similar identities have been found by Watson [20]. Using the theory of modular functions an analog for $p(11n+6)$ was found by Fine [8, (3.25)] and extended to higher powers of 11 by Atkin [4]. However, these other identities are of a different nature than (3.2) and do not yield obvious cranks. Further, Fine's identity for $t = 11$ and Ramanujan's identity for $t = 5$ or $t = 7$ are dissimilar so it is surprising that an identity like (3.2) holds for all three values $t = 5$, 7 and 11.

Finally we come to the cranks. An obvious statistic, which is increased by 1 mod 5 each time the α's are cyclically permuted is $\sum_{i=0}^{4} i\alpha_i$. So in terms of the $\vec{\alpha}$ basis the crank is clear, but in terms of the $\vec{n}$ basis (given combinatorially in §2) the crank is not clear.

Theorem 2. *A crank statistic for partitions λ of $5n+4$, $7n+5$, or $11n+6$ is given by the following algorithm.*

(1) *Find the t-core $\tilde{\lambda}$ of λ, ($t=5,7$, or 11) by Bijection 1.*
(2) *Find $\phi_2(\tilde{\lambda})=\vec{n}$ by Bijection 2.*
(3) *Let crank(λ) be the following* mod t *linear combination:*

$(t=5)$ $4n_0+n_1+n_3+4n_4$ *for* $5n+4$,
$(t=7)$ $4n_0+2n_1+n_2+n_4+2n_5+4n_6$ *for* $7n+5$,
$(t=11)$ $4n_0+9n_1+5n_2+3n_3+n_4+n_6+3n_7+5n_8+9n_9+4n_{10}$ *for* $11n+6$.

The new crank is not equal to the rank or the old crank. One can verify this on the 5 partitions of 9 which are 5-cores: 621, 522, 51111, 33111, and 321111.

4. More cranks.

The cranks in §3 depend only upon the t-core $\tilde{\lambda}$ of λ, and thus use Bijections 1 and 2. In this section we give two simpler ways (Proposition 1 and Theorem 3) of computing crank(λ), (one which is as a polynomial in the parts of λ), thus avoiding Bijections 1 and 2.

First, let $r_k(\lambda)$ be the number of cells in the t-residue diagram of λ which are labeled $k \mod t$. Suppose that λ is a t-core. A relation between the r_k's and the n_i's can be found by counting the cells in a row ending with i in region n_i. It is

$$r_k=\sum_{i=0}^{t-1}(n_i^2/2+n_i\chi(i\geq k)) \tag{4.1}$$

where

$$\chi(S)=\begin{cases} 1/2 & \text{if S is true,}\\ -1/2 & \text{if S is false.}\end{cases}$$

It is easy to see that (4.1) implies

$$n_k=r_k-r_{k+1}, \tag{4.2}$$

so that the relation in (4.1) for r_0 for t-cores becomes

$$r_0=\sum_{i=0}^{t-1}(r_i^2-r_ir_{i+1}).$$

The cranks in Theorem 2 are defined by a linear combination of the n_i's. Using (4.2), we can find mod t equivalent linear combinations of the r_k's.

Proposition 1. *The following* mod t *linear combinations of the t-residue numbers r_k are crank statistics:*

(1) $r_1+2r_2-2r_3-r_4$ *for* $5n+4$,
(2) $2r_1+r_2+r_3-r_4-r_5-2r_6$ *for* $7n+5$,
(3) $-6r_1-4r_2-2r_3-2r_4-r_5+r_6+2r_7+2r_8+4r_9+6r_{10}$ *for* $11n+6$.

Proof. We already know from Theorem 2 that certain linear combinations give the crank for t-cores, e.g. $(2,-1,1,-2)$ for $t=5$. If we multiply this linear combination by 3, we obtain (1) for t-cores. We reverse the sign to obtain (2), and (3) is immediate from Theorem 2. Now consider (1)-(3) as definitions of a crank for any partition λ, not just t-cores. Removing a rim hook of length t from λ reduces each r_k by one. The above linear combinations do not change, since the sum of all of the coefficients is zero. Thus the crank can be defined by Proposition 1 for all partitions λ. $\square$

Next, we give a definition of a crank based upon the parts of $\lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$.

Theorem 3. *For $t = 5$, 7 or 11, let $p_t(x) = (x - (t-1)/2)^{t-3} \mod t$. A crank statistic for partitions $\lambda : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ of $tn + r$ (where $r = 4$, 5, and 6, respectively) is given by*

$$crank(\lambda) = \sum_{i=1}^{m} (p_t(\lambda_i - i) - p_t(i-1)) \mod t.$$

Proof. Let a_i be the coefficient of r_i in Proposition 1. Recall that $a_1 + a_2 + \cdots + a_{t-1} = 0$.

First we evaluate the contribution to the crank in Proposition 1 from the first row of λ. The t-residue diagram of λ has an $x \equiv \lambda_1 - 1 \mod t$ at the end of the first row. Then the contribution of this row to the crank is

$$a_1 + a_2 + \cdots + a_x. \tag{4.3}$$

It remains to find a polynomial whose values at $x = 0, 1, \ldots, t-1$ agree with (4.3) mod t. Here are such polynomials:

(1) $q_5(x) = 3(p_5(x) - 4)$ for $t = 5$,
(2) $q_7(x) = -(p_7(x) - 4)$ for $t = 7$, and
(3) $q_{11}(x) = p_{11}(x) - 4$ for $t = 11$.

Thus we have shown that the first row contributes $q_t(x) \equiv q_t(\lambda_1 - 1) - q_t(0) \mod t$.

Consider row i of the t-residue diagram of λ: the first entry is $1 - i \mod t$, and the final entry is $\lambda_i - i \mod t$. The contribution of this row to the crank is

$$\begin{aligned} &a_1 + a_2 + \cdots + a_{\lambda_i - i} + (a_{t+1-i} + \cdots + a_{t-1}) \equiv \\ &\qquad a_1 + a_2 + \cdots + a_{\lambda_i - i} - (a_1 + \cdots + a_{t-i}) \equiv q_t(\lambda_i - i) - q_t(t-i) \mod t. \end{aligned}$$

Thus

$$crank(\lambda) = \sum_{i=1}^{m} (q_t(\lambda_i - i) - q_t(t-i)) \mod t.$$

Since $q_t(t-1-i) \equiv q_t(i) \mod t$, and $q_t(i)$ is a linear function of $p_t(i)$, the result follows. $\square$

5. Associated q-series.

From (2.1) there is an explicit generating function for the partitions of n which are t-cores. We investigate these functions for $t = 5, 7$, and 11, and give explicit formulas for $a_5(n)$ and $a_7(n)$ in Theorems 4 and 5.

The five cycle in §3 implies that $a_5(5n+4) \equiv 0 \mod 5$. However, much more is true,

$$a_5(5n+4) = 5a_5(n). \tag{5.1}$$

we give two proofs of (5.1): one which is analytic, and proves more (Theorem 4), and another which is a bijection.

Theorem 4. *Let $n+1 = 5^c p_1^{a_1} \cdots p_s^{a_s} q_1^{b_1} \cdots q_t^{b_t}$ be the prime factorization of $n+1$ into primes $p_i \equiv 1,4 \mod 5$, and $q_j \equiv 2,3 \mod 5$. Then*

$$a_5(n) = 5^c \prod_{i=1}^{s} \frac{p_i^{a_i+1}-1}{p_i-1} \prod_{j=1}^{t} \frac{q_j^{b_j+1}+(-1)^{b_j}}{q_j+1}.$$

Proof. Clearly (2.1) implies

$$\sum_{n=0}^{\infty} a_5(n) q^n = \frac{(q^5\,;q^5)_\infty^5}{(q\,;q)_\infty}.$$

However an identity of Ramanujan ([2, (3.46)] or [6]) is

$$q\frac{(q^5\,;q^5)_\infty^5}{(q\,;q)_\infty} = \sum_{n=0}^{\infty} \left(\frac{n}{5}\right) \frac{q^n}{(1-q^n)^2}. \tag{5.2}$$

where $\left(\frac{a}{b}\right)$ is the Legendre symbol. Clearly the right side of (5.2) is

$$\sum_{n=1}^{\infty} \sum_{d|n} \left(\frac{d}{5}\right) \frac{n}{d} q^n.$$

so

$$a_5(n-1) = \sum_{d|n} \left(\frac{d}{5}\right) \frac{n}{d}. \tag{5.3}$$

Since $\left(\frac{d}{5}\right)/d$ is multiplicative, (5.3) and [12, Theorem 265] implies that $a_5(n-1)$ is multiplicative. It remains to evaluate $a_5(p^a-1)$ for primes p, which can be done by (5.3). □

The combinatorial proof of (5.1) is explicit: given a partition $\tilde{\lambda}$ of n which is a 5-core, find five partitions θ_0, θ_1, θ_2, θ_3, θ_4, of $5n+4$ which are also 5-cores. Let $\phi_2(\tilde{\lambda}) = \vec{m}$. We now give the image under ϕ_2 of the five partitions of $5n+4$.

(1) $\phi_2(\theta_0) = (m_1+2m_2+2m_4+1, -m_1-m_2+m_3+m_4+1, 2m_1+m_2+2m_3, -2m_2-2m_3-m_4-1, -2m_1-m_3-2m_4-1)$,
(2) $\phi_2(\theta_1) = (2m_1+2m_4+1+m_3, -1-m_2-2m_3-2m_4, -2m_1-2m_2-m_4, m_1+2m_2+2m_3, -m_1+m_2+m_4-m_3)$,
(3) $\phi_2(\theta_2) = (m_1-m_2-m_4+m_3, 2m_2+m_4+1+2m_3, -m_1-2m_4-1-2m_3, -2m_1-2m_2-m_3, 2m_1+m_2+2m_4)$,
(4) $\phi_2(\theta_3) = (-2m_1-m_2-2m_4, -m_1-2m_2-2m_3, 2m_2+2m_4+1+m_3, m_1+m_2-m_4-1-m_3, 2m_1+2m_3+m_4)$,
(5) $\phi_2(\theta_4) = (-2m_1-m_4-2m_3, 2m_1+2m_2+m_3, m_1-m_2+m_4-m_3, m_2+2m_4+1+2m_3, -m_1-2m_2-2m_4-1)$.

We delete the verification of this claim. In fact, these five partitions θ_i form an orbit under the five cycle, and

$$crank(\theta_i) \equiv 3i \mod 5.$$

We denote by γ_n the $5-\text{to}-1$ map from the set of 5-cores of $5n+4$ to the set of 5-cores of n, $\gamma_n(\theta_i)=\tilde{\lambda}$ for $0\le i\le 4$.

For $t=7$, we have $a_7(7n+5)\equiv 0 \mod 7$. It is not true that $a_7(7n+5)=7a_7(n)$. There are however two multiplicative functions here. The identity which is analogous to (5.2) is due to Ramanujan [8, p. 159]

$$8q^2\frac{(q^7\,;q^7)_\infty^7}{(q\,;q)_\infty}+q(q^7\,;q^7)_\infty^3(q\,;q)_\infty^3=\sum_{n=1}^{\infty}\left(\frac{n}{7}\right)\frac{q^n(1+q^n)}{(1-q^n)^3}. \tag{5.4}$$

Let $a'(n)$, $b(n)$, $c(n)$ denote the coefficient of q^n in the expansion of each of the three functions in (5.4) so that

$$\begin{aligned}8a_7(n)&=a'(n+2),\\ a'(n)+b(n)&=c(n).\end{aligned}$$

We will give in Lemmas 1 and 2 explicit formulas for $b(n)$ and $c(n)$, resulting in an explicit formula for $a_7(n)$ in Theorem 5.

The proof of Theorem 4 analogously establishes the multiplicativity of $c(n)$. We delete the details. The resulting explicit formula is given in Lemma 1.

Lemma 1. *If $n=7^c p_1^{a_1}\cdots p_s^{a_s}q_1^{b_1}\cdots q_t^{b_t}$ is the prime factorization of n into primes $p_i\equiv 1,2,4 \mod 7$, and $q_j\equiv 3,5,6 \mod 7$, then*

$$c(n)=7^{2c}\prod_{i=1}^{s}\frac{p_i^{2a_i+2}-1}{p_i^2-1}\prod_{j=1}^{t}\frac{q_j^{2b_j+2}+(-1)^{b_j}}{q_j^2+1}.$$

To establish the multiplicativity of the sequence $b(n)$, we apply the theory of modular forms.

Proposition 2. *The sequence $b(n)$ is multiplicative, moreover*

$$b(p^m)=b(p)b(p^{m-1})-\left(\frac{p}{7}\right)p^2b(p^{m-2})$$

for any prime p and integer $m\ge 2$.

Proof. If $q=e^{2\pi iz}$, the generating function $B(z)$ for $b(n)$ (see (5.4)) can be considered a function of z in the upper half plane. In terms of the classical η-function it is $B(z)=(\eta(z)\eta(7z))^3$, which is a cusp form of level 7, weight 3, and character $\chi(n)=\left(\frac{n}{7}\right)$ [14, Prob. 12, 13, p.145]. Since this space of cusp forms is one-dimensional, $B(z)$ must be an eigenform for the Hecke operators T_n. This proves the multiplicativity of $b(n)$, and the three term recurrence follows from the Euler product for the associated Dirichlet series [14, p. 160 (5.11)]. □

To find an explicit formula for $b(n)$, we need only find $b(p)$ for all primes p. The three term recurrence in Proposition 2 gives $b(p^m)$, and multiplicativity gives $b(n)$ for all n. The next proposition gives the values of $b(p)$.

Proposition 3. *Suppose p is an odd prime. If $p \equiv 3, 5$ or $6 \mod 7$, then $b(p) = 0$. If $p \equiv 1, 2$ or $4 \mod 7$, let $p = x^2 + 7y^2$, where x and y are positive integers. Then $b(p) = 2(x^2 - 7y^2)$.*

Remark. The result also holds for $p = 2$ except in this case $x = y = \frac{1}{2}$.

Proof. From Jacobi's identity for $(q\,;q)_\infty^3$ [12, Thm 357] we find that

$$b(p) = \sum_{\substack{m,n\geq 0\\(2m+1)^2+7(2n+1)^2=8p}} (-1)^{m+n}(2m+1)(2n+1).$$

We need to determine all of the solutions (m, n) to

$$8p = (2m + 1)^2 + 7(2n + 1)^2, \tag{p1}$$

where $m, n \geq 0$. Clearly the right side is a quadratic residue mod 7, so if $p \equiv 3, 5$ or $6 \mod 7$, there are no solutions (m, n) to (p1) and $b(p) = 0$.

Thus we can assume that $p \equiv 1, 2$ or $4 \mod 7$. We first show in this case that p has a unique representation $p = x^2 + 7y^2$ for positive integers x, y. We work in $K = \mathbb{Q}(\sqrt{-7})$, whose ring of algebraic integers, $\mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$, is a unique factorization domain [12, Th. 246]. Since

$$1 = \left(\frac{p}{7}\right) = \left(\frac{-7}{p}\right) \qquad \text{(by quadratic reciprocity)},$$

p divides $a^2 + 7$ for some integer a. Since p does not divide $a \pm \sqrt{-7}$, p is not prime. Thus p splits

$$p = \beta\bar{\beta},$$

where $\beta = x + y\sqrt{-7}$ is prime in $\mathcal{O}_K$ and x, $y > 0$. Now, $x, y \in \mathbb{Z}$, otherwise $x = m + \frac{1}{2}$, $y = n + \frac{1}{2}$ (for some $m, n \in \mathbb{Z}$), $p = m(m + 1) + 7n(n + 1) + 2$ which is even and contradicts the assumption that p was odd. Hence $p = \beta\bar{\beta} = x^2 + 7y^2$ for some positive integers x, y. The uniqueness of the representation follows from unique factorization, and the only units being ± 1.

We next show that (p1) always has two solutions (m, n), whose contribution to $b(p)$ is $2(x^2 - 7y^2)$. (p1) is equivalent to $2p = \gamma\bar{\gamma}$, where

$$\gamma = \tfrac{(2m+1)+(2n+1)\sqrt{-7}}{2} = \tfrac{x_1+y_1\sqrt{-7}}{2}, x_1, y_1 > 0.$$

Let

$$2 = \left(\tfrac{1+\sqrt{-7}}{2}\right)\left(\tfrac{1-\sqrt{-7}}{2}\right) = \pi\bar{\pi}.$$

It follows that $2p$ factors as

$$2p = (\pi\beta)(\bar{\pi}\bar{\beta}) = (\pi\bar{\beta})(\bar{\pi}\beta) = \gamma\bar{\gamma}.$$

By unique factorization we must have $\gamma = \pm\pi\beta$, $\pm\bar{\pi}\beta$, $\pm\bar{\pi}\bar{\beta}$, or $\gamma = \pm\pi\bar{\beta}$. This allows eight possible solutions (x_1, y_1), six of which violate $x_1, y_1 > 0$. The two solutions can be explicitly given, depending upon which of the three inequalities $0 < y < 7y < x$, $0 < x < y < 7y$, or $0 < y < x < 7y$ occur. In each of these three cases we find $b(p) = 2(x^2 - 7y^2)$ as required. □

For $p \equiv 1, 2$ or $4 \mod 7$ note that Proposition 3 implies that $b(p) \neq 0$, since $b(p) = 0$ would imply that $x^2 = 7y^2$, and thus $p = 14y^2 \equiv 0 \mod 7$. We also note that $b(p) = 2(x^2 - 7y^2) = \beta^2 + \bar{\beta}^2$ where $p = \beta\bar{\beta}$, $\beta = x + y\sqrt{-7}$ and $x, y > 0$. Using the three term recurrence in Proposition 2 and the value of $b(p)$ in Proposition 3, the value of $b(p^m)$ is easily found. It is given in the next lemma.

Lemma 2. *Let $n = 7^c p_1^{a_1} \cdots p_s^{a_s} q_1^{b_1} \cdots q_t^{b_t}$ be the prime factorization of n into primes $p_i \equiv 1, 2, 4 \mod 7$, and $q_j \equiv 3, 5, 6 \mod 7$. Let $p_i = x_i^2 + 7y_i^2$ for some positive integers x_i, y_i in the case when p_i is odd and $x_1 = y_1 = \frac{1}{2}$ for the case $p_1 = 2$. Then*

$$b(n) = 0 \text{ if some } b_j \text{ is odd,}$$

$$b(n) = (-7)^c \prod_{i=1}^{s} \frac{\beta_i^{2a_i+2} - \bar{\beta}_i^{2a_i+2}}{\beta_i^2 - \bar{\beta}_i^2} \prod_{j=1}^{t} q_j^{b_j} \text{ otherwise,}$$

where $\beta_i = x_i + y_i\sqrt{-7}$.

Note that Lemma 2 implies that $b(p^m) = 0$ if, and only if, $p \equiv 3, 5$ or $6 \mod 7$ and m is odd. We can now state the explicit formula for $a_7(n)$.

Theorem 5. *Let $n + 2 = 7^c p_1^{a_1} \cdots p_s^{a_s} q_1^{b_1} \cdots q_t^{b_t}$ be the prime factorization of $n+2$ into primes $p_i \equiv 1, 2, 4 \mod 7$, and $q_j \equiv 3, 5, 6 \mod 7$, then*

$$a_7(n) = \frac{1}{8}(c(n + 2) - b(n + 2)).$$

where $c(n)$ and $b(n)$ are given in Lemmas 1 and 2 respectively.

We state relations which are corollaries of Theorems 4 and 5.

Corollary 1. *The following relations hold:*

(1) $a_5(5^\alpha m - 1) = 5^\alpha a_5(m - 1) \equiv 0 \mod 5^\alpha$,
(2) $a_7(7^\alpha m - 2) \equiv 0 \mod 7^\alpha$,
(3) $a_7(49n + 19) = 49a_7(7n + 1)$,
(4) $a_7(49n + 33) = 49a_7(7n + 3)$,
(5) $a_7(49n + 40) = 49a_7(7n + 4)$.

Proof. (1)–(5) are immediate from Theorems 4 and 5. □

For $a_{11}(n)$ there are identities for cusp forms which prove

$$a_{11}(11^\alpha m - 5) = 0 \mod 11^\alpha \tag{5.5}$$

and

$$a_{11}(43n - 5) = 0 \mod 44 \text{ for } (n, 43) = 1. \tag{5.6}$$

We delete the proofs of (5.5) and (5.6).

6. A crank for $p(25n + 24)$.

We can define a group G which acts on the set of partitions of $tn+r$: $G = G_1 \times S_t$, where G_1 is the automorphism group of the quadratic form in Bijection 2, and S_t is the symmetric group on t letters. For example, for $5n + 4$, $G = D_5 \times S_5$, where D_5 is the dihedral group of order 10. We have concentrated on the action of G_1 for congruences. We can also use S_t for congruences, and do so in this section to find a crank for

$$p(25n + 24) \equiv 0 \mod 25. \tag{6.1}$$

The group $G = D_5 \times S_5$ for $5n + 4$ contains two commuting five cycles which generate a subgroup of order 25. If each five cycle has no fixed points, then G will have no fixed points, and the orbits of G will have size 25. We have seen that the five cycle inside D_5 has no fixed points. Since the five cycle in S_5 could have fixed points, we consider two cases to define the crank, which will be an ordered pair of integers, each between 0 and 4. This ordered pair could be considered as the base 5 expansion for a number between 0 and 24.

Theorem 6. *A crank statistic for partitions λ of $25n+24$ is given by the following algorithm.*

(1) *Find $\phi_1(\lambda) = (\tilde{\lambda}, \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ given by Bijection 1.*
(2) *Let crank(λ) be the following ordered pair of integers:*

Suppose that $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$, so that $\tilde{\lambda}$ is a partition of $25s+24$ for some s. Let $\mu_1 = \gamma_{5s+4}(\tilde{\lambda})$ and $\mu_2 = \gamma_s(\mu_1)$ be partitions of $5s+4$ and s respectively (see §5). Put

$$crank(\lambda) = (crank(\mu_1), crank(\mu_2)). \tag{i}$$

Suppose that some $\lambda_i \neq \lambda_j$, so that the five cyclic permutations of $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ are distinct. Order these five tuples in lex order, and define the crank of the i th tuple to be $i-1$. Put

$$crank(\lambda) = (crank(\tilde{\lambda}), crank((\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4))). \tag{ii}$$

Proof. The vectors $\gamma_n(\theta_i) = \tilde{\lambda}$ in §5 form an orbit under $C_5 \leq G_1$, so that the cranks of these five partitions are $\{0, 1, 2, 3, 4\}$.

The second part of the definition similarly interprets $C_5 \times C_5$, as a subgroup of G. □

In Theorem 6(2)(ii) the lex order crank can be replaced by using following statistic. Find $\phi_1(\lambda)$, and if

$$|\lambda_0| + |\lambda_1| + |\lambda_2| + |\lambda_3| + |\lambda_4| \equiv 0 \mod 5,$$

define the crank as in Theorem 6(2)(i). Otherwise define

$$crank(\lambda) = (crank(\tilde{\lambda}), |\lambda_1| + 2|\lambda_2| + 3|\lambda_3| + 4|\lambda_4| \mod 5).$$

It is clear that the five cyclic permutations of the λ_i will give five distinct cranks for the second component.

7. Self-conjugate partitions.

In this section we consider the restriction of Bijections 1 and 2 to self-conjugate partitions. Let $asc_t(n)$ be the number of t-cores which are self-conjugate.

In Bijection 1, suppose that λ is self-conjugate. Then $\tilde{\lambda}$ must be self-conjugate, and $\lambda_i' = \lambda_{t-i-1}$, $0 \leq i \leq t-1$. Since $(-q\,;q^2)_\infty$ is the generating function for self-conjugate partitions, the self-conjugate version of (2.1) is

$$(-q\,;q^2)_\infty = \frac{1}{(q^{2t}\,;q^{2t})_\infty^{t/2}} \sum_{n=0}^{\infty} asc_t(n) q^n \qquad \text{for } t \text{ even}, \tag{7.1a}$$

and

$$(-q\,;q^2)_\infty = \frac{(-q^t\,;q^{2t})_\infty}{(q^{2t}\,;q^{2t})_\infty^{(t-1)/2}} \sum_{n=0}^{\infty} asc_t(n) q^n \qquad \text{for } t \text{ odd}. \tag{7.1b}$$

While there are congruences for $q_0(n)$, the number of self-conjugate partitions of n, e.g. $q_0(125n+99) \equiv 0 \mod 5$, it is not obvious how to prove these congruences from this approach. Nevertheless, for $t = 5$ there is a nice Lambert series, and a theorem analogous to Theorems 4 and 5.

The Lambert series is

$$q(-q\,;q^2)_\infty \frac{(q^{10}\,;q^{10})^2_\infty}{(-q^5\,;q^{10})_\infty} = \sum_{n=1}^{\infty} \chi_{20}(n) \frac{q^n}{1-q^n}, \tag{7.2}$$

where

$$\chi_{20}(n) = \begin{cases} 0 & \text{if } (n,10) > 1, \\ (-1)^{\frac{1}{2}(n-1)} & \text{otherwise.} \end{cases}$$

Thus only odd n occur in (7.2), and the appropriate signs for odd n mod 20 are $+-0-+-+0+-$. (7.2) follows from the ${}_6\psi_6$ summation formula [2, (3.1)]. The identity (7.2) also follows from an identity of Ramanujan [21, (1) p.60] and Jacobi's formula for $r_2(n)$, the number of representations of n as a sum of two squares [12, Thm 278]. The functions $r_2(n)$ and $asc_5(n)$ are related by

$$r_2(n) - r_2(\frac{n}{5}) = 4asc_5(n-1). \tag{7.3}$$

As before we obtain from (7.2) the following theorem.

Theorem 7. *Let $n+1 = 2^c 5^d p_1^{a_1} \cdots p_s^{a_s} q_1^{b_1} \cdots q_t^{b_t}$ be the prime factorization of $n+1$ into primes $p_i \equiv 1 \mod 4$, and $q_j \equiv 3 \mod 4$. Then*

$$asc_5(n) = \prod_{i=1}^{s}(a_i+1) \prod_{j=1}^{t}(1+(-1)^{b_j})/2.$$

A surprising corollary results.

Corollary 2. *The following relations hold:*

(1) $asc_5(2n+1) = asc_5(n)$,
(2) $asc_5(5n+4) = asc_5(n)$,
(3) $asc_5(n) = 0$ *if and only if there exists a prime $q \equiv 3 \mod 4$ and an odd integer b such that q^b exactly divides $n+1$.*

There is a nice combinatorial proof of Corollary 2(2). Suppose $\tilde{\lambda}$ is a self-conjugate partition of $5n+4$ which is a 5-core. It is easy to see that only θ_0 of §4 is self-conjugate. Thus, the restriction of the map γ_n to self-conjugate partitions is a bijection.

A similar bijection exists for Corollary 2(1). Just replace (n_1, n_2) in (7.4) by $(-n_1+n_2, -n_1-n_2-1)$. We do not have a combinatorial proof of Corollary 2(3).

In Bijection 2, clearly we have $n_i = -n_{t-1-i}$, so the generating function becomes

$$\sum_{n=0}^{\infty} asc_t(n) q^n = \sum_{\vec{n} \in \mathbb{Z}^{\lfloor t/2 \rfloor}} q^{t\|\vec{n}\|^2 + \vec{c}\cdot\vec{n}}. \tag{7.4}$$

where

$$\vec{c} = \begin{cases} (2,4,\cdots,t-1) & \text{for } t \text{ odd,} \\ (1,3,\cdots,t-1) & \text{for } t \text{ even,} \end{cases}$$

and $\lfloor t/2 \rfloor$ is the integral part of $t/2$. Using (7.4) dissections of $(-q\,;q^2)_\infty$ due to Kolberg [16] can be easily derived.

8. Partitions with distinct parts.

A bijection analogous to Bijection 1 for partitions with distinct parts has been given by Morris and Yaseen [17]. In this section we give the appropriate version of Bijections 1 and 2 for these partitions.

We begin with notation. Let $pd(n)$ be the number of partitions λ of n with distinct parts. Given such a λ, the shifted Ferrers diagram of λ, $S(\lambda)$, is the Ferrers diagram of λ with the i th row shifted to the right by $i-1$ cells. The doubled partition of λ, denoted $\lambda\lambda$, is defined by adding λ_i cells to the $i-1$ st column of $S(\lambda)$. For example, if $\lambda = 3\,1$, then $\lambda\lambda = 4\,3\,1$. We let DD denote the set of doubled distinct partitions $\lambda\lambda$. We define the t-core of $\lambda\lambda$ as the usual t-core of $\lambda\lambda$. Let DD_{t-core} be the set of partitions which are t-cores of doubled distinct partitions, and let $add_t(n)$ be the number of these which are partitions of n. We state without proof that $DD_{t-core} \subset DD$, thus $add_t(n) = 0$ if n is odd.

By restricting Bijection 1, we find a bijection for doubled distinct partitions. We must take the conjugate of the λ_0 in ϕ_1 to find λ_0 for ϕ_3.

Bijection 3. *There is a bijection* $\phi_3 : DD \to DD_{t-core} \times DD \times P \times \cdots \times P$,

$$\phi_1(\lambda\lambda) = (\tilde{\lambda\lambda}, \lambda_0, \lambda_1, \ldots, \lambda_{t-1}),$$

such that $2|\lambda| = |\lambda\lambda| = |\tilde{\lambda\lambda}| + t\sum_{i=0}^{t-1}|\lambda_i|$, *and* $\lambda_{t-i} = \lambda_i'$, *for* $1 \le i \le t-1$.

The generating function identities equivalent to Bijection 3 is

$$\sum_{n=0}^{\infty} pd(n)q^{2n} = \frac{(-q^{2t};q^{2t})_\infty}{(q^{2t};q^{2t})_\infty^{(t-1)/2}} \sum_{n=0}^{\infty} add_t(n)q^n \qquad \text{for } t \text{ odd.} \tag{8.1a}$$

and

$$\sum_{n=0}^{\infty} pd(n)q^{2n} = \frac{(-q^{t};q^{t})_\infty}{(q^{2t};q^{2t})_\infty^{(t-2)/2}} \sum_{n=0}^{\infty} add_t(n)q^n \qquad \text{for } t \text{ even.} \tag{8.1b}$$

Bijection 4. *There is a bijection* $\phi_4 : DD_{t-core} \to \{\vec{n} = (n_0, n_1, \ldots, n_{t-1}) : n_i \in \mathbb{Z}, n_0 = 0, n_i = -n_{t-i}, 1 \le i \le t-1\}$, *where*

$$|\tilde{\lambda\lambda}| = t\|\vec{n}\|^2/2 + \vec{b}\cdot\vec{n}, \qquad \vec{b} = (0, 1, \cdots, t-1).$$

The generating function identity equivalent to Bijection 4 is

$$\sum_{n=0}^{\infty} add_t(n)q^n = \sum_{\vec{n}\in\mathbb{Z}^{\lfloor (t-1)/2\rfloor}} q^{t\|\vec{n}\|^2 + \vec{d}\cdot\vec{n}}. \tag{8.2}$$

where

$$\vec{d} = \begin{cases} (1, 3, \cdots, t-2) & \text{for } t \text{ odd,} \\ (2, 4, \cdots, t-2) & \text{for } t \text{ even.} \end{cases}$$

9. Remarks.

Naturally it is desirable to prove the general congruence for powers of 5, 7, and 11 found in [4, 20]. It would be natural to assume that the symmetry group of the quadratic form for $t = 5^a 7^b 11^c$ contains a cycle of the appropriate length. We have not been able to find this cycle. Our crank for $p(25n+24)$ was inspired by Bailey's ([6], [2, p. 465]) proof of Ramanujan's congruence for $p(5n+4)$. In fact Bailey's method also yields the congruence for $p(25n+24)$. It may be possible to extend our methods to obtain a crank for $p(49n+47)$ and $p(121n+116)$. First we need to find a crank for $a_7(49n+47)$ and $a_{11}(121n+116)$. Our present method does not succeed in these cases since there is no analog of (5.1). It may be easier to find a mod 49 crank for $p(49n+19)$, $p(49n+33)$, $p(49n+40)$ in view of Corollary 1(3),(4),(5).

In [11] the symmetry groups of the forms associated with $p(tn+r)$ for $t \leq 8$ and all r were computed. The symmetry groups for $p(11n+r)$ for $r \neq 6$ need to be found. There is reason to believe these groups are larger than C_2. It seems that an unusually high proportion of the coefficients of $a_{11}(n)$ are divisible by 11 so there may be more 11-cycles. A higher proportion of the coefficients are divisible by 5. In fact, using modular forms which appear in an analog of (5.4) for $t = 11$, we can prove the following theorem.

Theorem 8. *If* $\left(\frac{n}{11}\right) = 1$ *and* n *not divisble by* 5, *then* $a_{11}(n-5) = 0 \mod 5$.

We can prove by a bijection from (7.4) that

$$asc_7(4n+6) = asc_7(n). \tag{9.1}$$

Further, by utilizing Gauss' result for sums of three squares [12, p.311] it can be shown that

$$asc_7(n) = 0 \qquad \text{if and only if } n+2 = 4^a(8m+1). \tag{9.2}$$

(This observation was made by Doug McDoniel.)

Kolberg [15, 16] has given several infinite products for dissections of $p(n)$, $q(n)$ and $q_0(n)$. Many of these results (for $q(n)$ and $q_0(n)$) follow easily from (7.4), (8.2) and the Jacobi triple product formula.

John Stembridge informed the authors of the existence of [17, 18] for distinct partitions. It should be noted that the definition of t-core given in §8 for t-even does not agree with [17, 18].

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